Homework 7 Solutions

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Section 7.3, Problem 9(a): Find the first two iterations of the SOR method with $\omega = 1.1$ for the following linear system, using $\mathbf{x}^{(0)} = \mathbf{0}$:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned} \tag{1}$$

Solution: For general linear system $A\mathbf{x} = \mathbf{b}$ in n equations and n unknowns, we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Successive Over-Relaxation (SOR) method for such system can be written as

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right], \tag{2}$$

for some ω and $i = 1, 2, \ldots, n$.

For 3×3 system in (1), we have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

This linear system has the unique solution $\mathbf{x} = (0.0351, -0.2368, 0.6579)$.
For 3×3 system, equations in (2) are written and simplified as

$$\begin{aligned} x_1^{(k)} &= (1-\omega)x_1^{(k-1)} + \frac{\omega}{a_{11}}\left[b_1 - \sum_{j=2}^{3} a_{1j}x_j^{(k-1)}\right] \\ &= (1-\omega)x_1^{(k-1)} + \frac{\omega}{a_{11}}\left[b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)}\right], \\ x_2^{(k)} &= (1-\omega)x_2^{(k-1)} + \frac{\omega}{a_{22}}\left[b_2 - \sum_{j=1}^{1} a_{2j}x_j^{(k)} - \sum_{j=3}^{3} a_{2j}x_j^{(k-1)}\right] \\ &= (1-\omega)x_2^{(k-1)} + \frac{\omega}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k-1)}\right], \\ x_3^{(k)} &= (1-\omega)x_3^{(k-1)} + \frac{\omega}{a_{33}}\left[b_3 - \sum_{j=1}^{2} a_{3j}x_j^{(k)}\right] \\ &= (1-\omega)x_3^{(k-1)} + \frac{\omega}{a_{33}}\left[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}\right]. \end{aligned}$$

Plugging in the values into these equations, we obtain

$$
\begin{aligned}
x_1^{(k)} &= -0.1x_1^{(k-1)} + \frac{1.1}{3}\left[1 + x_2^{(k-1)} - x_3^{(k-1)}\right], \\
x_2^{(k)} &= -0.1x_2^{(k-1)} + \frac{1.1}{6}\left[-3x_1^{(k)} - 2x_3^{(k-1)}\right], \\
x_3^{(k)} &= -0.1x_3^{(k-1)} + \frac{1.1}{7}\left[4 - 3x_1^{(k)} - 3x_2^{(k)}\right].
\end{aligned}
$$

Using the initial condition $\mathbf{x}^{(0)} = \mathbf{0}$, the first iteration gives:

$$
\begin{aligned}
x_1^{(1)} &= -0.1x_1^{(0)} + \frac{1.1}{3}\left[1 + x_2^{(0)} - x_3^{(0)}\right] = 0.36666666666667, \\
x_2^{(1)} &= -0.1x_2^{(0)} + \frac{1.1}{6}\left[-3x_1^{(1)} - 2x_3^{(0)}\right] = -0.20166666666667, \\
x_3^{(1)} &= -0.1x_3^{(0)} + \frac{1.1}{7}\left[4 - 3x_1^{(1)} - 3x_2^{(1)}\right] = 0.55078571428571.
\end{aligned}
$$

The second iteration gives:

$$
\begin{aligned}
x_1^{(2)} &= -0.1x_1^{(1)} + \frac{1.1}{3}\left[1 + x_2^{(1)} - x_3^{(1)}\right] = 0.05410079365079, \\
x_2^{(2)} &= -0.1x_2^{(1)} + \frac{1.1}{6}\left[-3x_1^{(2)} - 2x_3^{(1)}\right] = -0.21154353174603, \\
x_3^{(2)} &= -0.1x_3^{(1)} + \frac{1.1}{7}\left[4 - 3x_1^{(2)} - 3x_2^{(2)}\right] = 0.64771586224490.
\end{aligned}
$$

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Section 7.3, Problem 17: The linear system

$$\begin{aligned} &2x_1 - x_2 + x_3 = -1, \\ &2x_1 + 2x_2 + 2x_3 = 4, \\ &-x_1 - x_2 + 2x_3 = -5 \end{aligned} \tag{3}$$

has the solution $(1, 2, -1)^T$.
a) Show that $\rho(T_j) = \frac{\sqrt{5}}{2} > 1$.
c) Show that $\rho(T_g) = \frac{1}{2}$.

Solution:
a) A general $n \times n$ linear system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Jacobi method is written in the form $\mathbf{x^{(k)}} = T\mathbf{x^{(k-1)}} + \mathbf{c}$ by splitting A. Let D be the diagonal matrix whose diagonal entries are those of A, $-L$ be the strictly lower-triangular part of A, and $-U$ be the strictly upper-triangular part of A. Hence,

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \\ &= D - L - U. \end{aligned}$$

The equation $A\mathbf{x} = \mathbf{b}$, or

$$(D - L - U)\mathbf{x} = \mathbf{b}, \tag{4}$$

is then transformed into

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b},$$

and, if D^{-1} exists,

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This results in the matrix form of the Jacobi iterative technique:

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}.$$

Introducing $T_j = D^{-1}(L + U)$ and $c_j = D^{-1}\mathbf{b}$ (here, "j" stands for "Jacobi"), the Jacobi method has the form

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j.$$

For system in (3), we have

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The inverse of D is

$$D^{-1} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

and

$$T_j = D^{-1}(L+U) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & -0.5 \\ -1 & 0 & -1 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$

The **spectral radius** $\rho(T_j)$ of matrix T_j is defined by

$$\rho(T_j) = \max|\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } T_j.$$

Eigenvalues of T_j are

$$\lambda_{1,2} = \pm\frac{\sqrt{5}}{2}i, \quad \lambda_3 = 0.$$

Thus, $\rho(T_j) = |\pm\frac{\sqrt{5}}{2}i| = \frac{\sqrt{5}}{2} > 1.$ ✓

c) Equation (4) can be written as

$$(D-L)\mathbf{x} = U\mathbf{x} + \mathbf{b},$$

which gives the Gauss-Seidel method:

$$(D-L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

or

$$\mathbf{x}^{(k)} = (D-L)^{-1}U\mathbf{x}^{(k-1)} + (D-L)^{-1}\mathbf{b}.$$

Introducing $T_g = (D-L)^{-1}U$ and $c_g = (D-L)^{-1}\mathbf{b}$ (here, "g" stands for "Gauss-Seidel"), the Gauss-Seidel method has the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g.$$

For system in (3), we have

$$D-L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}.$$

The inverse of $D-L$ is

$$(D-L)^{-1} = \begin{bmatrix} 0.5 & 0 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0.25 & 0.5 \end{bmatrix},$$

and

$$T_g = (D-L)^{-1}U = \begin{bmatrix} 0.5 & 0 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & -0.5 \end{bmatrix}.$$

Eigenvalues of T_g are

$$\lambda_{1,2} = -\frac{1}{2}, \quad \lambda_3 = 0.$$

Thus, $\rho(T_g) = \max\big(|\lambda_1|, |\lambda_2|, |\lambda_3|\big) = \frac{1}{2}.$ ✓

Section 7.3, Problem 21(b): Consider the following bounds:

$$||\mathbf{x}^{(k)} - \mathbf{x}|| \leq ||T||^k ||\mathbf{x}^{(0)} - \mathbf{x}|| \tag{5}$$

and

$$||\mathbf{x}^{(k)} - \mathbf{x}|| \leq \frac{||T||^k}{1 - ||T||} ||\mathbf{x}^{(1)} - \mathbf{x}^{(0)}||, \tag{6}$$

where T is an $n \times n$ matrix with $||T|| < 1$ and

$$\mathbf{x}^{(\mathbf{k})} = T\mathbf{x}^{(\mathbf{k-1})} + \mathbf{c}, \qquad k = 1, 2, \ldots,$$

with $\mathbf{x}^{(0)}$ arbitrary, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.
Apply the bounds to Exercise 2(c), if possible, using the L_∞ norm.

Solution: In Section 7.3, Exercise 2(c), we had

$$\begin{aligned}
\mathbf{x} &= (-0.75342, 0.041096, -0.28082, 0.69178), \\
\mathbf{x}^{(0)} &= (0, 0, 0), \\
\mathbf{x}^{(1)} &= (-0.5, -0.25, 0, 0.33333), \\
\mathbf{x}^{(2)} &= (-0.52083, -0.041667, -0.21667, 0.41667).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{x}^{(2)} - \mathbf{x} &= (0.2326, -0.0828, 0.0642, -0.2751), \\
\Rightarrow \quad ||\mathbf{x}^{(2)} - \mathbf{x}||_\infty &= 0.2751,
\end{aligned}$$

and

$$\begin{aligned}
||\mathbf{x}^{(0)} - \mathbf{x}||_\infty &= 0.75342, \\
||\mathbf{x}^{(1)} - \mathbf{x}^{(0)}||_\infty &= 0.5.
\end{aligned}$$

Also,

$$T = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{5} & 0 & -\frac{1}{5} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix} \quad \Rightarrow \quad ||T||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |t_{ij}| = 1.0,$$

$$\Rightarrow \quad ||T||_\infty^2 = 1.0.$$

$$\begin{aligned}
||T||_\infty^2 ||\mathbf{x}^{(0)} - \mathbf{x}||_\infty &= 1.0 \cdot 0.75342 = 0.75342, \\
\frac{||T||_\infty^2}{1 - ||T||_\infty} ||\mathbf{x}^{(1)} - \mathbf{x}^{(0)}|| &= \frac{1.0}{1 - 1} \cdot 0.5 = \infty.
\end{aligned}$$

Thus, both inequalities

$$||\mathbf{x}^{(2)} - \mathbf{x}||_\infty \leq ||T||_\infty^2 ||\mathbf{x}^{(0)} - \mathbf{x}||_\infty$$

and

$$||\mathbf{x}^{(2)} - \mathbf{x}||_\infty \leq \frac{||T||_\infty^2}{1 - ||T||_\infty} ||\mathbf{x}^{(1)} - \mathbf{x}^{(0)}||_\infty$$

hold. $\checkmark$

Section 7.3, Problem 22: Show that if A is strictly diagonally dominant, then $||T_j||_\infty < 1$.

Solution: The $n \times n$ matrix A is said to be strictly diagonally dominant when

$$|a_{ii}| > \sum_{j=1,\, j\neq i} |a_{ij}|, \tag{7}$$

holds for each $i = 1, 2, \ldots, n$.
Matrix T_j is defined as

$$T_j = D^{-1}(L+U),$$

where D is the diagonal matrix whose diagonal entries are those of A, $-L$ is the strictly lower-triangular part of A, and $-U$ is the strictly upper-triangular part of A:

$$\begin{aligned}
D &= \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}, \\
L &= \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix}, \\
U &= \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
T_j = D^{-1}(L+U) &= \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{a_{nn}} \end{bmatrix} \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{n-1,n} \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots & -\frac{a_{2n}}{a_{22}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & -\frac{a_{n-1,n}}{a_{n-1,n-1}} \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots & -\frac{a_{n,n-1}}{a_{nn}} & 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$||T_j||_\infty = \max_{1\le i\le n} \sum_{j=1,j\neq i}^{n} \left| -\frac{a_{ij}}{a_{ii}} \right| = \max_{1\le i\le n} \sum_{j=1,j\neq i}^{n} \frac{|a_{ij}|}{|a_{ii}|} = \max_{1\le i\le n} \left\{ \frac{1}{|a_{ii}|} \sum_{j=1,j\neq i}^{n} |a_{ij}| \right\}.$$

From (7), we have $\displaystyle \frac{1}{|a_{ii}|} \sum_{j=1,j\neq i}^{n} |a_{ij}| < 1$ for all i, and hence, $||T_j||_\infty < 1$. ✓

Section 7.4, Problem 2: Compute the condition numbers of the following matrices relative to $||\cdot||_\infty$.

a) $\begin{bmatrix} 0.03 & 58.9 \\ 5.31 & -6.10 \end{bmatrix}$;

c) $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

Solution:
The **condition number** of the nonsingular matrix A relative to a norm $||\cdot||$ is

$$K(A) = ||A|| \cdot ||A^{-1}||.$$

In particular, the condition number of A relative to $||\cdot||_\infty$ is

$$K(A) = ||A||_\infty \cdot ||A^{-1}||_\infty,$$

where

$$||A||_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

a) We have

$$A = \begin{bmatrix} 0.03 & 58.9 \\ 5.31 & -6.10 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 0.0195 & 0.1882 \\ 0.0170 & -0.0001 \end{bmatrix},$$

and

$$\begin{aligned} ||A||_\infty &= 58.93, \\ ||A^{-1}||_\infty &= 0.2077. \end{aligned}$$

Thus, the condition number of A relative to $||\cdot||_\infty$ is

$$K(A) = ||A||_\infty \cdot ||A^{-1}||_\infty = 58.93 \cdot 0.2077 = 12.24. \quad \checkmark$$

c) We have

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix},$$

and

$$\begin{aligned} ||A||_\infty &= 3.0, \\ ||A^{-1}||_\infty &= 4.0. \end{aligned}$$

Thus, the condition number of A relative to $||\cdot||_\infty$ is

$$K(A) = ||A||_\infty \cdot ||A^{-1}||_\infty = 3.0 \cdot 4.0 = 12.0. \quad \checkmark$$

Section 7.4, Problem 4(a): The following linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 0.03x_1 + 58.9x_2 &= 59.2, \\ 5.31x_1 - 6.10x_2 &= 47.0. \end{aligned}$$

has $\mathbf{x} = (10, 1)^T$ as the actual solution and $\tilde{\mathbf{x}} = (30, 0.990)^T$ as an approximate solution. Using the results of Exercise 2(a), compute

$$||\mathbf{x} - \tilde{\mathbf{x}}||_\infty \quad \text{and} \quad K_\infty(A)\frac{||\mathbf{b} - A\tilde{\mathbf{x}}||_\infty}{||A||_\infty}.$$

Solution: We have $\mathbf{x} - \tilde{\mathbf{x}} = (-20, 0.010)$, and thus,

$$||\mathbf{x} - \tilde{\mathbf{x}}||_\infty = 20. \quad \checkmark$$

Also,

$$\begin{aligned} \mathbf{b} - A\tilde{\mathbf{x}} &= \begin{bmatrix} 59.2 \\ 47.0 \end{bmatrix} - \begin{bmatrix} 0.03 & 58.9 \\ 5.31 & -6.10 \end{bmatrix} \begin{bmatrix} 30 \\ 0.990 \end{bmatrix} \\ &= \begin{bmatrix} 59.2 \\ 47.0 \end{bmatrix} - \begin{bmatrix} 59.211 \\ 153.261 \end{bmatrix} \\ &= \begin{bmatrix} -0.0110 \\ -106.261 \end{bmatrix}, \\ ||\mathbf{b} - A\tilde{\mathbf{x}}||_\infty &= 106.261, \\ ||A||_\infty &= \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| = 58.93. \end{aligned}$$

Thus,

$$K_\infty(A)\frac{||\mathbf{b} - A\tilde{\mathbf{x}}||_\infty}{||A||_\infty} = 12.24 \cdot \frac{106.261}{58.93} = 22.07. \quad \checkmark$$