REAL QUADRATIC ANALOGUES OF TRACES OF SINGULAR INVARIANTS

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1. Introduction

There are numerous connections between quadratic fields and modular forms. One of the most beautiful is provided by the theory of singular invariants, which are the values of the classical j-function

$$j(\tau) = q^{-1} + 744 + 196884\,q + 21493760\,q^2 + \cdots \qquad (q = e(\tau))$$

at imaginary quadratic irrationalities.[1] It was observed recently by Kaneko, and independently by the authors, that it is possible to define real quadratic analogues of singular invariants though cycle integrals of the j-function. Although their theory is still in its infancy, various results and conjectures about them have been given in [7] and [4]. In this paper we will explore them further and give some new applications of the results of [4]. In particular, we will express certain sums over classes ("traces") of these invariants as regularized inner products of weakly holomorphic modular forms of weight 3/2. We will also realize certain modular integrals of weight 2 having rational period functions as Shimura-type lifts of such forms. These occur when the discriminant is positive, which is a case not allowed in the usual theory of the Shimura lift for weight 3/2.

2. Statement of results

Unless otherwise specified, in this paper d always denotes a *discriminant*, which means that it is a non-zero integer with $d \equiv 0, 1 \pmod 4$. It is called *fundamental* if it is the discriminant of the number field $\mathbb{Q}(\sqrt{d})$. Every discriminant d is a unique square multiple of a fundamental discriminant. Suppose that d is a non-square discriminant. For each such d let $\mathcal{Q}(d)$ be the set of all complex numbers of the form

$$w = \frac{-b + \sqrt{d}}{2a}, \qquad \text{where} \qquad d = b^2 - 4ac$$

with relatively prime $a, b, c \in \mathbb{Z}$. When $d < 0$ we assume that $a > 0$ and that $\sqrt{d} \in \mathcal{H}$, the upper half-plane. The modular group $\Gamma = PSL(2, \mathbb{Z})$ splits $\mathcal{Q}(d)$ into equivalence classes through its natural linear fractional action. The set of classes $\Gamma\backslash\mathcal{Q}(d)$ forms a finite abelian group of order $h(d)$, which is the *class number.* The group operation can be defined through that of associated binary quadratic forms, where we associate to w the form $ax^2 + bxy + cy^2$. The identity class is represented by $\frac{\sqrt{d}}{2}$ if d is even and by $\frac{-1+\sqrt{d}}{2}$ if d is odd. The isotropy group $\Gamma_w = \{\gamma \in \Gamma; gw = w\}$ consists of all transformations

$$\gamma = \pm \begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2} \end{pmatrix} \tag{2.1}$$

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[1]The term singular moduli is often used for these, although that actually refers to values of the modulus $k(\tau)$ of an elliptic integral. Given the fundamental nature of these objects, it seems useful to retain the distinction made by some of the early writers (see [12]).

where (t, u) is an integral solution to the Pell equation $t^2 - du^2 = 4$. When $d < 0$ this group is trivial unless $d = -3, -4$, in which case it has order 2 or 3, respectively. When $d > 0$ it is infinite cyclic with generator in (2.1) coming from $t, u > 0$ with t minimal. In all cases the *regulator* $R(d)$, which is a certain co-volume of Γ_w, depends only on d and is given for any $w \in \mathcal{Q}(d)$ by

$$R(d) = \begin{cases} 2\pi(\#\Gamma_w)^{-1} & \text{if } d < 0 \\ 2\log \varepsilon_d & \text{if } d > 0, \end{cases}$$

where $\varepsilon_d = \frac{t+u\sqrt{d}}{2}$. If $d \neq 1$ is fundamental we have the elegant class number formula of Dirichlet

$$R(d)h(d) = |d|^{\frac{1}{2}} L(1, \chi_d), \tag{2.2}$$

where χ_d is the Kronecker symbol. For positive fundamental $d > 1$ the size of $R(d)$ is both erratic and mysterious as d varies. This makes the corresponding behavior of $h(d)$ even more inaccessible than that of $L(1, \chi_d)$.

It is useful to define the general Hurwitz function $h^*(d)$ for non-square discriminants d by

$$h^*(d) = \tfrac{1}{2\pi} \sum_{\ell^2 | d} R(d/\ell^2) h(d/\ell^2). \tag{2.3}$$

For $d < 0$ we have that $h^*(d) = H(|d|)$, where $H(n)$ is the usual Hurwitz class number. By convention $H(0) = -1/12$ and $H(n) = 0$ for $n \equiv 1, 2 \pmod 4$. Early on it was realized that there is a connection between $H(n)$ and modular forms of weight 3/2. Let θ be the classical Jacobi theta function

$$\theta(\tau) = 1 + 2q + 2q^4 + 2q^9 + \cdots, \tag{2.4}$$

which is a modular form of weight 1/2 for $\Gamma_0(4)$, the usual congruence subgroup of Γ. Here as usual $q = e(\tau) = e^{2\pi i \tau}$ for $\tau = x + iy \in \mathcal{H}$. Then

$$\theta(\tau)^3 = \sum_{n \geq 0} r_3(n) q^n = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + \cdots,$$

where $r_3(n)$ is the number of representations of n as the sum of three squares. Now $\theta(\tau)^3$ is a modular form of weight 3/2 for $\Gamma_0(4)$ and its coefficients are related to $H(n)$ by the famous result of Gauss, which states that for all $n \geq 0$ we have

$$r_3(n) = 12(H(4n) - 2H(n)).$$

Zagier [13] showed that if we allow a modular form to be harmonic we get a generating series for $H(n)$ itself. Specifically, the function

$$g_0(\tau) = \sum_{n \geq 0} H(n) q^n + y^{-1/2} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 y) q^{-n^2} \tag{2.5}$$

has weight 3/2 for $\Gamma_0(4)$, meaning it transforms like θ^3. Here

$$\beta(y) = \tfrac{1}{16\pi} \int_1^\infty t^{-3/2} e^{-yt} dt$$

for $y > 0$ is an incomplete gamma function. The Fourier coefficients of g_0 are supported on integers n with $n \equiv 0, 3 \pmod 4$, and there are no non-zero holomorphic forms of weight 3/2 for $\Gamma_0(4)$ with this property. If we allow poles in the cusps there are infinitely many and the space $M^!_{3/2}$ of all such forms has a natural basis also found by Zagier [14]. Its elements are parameterized by positive discriminants (squares allowed). The first one is given explicitly in terms of the usual modular forms E_4 and Δ by

$$g_1(\tau) = \theta(\tau + \tfrac{1}{2}) \frac{E_4(4\tau)}{\Delta(4\tau)^{1/4}} = q^{-1} - 2 + 248\, q^3 - 492\, q^4 + 4119\, q^7 - 7256\, q^8 + \cdots . \tag{2.6}$$

For each $d > 0$ there is a unique form $g_d \in M^!_{3/2}$ with Fourier series of the form

$$g_d(\tau) = q^{-d} + \sum_{0 \le n \equiv 0,3(4)} B(d,n) q^n. \tag{2.7}$$

As in the case $d = 1$, the coefficients $B(d,n)$ are all integers and Zagier discovered that for fixed $d > 0$ they are analogous to the Hurwitz class number $H(n)$. This is illustrated by the simplest case when $d = 1$, $n > 4$ and $-n$ is fundamental where

$$B(1,n) = \sum_{w \in \Gamma \backslash \mathcal{Q}_{-n}} j_1(w), \tag{2.8}$$

with $j_1 = j - 744$ being the j-function normalized to have constant term 0. The fact that $B(1,n)$ is an integer reflects the classical result that $j_1(w)$ is an algebraic integer and the the sum over $\Gamma \backslash \mathcal{Q}(-n)$ gives its algebraic trace. For every d and n such a formula holds for $B(n,d)$, but it involves character twists and more general modular functions.

In this paper we will consider the regularized Petersson inner product of two different g_d's. Recall that for two modular forms f, g of weight 3/2 for $\Gamma_0(4)$ with singularities only in the cusps we can define

$$\langle f, g \rangle = \lim_{Y \to \infty} \int_{\mathcal{F}_4(Y)} f(\tau) \overline{g(\tau)} y^{3/2} \tfrac{dx\, dy}{y^2} \tag{2.9}$$

where $\mathcal{F}_4(Y)$ is the standard truncated fundamental domain for $\Gamma_0(4)$ obtained by removing Y-neighborhoods of the cusps and given explicitly below. Of course this might not converge but when it does its value can be interesting.

Theorem 1. *For $d > 1$ fundamental and g_0, g_d defined in (2.5) and (2.7) we have*

$$\langle g_d, g_0 \rangle = -\tfrac{3}{4\pi} d^{-\frac{1}{2}} (\log \varepsilon_d)\, h(d).$$

In view of the class number formula (2.2) this can be written

$$L(1, \chi_d) = -\tfrac{8\pi}{3} \langle g_d, g_0 \rangle.$$

It is a pleasant surprise that the inner product of the "imaginary quadratic" functions g_0 and g_d contains real quadratic information! Note that it follows that g_0 and g_d are never orthogonal for $d > 1$ fundamental. Theorem 1 should be compared to the weight 1/2 result of Borcherds [3, Cor. 9.6 p. 530].

What about $\langle g_{d_1}, g_{d_2} \rangle$ when $d_1, d_2 > 0$? Looking at (2.8) for a clue, to give a formula here we first must show how to extend the domain of the modular function j_1 to include real quadratic numbers. This can be done by setting for $d > 0$

$$j_1(w) = j_1(\tfrac{-b+\sqrt{d}}{2a}) = \tfrac{1}{2R(d)} \int_{-\alpha}^{\alpha} j_1(-\tfrac{b}{2a} + \tfrac{i\sqrt{d}}{2|a|}\, e^{i\theta}) \tfrac{d\theta}{\cos\theta}, \tag{2.10}$$

where

$$\alpha = 2 \tan^{-1} (\tfrac{u\sqrt{d}}{t}),$$

with t, u defined below (2.1). As will be shown below, this extension j_1 is still Γ-invariant. It is obvious from (2.10) that for $w' = \frac{b+\sqrt{d}}{-2a}$ we have $j(w) = j(w') = \overline{j(-w)}$. This observation was first made by Kaneko in [7], who used a different looking but equivalent definition of $j(w)$.

Suppose that D is a fundamental discriminant. It is known that the real characters χ of $\Gamma\backslash\mathcal{Q}(D)$, the *genus characters*, are in one-to-one correspondence with distinct (unordered) factorizations $D = dd'$ of D into fundamental discriminants. The value of χ can be computed unambiguously from a representative $w = \frac{-b+\sqrt{D}}{2a}$ of $\Gamma\backslash\mathcal{Q}(D)$ by

$$\chi(w) = \begin{cases} (\frac{d}{a}) & \text{if } (a,d) = 1 \\ (\frac{d'}{a}) & \text{if } (a,d') = 1. \end{cases}$$

Clearly $\chi(w) = \chi(w') = \chi(-w)$. Using the definition of $j(w)$ for real quadratic w given above we have the following evaluation.

Theorem 2. *Suppose that d, d' are distinct positive fundamental discriminants. Then*

$$\langle g_d, g_{d'}\rangle = \tfrac{3}{4\pi} D^{-\frac{1}{2}} \log \varepsilon_D \sum_{w\in\Gamma\backslash\mathcal{Q}(D)} \chi(w)\, j_1(w),$$

where $D = dd'$ and χ is the corresponding genus character of $\Gamma\backslash\mathcal{Q}(D)$.

In particular we have

$$\langle g_d, g_1\rangle = \tfrac{3}{4\pi} d^{-\frac{1}{2}} \log \varepsilon_d \sum_{w\in\Gamma\backslash\mathcal{Q}(d)} j_1(w).$$

Theorems 1 and 2 together yield a kernel function for a Shimura-type lift that maps weakly holomorphic modular forms of weight 3/2 to modular integrals of weight 2 with rational period functions. Suppose that $d < 0$ is a *negative* fundamental discriminant. The d^{th} Shimura lift of g_1 is defined by

$$\mathscr{S}_d\, g_1(\tau) = \sum_{m,n>0} \chi_d(m) B(1, n^2|d|)\, q^{mn}. \tag{2.11}$$

A slight generalization of a result of Borcherds shows that $\mathscr{S}_d\, g_1(\tau)$ is a meromorphic modular form of weight 2 with a simple pole at each point $w \in \mathcal{Q}(d)$. To get the real quadratic analogue of this result we will define a kernel function of the same shape as that used by Kohnen and Zagier (see [10] and [9]) for the ordinary Shimura lift on cusp forms. As a d^{th} Shimura lift, we must use a "forbidden" positive discriminant d for weight 3/2. Consider the simplest case $d = 1$ and let

$$\Omega_M(z,\tau) = g_0(\tau) - \sum_{1\le m\le M} \sum_{n|m} n g_{n^2}(\tau) e(mz). \tag{2.12}$$

Then we have the following result.

Theorem 3. *For a positive fundamental discriminant d and $\tau \in \mathcal{H}$ the function $\mathscr{S}_1 g_d(\tau)$ defined by*

$$\mathscr{S}_1 g_d(\tau) \equiv \lim_{M\to\infty} \langle \tfrac{8\pi}{3} \Omega_M(\tau, \cdot), g_d\rangle$$

exists and is a holomorphic function on $\mathcal{H}$. It is a modular integral of weight 2 with a rational period function:

$$\mathscr{S}_1 g_d(\tau) - \tau^{-2} \mathscr{S}_1 g_d(-\tfrac{1}{\tau}) = 2d^{-\frac{1}{2}} \sum_{\substack{w\in\mathcal{Q}(d)\\ ww'<0}} \operatorname{sgn}(w)(\tau - w)^{-1}. \tag{2.13}$$

Note that the period function has simple poles occurring at certain points $w \in \mathcal{Q}(d)$, in analogy with the imaginary quadratic case. The existence of a modular integral having this rational period function was first shown by Knopp [8], but its connection with weakly holomorphic modular forms of half-integral weight was previously unknown. Furthermore, its Fourier coefficients have remained rather mysterious. We will see that the m^{th} Fourier coefficient of $\mathscr{S}_1 g_d(\tau)$ is given by

$$-d^{-\frac{1}{2}} R(d) \sum_{w \in \Gamma \backslash \mathcal{Q}(d)} j_1 | T_m(w),$$

where T_m is the usual weight zero Hecke operator. We remark that Theorems 1–3 can be generalized to include some non-fundamental discriminants.

3. Preliminaries

First we will review some basic facts about modular forms. Recall the Jacobi theta function $\theta(\tau)$, which was defined in (2.4). Set

$$j_\theta(\gamma, \tau) = \theta(\gamma\tau)/\theta(\tau) \quad \text{for } \gamma \in \Gamma_0(4). \tag{3.1}$$

As usual, for non-zero $z \in \mathbb{C}$ and $v \in \mathbb{R}$ we define $z^v = |z|^v \exp(iv \arg z)$ with $\arg z \in (-\pi, \pi]$. We have the explicit evaluation [11, p. 447]

$$j_\theta(\gamma, \tau) = (c\tau + a)^{1/2} \varepsilon_a^{-1} \left(\tfrac{c}{a}\right) \quad \text{for} \quad \gamma = \pm \left(\begin{smallmatrix} * & * \\ c & a \end{smallmatrix}\right) \in \Gamma_0(4), \tag{3.2}$$

where $\left(\frac{c}{a}\right)$ is the extended Kronecker symbol and

$$\varepsilon_a = \begin{cases} 1 & \text{if } a \equiv 1 \pmod 4 \\ i & \text{if } a \equiv 3 \pmod 4. \end{cases}$$

For $k \in \frac{1}{2}\mathbb{Z}$ say that f defined on $\mathcal{H}$ has weight k for $\Gamma_0(4)$ (or just has weight k, when the group is clear) if

$$f(\gamma\tau) = j_\theta(\gamma, \tau)^{2k} f(\tau) \tag{3.3}$$

for all $\gamma \in \Gamma_0(4)$. For $k \in 2\mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathbb{R})$ it is usual to write

$$(f|_k \gamma)(\tau) = (c\tau + d)^{-k} f(\tau)$$

and for such k we see from (3.2) that f has weight k for $\Gamma_0(4)$ if and only if $f|_k \gamma = f$ for all $\gamma \in \Gamma_0(4)$.

The group $\Gamma_0(4)$ has 3 inequivalent cusps represented by $i\infty, 0$ and $1/2$. Let $\mathcal{F}_4$ be the fundamental domain for $\Gamma_0(4)$ shown in Figure 1. Let $\mathcal{F}_4(Y)$ be this domain truncated at cusp $i\infty$ by the line $\mathrm{Im}(\tau) = Y$, at cusp $1/2$ by the circle $|\tau - (\frac{1}{2} + \frac{i}{8Y})| = \frac{1}{8Y}$, and at cusp 0 by the circle $|\tau - \frac{i}{8Y}| = \frac{1}{8Y}$. Consider the scaling matrices in $\sigma_0, \sigma_{1/2} \in \mathrm{PSL}(2, \mathbb{R})$ given by

$$\sigma_0 = \pm \begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_{\frac{1}{2}} = \pm \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

One checks that σ_0 maps $i\infty$ to 0 and $\mathrm{Im}(\tau) = Y$ to $|\tau - \frac{i}{8Y}| = \frac{1}{8Y}$ and that $\sigma_{1/2}$ maps $i\infty$ to $1/2$ and $\mathrm{Im}(\tau) = Y$ to $|\tau - (\frac{1}{2} + \frac{i}{8Y})| = \frac{1}{8Y}$, both with unchanged orientation. The next result follows easily.

Lemma 1. *Suppose that $f : \mathcal{H} \to \mathbb{C}$ is continuous and that $f|_2 \gamma = f$ for all $\gamma \in \Gamma_0(4)$. Then for $Y \geq 2$ we have*

$$\int_{\partial \mathcal{F}_4(Y)} f(\tau) d\tau = -\int_{-1/2+iY}^{1/2+iY} \Big(f(\tau) + f|_2 \sigma_0(\tau) + f|_2 \sigma_{\frac{1}{2}}(\tau) \Big) \, d\tau,$$

where the first integral is taken in the positive direction around the boundary of $\mathcal{F}_4(Y)$.

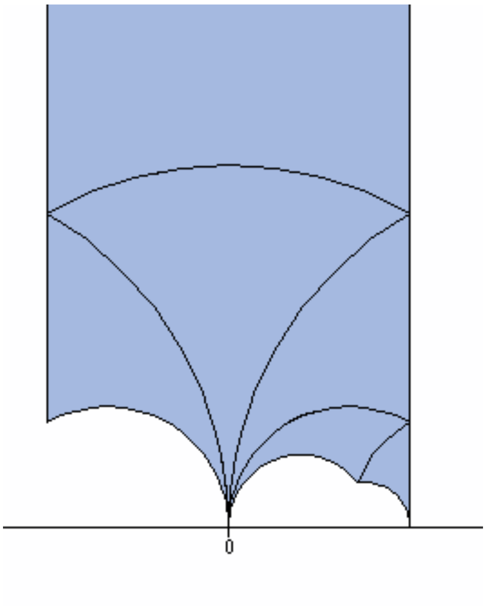

FIGURE 1. The fundamental domain $\mathcal{F}_4$ for $\Gamma_0(4)$.

If f of weight k for $\Gamma_0(4)$ is smooth, for example, it will have a Fourier expansion in each cusp. For the cusp at $i\infty$ we have *the* Fourier expansion

$$f(\tau) = \sum_n a(n,y)e(nx) \tag{3.4}$$

which, if f is holomorphic, has $a(n,y) = a(n)e(niy)$. Set

$$f^{\rm e}(\tau) = \sum_{n\equiv 0(2)} a(n, \tfrac{y}{4})e(\tfrac{nx}{4}) \quad \text{and} \quad f^{\rm o}(\tau) = \sum_{n\equiv 1(2)} a(n, \tfrac{y}{4})e(\tfrac{n}{8})e(\tfrac{nx}{4}). \tag{3.5}$$

Suppose that $k \in \frac{1}{2} + \mathbb{Z}$ and that the Fourier coefficients $a(n,y)$ satisfy the *plus space condition*, meaning that they vanish unless $(-1)^{k-1/2}n \equiv 0, 1 \pmod 4$ An easy extension of arguments given in [10, p.190] shows that such an f satisfies

$$(\tfrac{2\tau}{i})^{-k} f(-\tfrac{1}{4\tau}) = \alpha f^{\rm e}(\tau) \qquad \text{and} \qquad (\tfrac{2\tau+1}{i})^{-k} f(\tfrac{\tau}{2\tau+1}) = \alpha f^{\rm o}(\tau) \tag{3.6}$$

where

$$\alpha = (-1)^{\lfloor \frac{2k+1}{4} \rfloor} 2^{-k+\frac{1}{2}}.$$

In particular, the behavior of such an f at the cusps 0 and 1/2 is determined by that at $i\infty$. Thus to check that a form is weakly holomorphic, meaning it is holomorphic on $\mathcal{H}$ and meromorphic in the cusps, one only needs look at the Fourier expansion at $i\infty$. As is now standard, we denote by $M_k^!$ the space of weakly holomorphic modular forms of weight k for $\Gamma_0(4)$ whose Fourier coefficients satisfy the plus space condition.

We need the differential operator ξ_k defined for any $k \in \mathbb{R}$ by

$$\xi_k f(\tau) = 2iy^k \overline{f_{\bar\tau}(\tau)}. \tag{3.7}$$

Clearly $\xi_k f = 0$ if and only if f is holomorphic. The operator ξ is related to the weight k Laplacian via

$$-\xi_{2-k} \circ \xi_k = \Delta_k = -y^2(\partial_x^2 + \partial_y^2) + iyk(\partial_x + i\partial_y). \tag{3.8}$$

When $k \in \frac{1}{2}\mathbb{Z}$ it is readily checked that if f has weight k for $\Gamma_0(4)$ then $\xi_k f$ has weight $2-k$ for $\Gamma_0(4)$.

Lemma 2. *Suppose $k \in \frac{1}{2}+\mathbb{Z}$ and that g is holomorphic on $\mathcal{H}$ of weight k for $\Gamma_0(4)$ whose Fourier expansion satisfies the plus space condition. Suppose that h is a smooth function of weight $2-k$ for $\Gamma_0(4)$ whose Fourier expansion satisfies the plus space condition for weight $2-k$. Then, for $Y \geq 2$ we have*

$$\int_{\mathcal{F}_4(Y)} g(\tau)\overline{\xi_{2-k}h(\tau)}y^k \,\tfrac{dxdy}{y^2} = \int_{-1/2+iY}^{1/2+iY} \big(g(\tau)h(\tau) + \tfrac{1}{2}g^{\mathrm{e}}(\tau)h^{\mathrm{e}}(\tau) + \tfrac{1}{2}g^{\mathrm{o}}(\tau)h^{\mathrm{o}}(\tau)\big)\, d\tau.$$

Proof. By (3.7) and the identity $d\tau\, d\bar{\tau} = 2idx\,dy$ we have

$$\int_{\mathcal{F}_4(Y)} g(\tau)\overline{\xi_{2-k}h(\tau)}y^k \,\tfrac{dxdy}{y^2} = \int_{\mathcal{F}_4(Y)} g(\tau)h_{\bar{\tau}}(\tau)\, d\tau\, d\bar{\tau}.$$

We now apply Stoke's theorem in the form

$$\int_{\mathcal{F}_4(Y)} g(\tau)h_{\bar{\tau}}(\tau) + h(\tau)g_{\bar{\tau}}(\tau)\, d\tau\, d\bar{\tau} = -\oint_{\partial\mathcal{F}_4(Y)} g(\tau)h(\tau)d\tau,$$

in which the second term of the first integral vanishes as g is holomorphic. Now $f = gh$ has weight 2 for $\Gamma_0(4)$ and so satisfies $f|_2\gamma = f$ for all $\gamma \in \Gamma_0(4)$. Thus by Lemma 1 we have

$$\int_{\mathcal{F}_4(Y)} g(\tau)\overline{\xi_{2-k}h(\tau)}y^k \,\tfrac{dxdy}{y^2} = \int_{-1/2+iY}^{1/2+iY} \Big(f(\tau) + f|_2\sigma_0(\tau) + f|_2\sigma_{\frac{1}{2}}(\tau)\Big)\, d\tau.$$

We are reduced to proving the following identities, which follow easily from (3.6).

$$(gh)|_2\,\sigma_0 = \tfrac{1}{2}\, g^{\mathrm{e}}h^{\mathrm{e}}$$
$$(gh)|_2\,\sigma_{\frac{1}{2}} = \tfrac{1}{2}\, g^{\mathrm{o}}h^{\mathrm{o}}.$$

□

4. Inner products

First we prove Theorem 1. In fact we have a more general result. Recall the general Hurwitz number $h^*(d)$ defined in (2.3). Theorem 1 is a special case of the following result.

Proposition 1. *Suppose that d is a positive non-square discriminant. Then*

$$\langle g_d, g_0\rangle = -\tfrac{3}{4}d^{-\frac{1}{2}}h^*(d).$$

Proof. Recall the function g_0, which was defined in (2.5). It was shown in [4] that there is a real analytic function $h(\tau)$ having weight 1/2 for $\Gamma_0(4)$ with

(4.1) $$\xi_{\frac{1}{2}}\, h(\tau) = -2g_0(\tau).$$

Let $\mathcal{P}$ denote the set of all positive non-square discriminants and $\mathcal{P}^c$ the rest of the discriminants. The Fourier expansion of h can be written

(4.2) $$h(\tau) = \sum_{n\in\mathcal{P}} n^{-\frac{1}{2}}h^*(n)q^n + \sum_{n\in\mathcal{P}^c} a(n,y)e(nx)$$

where the function defined by the second sum is $\ll y^{1/2}$ for $y \geq 2$. By Lemma 2 we have

$$\langle g_d, g_0\rangle = -\tfrac{1}{2}\lim_{Y\to\infty}\int_{-1/2+iY}^{1/2+iY} \big(g_d(\tau)h(\tau) + \tfrac{1}{2}g_d^{\mathrm{e}}(\tau)h^{\mathrm{e}}(\tau) + \tfrac{1}{2}g_d^{\mathrm{o}}(\tau)h^{\mathrm{o}}(\tau)\big)\, d\tau.$$

Since $g_d(\tau) = q^{-d} + O(q)$ from (2.7) when d is not a square, we easily deduce Proposition 1 upon using (3.5) to check that when d is even we get a contribution from $g_d^{\mathrm{e}}(\tau)h^{\mathrm{e}}(\tau)$ while when d is odd we get one from $g_d^{\mathrm{o}}(\tau)h^{\mathrm{o}}(\tau)$. □

It is easy to check that

$$\xi_{\frac{3}{2}}\, g_0(\tau) = -\tfrac{1}{16\pi}\theta(\tau).$$

Thus by (3.8) and (4.1) we see that h satisfies the inhomogeneous equation

$$\Delta_{\frac{1}{2}} h(\tau) = -\tfrac{1}{8\pi}\theta(\tau).$$

In order to prove Theorems 2 and 3 we need to employ *weakly harmonic* modular forms of weight $1/2$ for $\Gamma_0(4)$. Suppose that f is a real analytic function on $\mathcal{H}$ of weight k for $\Gamma_0(4)$ that is harmonic on $\mathcal{H}$ in the sense that

$$\Delta_k f = 0.$$

Such an f will have a Fourier expansion at $i\infty$ each of whose terms has at most linearly exponential growth. Such an f is called weakly harmonic if it has only finitely many such terms. If the Fourier expansion satisfies the plus space condition then by (3.6) its growth in the other cusps is at most linearly exponential. The space of all such forms is denoted by $H^!_k$. It is clear from (3.8) that $M^!_k \subset H^!_k$.

It follows from [4, Thm 1] that $H^!_{1/2}$ has a natural basis $\{h_d\}_{d\equiv 0,1(4)}$ where $\{h_d\}_{d\leq 0}$ is the *Borcherds basis* for $M^!_{1/2}$ with $h_d(\tau) = q^{|}d| + \mathrm{O}(q)$ for each $d \leq 0$. For $d > 0$ we have that h_d satisfies $\xi_{\frac{1}{2}} h_d = 2d^{\frac{1}{2}} g_d$ and its Fourier expansion has the form

$$h_d(\tau) = a_d(d, y)e(dx) - 4y^{\frac{1}{2}}\delta_{\square,d} + \sum_{\substack{0<n\equiv 0,1(4)\\ n\neq d}} n^{-\frac{1}{2}} a_d(n)q^n + \sum_{\substack{n\equiv 0,1(4)\\ n<0}} a_d(n, y)e(nx)$$

where $a_d(d, y) \sim \frac{1}{2\pi}(dy)^{-\frac{1}{2}} e^{2\pi dy}$ as $y \to \infty$ is the lone exponentially growing term and the function defined by the second sum is bounded for $y \geq 2$. An argument like that in the previous proof yields the following result.

Proposition 2. *Suppose that n, d are distinct positive discriminants, not both squares. Then*

$$\langle g_n, g_d\rangle = \tfrac{3}{4}(nd)^{-\frac{1}{2}} a_d(n).$$

It was shown in [4] that $a_d(n) = a_n(d)$ so it follows that $\langle g_n, g_d\rangle \in \mathbb{R}$.

If $d = n$ or both are squares the regularized inner product will not be finite. However, one can isolate the growing terms and subtract them in order to compute $a_d(n)$.

5. Values of modular functions at real quadratic numbers

Now we apply more results of [4] to deduce Theorems 2 and 3. For any modular function $f \in \mathbb{C}[j]$ we define $f(w)$ for $w \in \mathcal{Q}(d)$ with $d > 0$ not a square by

$$f\left(\tfrac{-b+\sqrt{d}}{2a}\right) = \tfrac{1}{2R(d)} \int_{-\alpha}^{\alpha} f\left(-\tfrac{b}{2a} + \tfrac{i\sqrt{d}}{2|a|}\, e^{i\theta}\right) \tfrac{d\theta}{\cos\theta}, \tag{5.1}$$

where $\alpha = 2\tan^{-1}\left(\frac{u\sqrt{d}}{t}\right)$ and $t, u \in \mathbb{Z}^+$ solve $t^2 - du^2 = 4$ with u minimal. If $w = \frac{-b+\sqrt{d}}{2a}$, a computation shows that for $Q(x, y) = ax^2 + bxy + cy^2$ we have

$$R(d)f(w) = \int_{C_Q} f(\tau)d\tau_Q \qquad \text{where} \qquad d\tau_Q = \tfrac{\sqrt{d}}{Q(\tau,1)}d\tau$$

and C_Q is any smooth curve from $z = -\frac{b}{2a} + \frac{i\sqrt{d}}{2|a|} \in \mathcal{H}$ to $g_Q z$, where

$$g_Q = \pm\begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2}\end{pmatrix}.$$

It follows from [4] that $f(\gamma w) = f(w)$ for all $\gamma \in \Gamma$. Observe that under this extension a constant function remains constant. It is well-known that $\mathbb{C}[j]$, has a unique basis $\{j_m\}_{m\geq 0}$ of the form

$$j_m(\tau) = q^{-m} + \sum_{n\geq 1} c_m(n)q^n. \tag{5.2}$$

This j_m can be obtained from j_1 by applying the m-th Hecke operator T_m or defined recursively. The following is a special case of the main result of [4].

Proposition 3. *Suppose that d, d' are distinct positive fundamental discriminants and that $m \geq 1$. Then*

$$\sum_{n|m} \chi_d(\tfrac{m}{n}) a_{d'}(n^2 d) = \tfrac{R(D)}{2\pi} \sum_{w\in\Gamma\backslash\mathcal{Q}(D)} \chi(w) j_m(w), \tag{5.3}$$

where $D = dd'$ and χ is the associated genus character.

Theorem 2 now follows from Proposition 2 and Proposition 3 with $m = 1$.

Turning now to the proof of Theorem 3, recall that we defined

$$\Omega_M(z,\tau) = g_0(\tau) - \sum_{1\leq m\leq M} \sum_{n|m} n g_{n^2}(\tau) e(mz).$$

By Propositions 1 and 2 we have for any $d > 0$ not a square that

$$\tfrac{4}{3} d^{\frac{1}{2}} \langle \Omega_M(z,\cdot), g_d\rangle = -h^*(d) - \sum_{1\leq m\leq M} \sum_{n|m} a_d(n^2) e(mz). \tag{5.4}$$

It follows from Proposition 3 that for any $m \geq 1$ and fundamental $d > 0$

$$\sum_{n|m} a_d(n^2) = \tfrac{R(d)}{2\pi} \sum_{w\in\Gamma\backslash\mathcal{Q}(d)} j_m(w).$$

Using this together with (5.4) we obtain

$$\tfrac{8\pi\sqrt{d}}{3} \langle \Omega_M(\tau,\cdot), g_d\rangle = -R(d) \sum_{0\leq m\leq M} \sum_{w\in\Gamma\backslash\mathcal{Q}(d)} j_m(w) q^m.$$

It follows from [4, Prop. 9] that the sum on the right hand side converges to the holomorphic function

$$F_d(\tau) = -R(d) \sum_{m\geq 0} \sum_{w\in\Gamma\backslash\mathcal{Q}(d)} j_m(w) q^m \tag{5.5}$$

as $M \to \infty$. In order to compute these values we integrate term by term in Faber's generating function:

$$\sum_{m\geq 0} j_m(z) q^m = \frac{j'(\tau)}{j(z) - j(\tau)}, \quad \text{where} \quad j'(\tau) = \frac{1}{2\pi i} \frac{dj}{d\tau}, \tag{5.6}$$

which converges uniformly on compact subsets in $\{z \in \mathcal{H};\ \mathrm{Im}\, z < \mathrm{Im}\, \tau\}$ for fixed $\tau \in \mathcal{H}$ (see e.g. [1] or [5]). This gives a representation of F_d as a certain sum of contour integrals. By means of an elementary argument involving deformations of these contours, we prove in [4] that for any positive non-square discriminant d the function F_d satisfies

$$F_d(\tau) - \tau^{-2} F_d(-\tfrac{1}{\tau}) = 2\sqrt{d} \sum_{\substack{c<0<a \\ b^2-4ac=d}} (a\tau^2 + b\tau + c)^{-1},$$

from which Theorem 3 follows.

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