7.2 The Continuous Case. The simplest extension of the minimax theorem to the continuous case is to assume that $X$ and $Y$ are compact subsets of Euclidean spaces, and that $A(x, y)$ is a continuous function of $x$ and $y$. To conclude that optimal strategies for the players exist, we must allow arbitrary distribution functions on $X$ and $Y$. Thus if $X$ is a compact subset of $m$-dimensional space $\mathcal{R}^m$, $X^*$ is taken to be the set of all distributions on $\mathcal{R}^m$ that give probability 0 to the complement of $X$. Similarly if $Y$ is $n$-dimensional, $Y^*$ is taken to be the set of all distributions on $\mathcal{R}^n$ giving weight 0 to the complement of $Y$. Then $A$ is extended to be defined on $X^* \times Y^*$ by

$$A(P, Q) = \int \int A(x, y) \, dP(x) \, dQ(y)$$

**Theorem 7.2.** If $X$ and $Y$ are compact subsets of Euclidean space and if $A(x, y)$ is a continuous function of $x$ and $y$, then the game has a value, $v$, and there exist optimal strategies for the players, that is, there is a $P_0 \in X^*$ and a $Q_0 \in Y^*$ such that

$$A(P, Q_0) \leq v \leq A(P_0, Q)$$

for all $P \in X^*$ and $Q \in Y^*$.

**Example 1.** Suppose Player I chooses $0 \leq x \leq 1$ and Player II choose $0 \leq y \leq 1$ and the payoff is $A(x, y) = g(|x - y|)$ where $g(z)$ is a continuous function defined on $[0, 1]$ such that $g(z) = g(1 - z)$. Examples of such $g$ are $g(z) = z(1 - z)$, $g(z) = \sin(\pi z)$, and $g(z) = |z - 1/2|$.

Here, $X = Y = [0, 1]$, and $X^* = Y^*$ is the set of probability distributions on the unit interval. Since $X$ and $Y$ are compact and $A(x, y)$ is continuous on $[0, 1]^2$, we have by Theorem 7.2, that the game has a value and the players have optimal strategies. Let us check that the optimal strategies for both players is the uniform distribution on $[0, 1]$. If Player II uses a uniform on $[0, 1]$ to choose $y$ and Player I uses the pure strategy $x \in [0, 1]$, the expected payoff to Player I is

$$\int_0^1 g(|x - y|) \, dy = \int_0^x g(x - y) \, dy + \int_x^1 g(y - x) \, dy$$

$$= \int_0^x g(1 - x + y) \, dy + \int_0^{1-x} g(z) \, dz$$

$$= \int_{1-x}^1 g(z) \, dz + \int_0^{1-x} g(z) \, dz = \int_0^1 g(z) \, dz$$

Since this is independent of $x$, Player II’s strategy is an equalizer strategy, guaranteeing her an average loss of at most $\int_0^1 g(z) \, dz$. Clearly, the same analysis gives Player I at least this amount if he chooses $x$ at random according to a uniform distribution on $[0, 1]$. So these strategies are optimal and the value is $v = \int_0^1 g(z) \, dz$. It may be noticed that this example is a continuous version of a Latin square game.

**A One-Sided Minimax Theorem.** In the way that Theorem 7.1 generalized the finite minimax theorem, we would like to generalize Theorem 7.2 to the case where $X$ is Euclidean, while allowing $y$ to be arbitrary. We can do this if we keep the compactness condition for Player I and assume that $A(x, y)$ is a continuous function of $x$ for all $y$. And even this can be weakened to assuming only that $A(x, y)$ is an upper semi-continuous function of $x$ for all $y$. 

II – 1
**Theorem 7.3.** If $X$ is a compact subset of Euclidean space, and if $A(x, y)$ is an upper semi-continuous function of $x$ for all $y \in Y$ and if $A$ is bounded below (or if $Y^*$ is the set of finite mixtures), then the game has a value, Player I has an optimal strategy in $X^*$, and for every $\epsilon > 0$ Player II has an $\epsilon$-optimal strategy giving weight to a finite number of points.

Similarly from Player II’s viewpoint, if $Y$ is a compact subset of Euclidean space, and if $A(x, y)$ is an lower semi-continuous function of $y$ for all $x \in X$ and if $A$ is bounded above (or if $X^*$ is the set of finite mixtures), then the game has a value and Player II has an optimal strategy in $Y^*$.

**Example 2.** Player I chooses a number in $[0,1]$ and Player II tries to guess what it is. Player I wins 1 if Player II’s guess is off by at least 1/3; otherwise, there is no payoff.

Thus, $X = Y = [0,1]$, and $A(x, y) = \begin{cases} 1 & \text{if } |x - y| \geq 1/3 \\ 0 & \text{if } |x - y| < 1/3. \end{cases}$ Although the payoff function is not continuous, it is upper semi-continuous in $x$ for every $y \in Y$. Thus the game has a value and Player I has an optimal mixed strategy.

If we change the payoff so that Player I wins 1 if Player II’s guess is off by more than 1/3, then $A(x, y) = \begin{cases} 1 & \text{if } |x - y| > 1/3 \\ 0 & \text{if } |x - y| \leq 1/3. \end{cases}$ This is no longer upper semi-continuous in $x$ for fixed $y$; instead it is lower semi-continuous in $y$ for each $x \in X$. This time, the game has a value and Player II has an optimal mixed strategy.

**Exercise 4.** Solve the two games of Example 2. Hint: Use domination to remove some pure strategies.

**Solution.** 4. (a) For the upper semi-continuous payoff, the value is 1/2. An optimal strategy for Player I is to choose 0 and 1 with probability 1/2 each. For any $0 < \epsilon < 1/6$, an optimal strategy for Player II is choose $1/3 - \epsilon$ and $2/3 + \epsilon$ with probability 1/2 each.

(b) For the lower-semi continuous payoff, the value is 1/2. An optimal strategy for Player II is to choose 1/3 and 2/3 with probability 1/2 each. Player I has an optimal strategy here too. It is the same as above, namely, to choose 0 and 1 with probability 1/2 each.