7.2 The Continuous Case. The simplest extension of the minimax theorem to the continuous case is to assume that X and Y are compact subsets of Euclidean spaces, and that $A(x,y)$ is a continuous function of x and y. To conclude that optimal strategies for the players exist, we must allow arbitrary distribution functions on X and Y. Thus if X is a compact subset of m-dimensional space $\mathcal{R}^m$, X^* is taken to be the set of all distributions on $\mathcal{R}^m$ that give probability 0 to the complement of X. Similarly if Y is n-dimensional, Y^* is taken to be the set of all distributions on $\mathcal{R}^n$ giving weight 0 to the complement of Y. Then A is extended to be defined on $X^* \times Y^*$ by

$$A(P,Q) = \int\int A(x,y)\,dP(x)\,dQ(y)$$

Theorem 7.2. *If X and Y are compact subsets of Euclidean space and if $A(x,y)$ is a continuous function of x and y, then the game has a value, v, and there exist optimal strategies for the players, that is, there is a $P_0 \in X^*$ and a $Q_0 \in Y^*$ such that*

$$A(P,Q_0) \le v \le A(P_0,Q) \qquad \text{for all } P \in X^* \text{ and } Q \in Y^*.$$

Example 1. Suppose Player I chooses $0 \le x \le 1$ and Player II choose $0 \le y \le 1$ and the payoff is $A(x,y) = g(|x-y|)$ where $g(z)$ is a continuous function defined on $[0,1]$ such that $g(z) = g(1-z)$. Examples of such g are $g(z) = z(1-z)$, $g(z) = \sin(\pi z)$, and $g(z) = |z - \frac{1}{2}|$.

Here, $X = Y = [0,1]$, and $X^* = Y^*$ is the set of probability distributions on the unit interval. Since X and Y are compact and $A(x,y)$ is continuous on $[0,1]^2$, we have by Theorem 7.2, that the game has a value and the players have optimal strategies. Let us check that the optimal strategies for both players is the uniform distribution on $[0,1]$. If Player II uses a uniform on [0,1] to choose y and Player I uses the pure strategy $x \in [0,1]$, the expected payoff to Player I is

$$\begin{aligned}\int_0^1 g(|x-y|)\,dy &= \int_0^x g(x-y)\,dy + \int_x^1 g(y-x)\,dy \\ &= \int_0^x g(1-x+y)\,dy + \int_0^{1-x} g(z)\,dz \\ &= \int_{1-x}^1 g(z)\,dz + \int_0^{1-x} g(z)\,dz = \int_0^1 g(z)\,dz\end{aligned}$$

Since this is independent of x, Player II's strategy is an equalizer strategy, guaranteeing her an average loss of at most $\int_0^1 g(z)\,dz$. Clearly, the same analysis gives Player I at least this amount if he chooses x at random according to a uniform distribution on [0,1]. So these strategies are optimal and the value is $v = \int_0^1 g(z)\,dz$. It may be noticed that this example is a continuous version of a Latin square game. ■

A One-Sided Minimax Theorem. In the way that Theorem 7.1 generalized the finite minimax theorem, we would like to generalize Theorem 7.2 to the case where X is Euclidean, while allowing y to be arbitrary. We can do this if we keep the compactness condition for Player I and assume that $A(x,y)$ is a continuous function of x for all y. And even this can be weakened to assuming only that $A(x,y)$ is an upper semi-continuous function of x for all y.

Theorem 7.3. *If X is a compact subset of Euclidean space, and if $A(x,y)$ is an upper semi-continuous function of x for all $y \in Y$ and if A is bounded below (or if Y^* is the set of finite mixtures), then the game has a value, Player I has an optimal strategy in X^*, and for every $\epsilon > 0$ Player II has an ϵ-optimal strategy giving weight to a finite number of points.*

Similarly from Player II's viewpoint, if Y is a compact subset of Euclidean space, and if $A(x,y)$ is an lower semi-continuous function of y for all $x \in X$ and if A is bounded above (or if X^* is the set of finite mixtures), then the game has a value and Player II has an optimal strategy in Y^*.

Example 2. Player I chooses a number in [0,1] and Player II tries to guess what it is. Player I wins 1 if Player II's guess is off by at least 1/3; otherwise, there is no payoff.

Thus, $X = Y = [0,1]$, and $A(x,y) = \begin{cases} 1 & \text{if } |x-y| \geq 1/3 \\ 0 & \text{if } |x-y| < 1/3 \end{cases}$. Although the payoff function is not continuous, it is upper semi-continuous in x for every $y \in Y$. Thus the game has a value and Player I has an optimal mixed strategy.

If we change the payoff so that Player I wins 1 if Player II's guess is off by more than 1/3, then $A(x,y) = \begin{cases} 1 & \text{if } |x-y| > 1/3 \\ 0 & \text{if } |x-y| \leq 1/3 \end{cases}$. This is no longer upper semi-continuous in x for fixed y; instead it is lower semi-continuous in y for each $x \in X$. This time, the game has a value and Player II has an optimal mixed strategy.

Exercise 4. Solve the two games of Example 2. Hint: Use domination to remove some pure strategies.

Solution. 4. (a) For the upper semi-continuous payoff, the value is 1/2. An optimal strategy for Player I is to choose 0 and 1 with probability 1/2 each. For any $0 < \epsilon < 1/6$, an optimal strategy for Player II is choose $1/3 - \epsilon$ and $2/3 + \epsilon$ with probability 1/2 each.

(b) For the lower-semi continuous payoff, the value is 1/2. An optimal strategy for Player II is to choose 1/3 and 2/3 with probability 1/2 each. Player I has an optimal strategy here too. It is the same as above, namely, to choose 0 and 1 with probability 1/2 each.