Stat 200C, Spring 2009 Ferguson

Solutions to Exercise Set 2.

3.4. (a) Since $\sum_0^\infty \binom{r+x-1}{x}(1-p)^r p^x = 1$, we have $\sum_0^\infty \binom{r+x-1}{x} p^x = (1-p)^{-r}$, for all p. Therefore,

$$\begin{aligned}\varphi(t) = \mathrm{E}e^{itX} &= \sum_{x=0}^{\infty} e^{itx}\binom{r+x-1}{x}(1-p)^r p^x \\ &= (1-p)^r \sum_{x=0}^{\infty}\binom{r+x-1}{x}(pe^{it})^x = \frac{(1-p)^r}{(1-pe^{it})^r}.\end{aligned}$$

(b) If $r \to \infty$ and $p \to 0$ in such a way that $rp \to \lambda$, then $(1-p)^r \to e^{-\lambda}$ and $(1-pe^{it})^r \to e^{-\lambda e^{it}}$. Therefore, $\varphi(t) \to e^{\lambda(e^{it}-1)}$. Since this is the characteristic function of the Poisson distribution, $\mathcal{P}(\lambda)$, we have $X \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda)$.

4.3. $\overline{X}_n - \bar{\mu}_n \xrightarrow{\text{q.m.}} 0$ because

$$\mathrm{E}(\overline{X}_n - \bar{\mu}_n)^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}(X_i, X_j) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 \to 0.$$

This implies that $\overline{X}_n - \bar{\mu}_n \xrightarrow{\mathrm{P}} 0$ from Theorem 1.

In Bernstein's Theorem, the expectations are taken to be zero but this may be done without loss of generality. The main difference is that the correlations are allowed to be different from 0. However, Bernstein's Theorem does not contain Chebyshev's because in the former the variances are required to be uniformly bounded above.

5.3. (a) Since the sum of independent Poisson is Poisson, we have that Z_n has a Poisson distribution with parameter $\sum_1^n \lambda_j$. Therefore, if $\sum_1^n \lambda_j \to \infty$ as $n \to \infty$, we have $(Z_n - \mathrm{E}Z_n)/\sqrt{\mathrm{Var}Z_n} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ from Exercise 2(d) of the Additional Exercises, Section 3. If $\sum_1^n \lambda_j$ does not converge to ∞, then $Z_n \xrightarrow{\mathcal{L}} Z \in \mathcal{P}(\sum_1^\infty \lambda_j)$.

(b) If $\lambda_n = 2^{n-1}$, then $\sum_1^n \lambda_j \to \infty$, so that $(Z_n - \mathrm{E}Z_n)/\sqrt{\mathrm{Var}Z_n} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. Yet, $B_n^2 = \sum_1^n 2^{n-1} = 2^n - 1$ and

$$\begin{aligned}\frac{1}{B_n^2}\sum_1^n \mathrm{E}[(X_j - \mathrm{E}X_j)^2 \mathrm{I}(|X_j - \mathrm{E}X_j| > \epsilon B_n)] &\geq \epsilon^2 \sum_1^n \mathrm{P}(|X_j - \mathrm{E}X_j| > \epsilon B_n) \\ &\geq \epsilon^2 \mathrm{P}(|X_n - \mathrm{E}X_n| > \epsilon B_n) \to \epsilon^2 \mathrm{P}(|\mathcal{N}(0,1)| > \epsilon\sqrt{2}) > 0\end{aligned}$$

since $(X_n - \mathrm{E}X_n)/2^{(n-1)/2} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.