Stat 200C, Spring 2009 Ferguson

Solutions to Exercise Set 1.

1.1. (a) If you go over the proof of Markov's Inequality, you will see that the the inequality will be strict unless the distribution of X gives all its mass to 0 and $\pm\epsilon$. To get $\mathrm{E}X = 0$, we take X to be symmetric about 0 and let $p = \mathrm{P}(X = \epsilon) = \mathrm{P}(X = -\epsilon)$ and $\mathrm{P}(X = 0) = 1 - 2p$. The expectation, $\mathrm{E}|X| = 2p\epsilon$, is 1 if $p = 1/2\epsilon$. For this choice of p, we have $\mathrm{P}(|X| \geq \epsilon) = 1/\epsilon$ and $\mathrm{E}|X|/\epsilon = 1/\epsilon$. Thus we have equality in Markov's Inequality.

(b) The function, $g(x) = \cosh(x) - 1$, is nonnegative, symmetric about 0, and increasing on $[0, \infty)$. Therefore, $\mathrm{E}g(X) \geq \mathrm{E}(g(X)\mathrm{I}(|X| \geq \epsilon)) \geq g(\epsilon)\mathrm{P}(|X| \geq \epsilon)$, as in the proof of Chebyshev's Inequality. This implies $\mathrm{P}(|X| \geq \epsilon) \leq \mathrm{E}(\cosh(X) - 1)/(\cosh(\epsilon) - 1)$.

1.5. First note that $\mathrm{Var}X_i = \mathrm{E}(X_i^2) - (\mathrm{E}X_i)^2 \leq \mathrm{E}(X_i^2) \leq \mathrm{E}X_i$ for all i, where the last inequality follows because $0 \leq X_i \leq 1$ with probability 1. This and independence implies $\mathrm{Var}S_n \leq \mathrm{E}S_n = \mu_n$. Chebyshev's inequality states $\mathrm{P}(|S_n - \mu_n| \geq \epsilon) \leq \mathrm{Var}(S_n)/\epsilon^2$. Replace ϵ by $\mu_n\epsilon$ to get $\mathrm{P}(|S_n - \mu_n| \geq \mu_n\epsilon) \leq 1/(\mu_n\epsilon^2)$. Since μ_n converges to infinity, this converges to zero, or equivalently, $\mathrm{P}(|S_n - \mu_n| < \mu_n\epsilon) \to 1$ as $n \to \infty$. This implies $\mathrm{P}(-\mu_n\epsilon < S_n - \mu_n) \to 1$, or $\mathrm{P}(\mu_n(1 - \epsilon) \leq S_n) \to 1$. Since, $\mu_n \to \infty$, this shows $\mathrm{P}(B < S_n) \to 1$ for any fixed B, so $S_n \xrightarrow{P} \infty$. In our case, $S_n \xrightarrow{P} \infty$ implies $S_n \xrightarrow{a.s.} \infty$ because the S_n are nondecreasing with probability 1, so that $\mathrm{P}(S_n > B) = \mathrm{P}(S_k > B \text{ for every } k \geq n)$.