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NOTE

Trees Associated with the Motzkin Numbers

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We consider plane rooted trees on $n+1$ vertices without branching points on odd levels. The number of such trees in equal to the Motzkin number M_n. We give a bijective proof of this statement.

1

Let $\mathscr{P}_n$ be the set of all plane rooted trees on $n+1$ unlabeled vertices with edges oriented from the root (see [1]). We say that a vertex v in a tree $T \in \mathscr{P}_n$ is a *branching point* if at least two edges in T go from v. The *level* of a vertex v is the number of edges in the shortest path between v and the root. Let $\mathscr{E}_n \subset \mathscr{P}_n$ denote the set of plane trees without branching points on odd levels. By $\mathscr{M}_n \subset \mathscr{P}_n$ denote the set of plane trees with at most two edges going from every vertex.

The *Motzkin number* M_n is the number of elements in $\mathscr{M}_n$. The generating function $M(x) = 1 + \sum_{n \geqslant 0} M_n x^{n+1}$ satisfies the following functional equation (see [1–3]):

$$M(x) = 1 + xM(x) + x^2 M^2(x).$$

This equation gives a recurrence relation for the Motzkin numbers.

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Note that $|\mathscr{P}_n|$ is the *Catalan number* $C_n = (1/(n+1))\binom{2n}{n}$ (see [1, 3]).

THEOREM 1. $|\mathscr{E}_n| = M_n$.

THEOREM 2. *The number of trees* $T \in \mathscr{E}_n$ *with* $k+1$ *vertices on even levels is equal to* $\binom{n}{2k} C_k$.

We get the known formula $M_n = \sum_{k \geqslant 0} \binom{n}{2k} C_k$ (see [2, 4]).

2

Our proofs of Theorems 1 and 2 are based on a bijection $\rho: \mathscr{E}_n \to \mathscr{M}_n$. Let $\mathscr{B}_n$ be the set of binary trees with n unlabeled vertices (see [1]). There is a simple bijection $\phi: \mathscr{P}_n \to \mathscr{B}_n$ (see e.g. [1, 3]). This bijection is clear from an example on Fig. 1.

Denote $\mathscr{E}\mathscr{B}_n = \phi(\mathscr{E}_n)$ and $\mathscr{M}\mathscr{B}_n = \phi(\mathscr{M}_n)$. Then a tree T is an element of $\mathscr{M}\mathscr{B}_n$ if and only if there are no chains of two left edges in T. We say that *level* of a vertex v in a binary tree is the number of right edges in the shortest path between the root and v. Then a binary tree T is an element of $\mathscr{E}\mathscr{B}_n$ if and only if there are no left edges in T which go from an odd level vertex. Let $\tau: \mathscr{B}_n \to \mathscr{B}_n$ be the involution which exchanges left and right edges in a tree $T \in \mathscr{B}_n$.

Now construct a bijection $\eta: \mathscr{E}\mathscr{B}_n \to \tau(\mathscr{M}\mathscr{B}_n)$. Let the map η changes all right edges in $T \in \mathscr{E}\mathscr{B}_n$ going from an odd level vertex to left edges. Then $\eta(T)$ does not have chains of two rights edges, otherwise, one of the edges in such a chain goes from an odd level vertex.

Conversely, construct the inverse map $\eta^{-1}: \tau(\mathscr{M}\mathscr{B}_n) \to \mathscr{E}\mathscr{B}_n$. Let $T \in \tau(\mathscr{M}\mathscr{B}_n)$. For every right edge (v, u) in T such that u has a child w (then (u, w) should be a left edge) we change (u, w) to a right edge. It is not difficult to see that we get a tree from $\mathscr{E}\mathscr{B}_n$ and this map is the inverse to η.

Hence η is a bijection between $\mathscr{E}\mathscr{B}_n$ and $\tau(\mathscr{M}\mathscr{B}_n)$. Now $\rho = \phi^{-1} \circ \tau \circ \eta \circ \phi$ is a bijection between $\mathscr{E}_n$ and $\mathscr{M}_n$. See Fig. 2, as an example. We have proved Theorem 1.

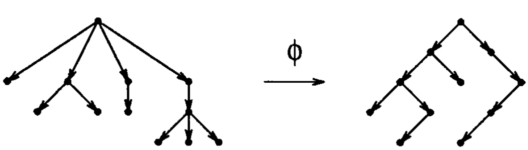

FIG. 1. Bijection $\phi: \mathscr{P}_n \to \mathscr{B}_n$.

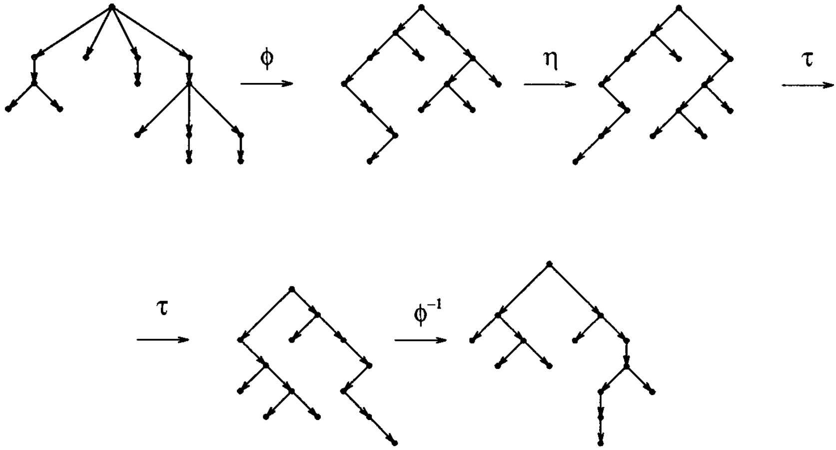

FIG. 2. Bijection $\rho\colon \mathscr{E}_n \to \mathscr{M}_n$.

The proof of Theorem 2 is based on the following property of the bijection ρ: The number of vertices on even levels of a plane tree $T \in \mathscr{E}_n$ is equal to the number of end points in the plane tree $\rho(T) \in \mathscr{M}_n$. Indeed, let a tree $T \in \mathscr{E}_n$ have $k+1$ vertices on even levels. Then $\phi(T)$ contains $k+1$ vertices which do not have a left child and which are either end points or lie on an even level. The bijection η maps these vertices to the vertices of $(\eta \circ \phi)(T)$ which do not have a left child. And $\phi^{-1} \circ \tau$ maps them to the end points of $\rho(T)$.

On the other hand, it is known (see [4]) that the number of trees $T \in \mathscr{M}_n$ with $k+1$ end points is equal to $\binom{n}{2k} C_k$. This completes the proof of Theorem 2.

Remark. The bijection ρ is an "unlabeled analogue" of a bijection from [5]. In this sense, the sequence of Motzkin numbers is an "unlabeled analogue" of the numbers of up-down (alternating) permutations.

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