ANALYTIC VARIATION OF TATE–SHAFAREVICH GROUPS

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Abstract. Let $K$ be a number field. For a prime $p$, we study the inductive limit of the $p$-ordinary part of the Tate-Shafarevich groups and the Selmer groups (over $K$) of modular Jacobians of level $Np^r$ as $r \to \infty$ for a fixed integer $N$ prime to $p$. We prove control theorems of the $p$-primary part of $\mathbb{III}_K(A_P)$ over $p$-adic analytic family of abelian varieties $A_P$. In particular, under mild conditions, we show that if $\mathbb{III}_K(A_{P_0})^{\text{ord}}$ is finite for one member $A_{P_0}$ of the analytic family and the Mordell–Weil rank of $A_{P_0}$ is $\leq 1$ over its Hecke field, then $\mathbb{III}_K(A_P)^{\text{ord}}$ is finite for almost all members $A_P$.

1. Introduction

Fix a prime $p$. Let $\text{Spec}(\mathbb{I})$ be an irreducible component of the $p$-ordinary big Hecke algebra $\mathfrak{h}$. Attached to $\mathbb{I}$ is the Mazur–Kitagawa $p$-adic $L$-function $L(k, s)$ for the weight variable $k$ and the cyclotomic variable $s$. One may regard, essentially, $L$ as an element of the affine ring of the irreducible component of Spec($\mathfrak{h}^{\text{ord}}$) covering Spec$(\mathbb{I})$ for the two variable nearly $p$-ordinary big Hecke algebra $\mathfrak{h}^{\text{ord}}$. We study in this paper the tower of modular curves $\{X_r\}_r$ whose Jacobians (or more precisely their $p$-ordinary part) correspond to the one variable $p$-adic $L$-function $k \mapsto L(2k + 2; \alpha k + 1)$ for a fixed pair of $p$-adic integers $\alpha, \delta \in \mathbb{Z}_p$. In other words, regarding $L$ as a function of the formal torus $\hat{\mathbb{G}}_m$ (as $\hat{\mathbb{G}}_m = \text{Spf}(\Lambda)$ for the Iwasawa algebra $\Lambda$), an abelian factor $A$ of $J_r := \text{Pic}^0(X_r)$ belonging to $\mathbb{I}$ corresponds to the L-value $L(\zeta^\delta, \zeta^\alpha)$ for a $p^t$-th root of unity $\zeta \in \hat{\mathbb{G}}_m$. In this introduction, for simplicity, we assume that $\alpha = 0$ and $\delta = 1$; so the corresponding $p$-adic $L$-function is $k \mapsto L(2k + 2, 1)$ concentrating to the weight variable, though one of the referees of this paper pointed out that the case $(\alpha, \delta) = (1, 1)$ is more interesting as $k \mapsto L(2k + 2, k + 1)$ interpolates the central critical values (so the function $k \mapsto L(2k + 2, k + 1)$ could be identically zero; see Section 18).

This case (and also a more general case of an arbitrary $(\alpha, \delta)$) will be taken care of in the main text, and all the results stated here for $(\alpha, \delta) = (0, 1)$ stand for general $(\alpha, \delta)$, starting with an exotic tower $\{X_r\}_r$ (depending on $(\alpha, \delta)$) different from $\{X_1(\mathbb{N}p^r)\}_{r}$. Because of our simplifying assumption $(\alpha, \delta) = (0, 1)$, the tower $X_r = X_1(\mathbb{N}p^r)/\mathbb{Q}$ is given by the compactified moduli of the classification problem of pairs $(E, \phi)$ of elliptic curves $E$ and an embedding $\phi : \mu_{N_p} \hookrightarrow E[\mathbb{N}p^r]$ as finite flat group schemes. Write $J_r/\mathbb{Q}$ for the Jacobian variety of $X_r$ whose origin is given by the infinity cusp $\infty \in X_r(\mathbb{Q})$ of $X_r$. We regard $J_r$ as the degree 0 component of the Picard scheme of $X_r$.

For a set of places $S$ of a number field $K$, write $K^S/K$ for the maximal extension unramified outside $S$. For a topological $\text{Gal}(K^S/K)$-module $M$ and $v \in S$, we write $H^\bullet(K^S/K, M)$ (resp. $H^\bullet(K_v/M)$ for the $v$-completion $K_v$ of $K$) for the continuous cohomology for the profinite group $\text{Gal}(K^S/K)$ (resp. $\text{Gal}(\overline{K_v}/K_v)$ for an algebraic closure $\overline{K_v}$ of $K_v$). Define

$$\mathbb{III}^j(K^S/K, M) = \text{Ker}(H^j(K^S/K, M) \to \prod_{v \in S} H^j(K_v, M)) \quad \text{for } j = 1, 2$$

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and put $\text{III}^0(K^S/K,M)_p := \text{III}^0(K^S/K,M) \otimes \mathbb{Z}_p$. Often we simply write $\text{III}$ for $\text{III}^0$. More generally, for a torsion or profinite module $M$, we define $M_p$ for the $p$-primary part of $M$ (so, $M_p$ is the maximal $p$-power torsion submodule $M[p^\infty]$ of $M$ if $M$ is torsion, and the maximal $p$-profinite quotient if $M$ is profinite). Throughout the paper, when $M$ is related to an abelian variety, we always assume that $S$ contains all finite places at which the abelian variety has bad reduction in addition to all $p$-adic and archimedean places of $K$. Often we write $\text{III}$ for $\text{III}^1$, and unless otherwise mentioned, we assume $S$ to be chosen finite.

In addition to the Mordell–Weil group $J_r(K) \otimes \mathbb{Z}_p/\mathbb{Z}_p$, we study the Tate–Shafarevich group $\text{III}_K(J_r)$, $\text{III}_K(K^S/K,J_r[p^\infty])$ and the Selmer group

$$\text{Sel}_r(J_r) = \text{Ker}(H^1(K^S/K,J_r[p^\infty]) \rightarrow \prod_{v \in S} H^1(K_v,J_r)).$$

The Tate–Shafarevich group and the Selmer group of an abelian variety are independent of $S$; so, we omitted $K^S/K$ from the notation. The Hecke operator $U(p)$ and its dual $U^*(p)$ acts on $\text{III}_K(J_r)$ and their $p$-adic limit $e = \lim_{n \to \infty} U(p)^{n!}$ and $e^* = \lim_{n \to \infty} U^*(p)^{n!}$ are well defined on the above groups $H$. We write $H^\text{ord} := (e(H))$. More generally, adding superscript or subscript “ord” (resp. “co-ord”), we indicate the image of $e$ (resp. $e^*$) depending on the situation.

By Picard functoriality, we have injective limits $G := \lim_{\to} J_r(p^{n!})$ (a $\Lambda$-BT group in the sense of [H14a]), $J_r^\text{ord}(K) = \lim_{\to} J^\text{ord}_r(K)$ for $J_r(K) \otimes \mathbb{Z}_p$, $\text{III}_K(J^\text{ord}) = \lim_{\to} \text{III}_K(J^\text{ord}_r)$, $\text{III}_K(G) = \lim_{\to} \text{III}_K(J(p^{n!})^\text{ord})$, and $\text{Sel}_r(J^\text{ord}) = \lim_{\to} \text{Sel}_r(J^\text{ord}_p)$. We study control under Hecke operators acting on these arithmetic cohomology groups. These groups, we call $\Lambda$-BT groups, $\Lambda$-MW, $\Lambda$-SS groups and $\Lambda$-Selmer groups in order. For any Shimura’s abelian factor $A_f \subset J_1(Np^r)$ associated to a Hecke eigenform $f \in S_2(\Gamma_1(Np^r))$ [IAT, Theorem 7.14], we can think of $\text{III}_K(A_f)^\text{ord}$. Let $h(N)$ be a big ordinary Hecke algebra of prime-to-$p$ level $N$, and pick a primitive connected component $\text{Spec}(T)$ of $\text{Spec}(h(N))$ in the sense of [H86a, §3]. Then points $P \in \text{Spec}(T)(\overline{\mathbb{Q}}_p)$ correspond one-to-one to $p$-adic Hecke eigenforms $f_P$ in a slope 0 analytic family.

In a densely populated subset $\Omega_T \subset \text{Spec}(T)(\overline{\mathbb{Q}}_p)$ (indexed by $(\zeta^\delta,\zeta^\alpha) = (\zeta,1)$ for $\zeta \in \mu_{p^\infty}$), $f_P$ is classical, new at all prime factors of $N$ and of weight 2. Write $Np^{r(P)}$ for the minimal level of $f_P$.

Let $A_P/Q$ be Shimura’s abelian factor of $J_1(Np^{r(P)})$ associated to $f_P$. Write $H_P$ for the subfield of $\text{End}(A_P/Q) \otimes \mathbb{Q}$ generated by the Hecke operators. In this introduction, for simplicity, we assume that $A_P$ for every $P \in \Omega_T$ has potential good reduction at $p$ and that $A_P$ for some $P \in \Omega_T$ has good reduction at $p$. We prove control theorems for these arithmetic cohomology groups which imply

**Theorem A.** Suppose $p > 2$, $|S| < \infty$ and that $T$ is a unique factorization domain.

1. If $\text{III}_K(K^S/K,A_P[p^{n!}]^\text{ord})$ is finite for a single point $P_0 \in \Omega_T$, then $\text{III}_K(K^S/K,A_P[p^{n!}]^\text{ord})$ is finite for all $P \in \Omega_T$ (Corollary 12.2).
2. If $\text{III}_K(A_P[p^{n!}]^\text{ord})$ is finite and $\dim_{H^0(I)} A_P(K) \otimes \mathbb{Q} \leq 1$ for a single point $P_0 \in \Omega_T$, then $\text{III}_K(A_P)^{\text{ord}}$ is finite for all $P \in \Omega_T$ (Corollary 13.6).
3. If $\text{III}_K(A_P[p^{n!}]^\text{ord})$ is finite and $\dim_{H^0(I)} A_P(K) \otimes \mathbb{Q} \leq 1$ for a single point $P_0 \in \Omega_T$, then for almost all $P \in \Omega_T$, $\dim_{H^0(P)} A_P(K) \otimes \mathbb{Q} = 0$ or 1 independent of $P$ (Corollary 13.6).
4. If $\text{Sel}_r(A_P)^{\text{ord}}$ is finite for a single point $P_0 \in \Omega_T$, then $\text{Sel}_r(A_P)^{\text{ord}}$ is finite for all $P \in \Omega_T$. Moreover if $\text{Sel}_r(A_P)^{\text{ord}} = 0$ for a single point $P_0 \in \Omega_T$ such that $A_{P_0}/Q$ has good reduction modulo $p$ with $A_{P_0}(\mathbb{F}_p) = 0$, $\text{Sel}_r(A_P)^{\text{ord}}$ is finite for all $P \in \Omega_T$ without exception (Corollary 10.5).

Here the words “almost all” means “except for finitely many”. More general statements covering “exotic modular towers” will be given as Corollaries indicated in the theorem. The ring $T$ is usually a power series ring of one variable over a discrete valuation ring (and hence a unique factorization domain; see Theorem 5.3).

Writing $p_1$ for the modular two-dimensional Galois representation associated to $\Gamma$ (see [GME, §4.3.1]), we can think of the Selmer group $\text{Sel}_q(p_1 \otimes \kappa)$ for a Galois character $\kappa$ with values in $\Gamma^\times$ (cf. [Gr94] or [HMI, §1.2.4]). The group $\text{Sel}_q(p_1 \otimes \kappa)$ has natural relation to the limit Mordell–Weil
The Tate-Shafarevich groups (and the Selmer groups) have precise control over a given $p$-adic analytic family relative to the “inductive” limit over the tower without completion (as treated in this paper). This is true even when the corresponding $p$-adic L-function is identically zero over the family because of the parity of the root number.

The Tate-Shafarevich part behaves well for the injective limit. Thus, possibly, the Tate–Shafarevich–Weil part seems to behave differently.

We note that this paper is independent of the results in [H14b] and [H15a] (except for elementary facts and definitions) as we reprove necessary facts for $J_{\infty}^{\text{ord}}(K)$ in a different manner from these papers; indeed, the proofs here are much simpler. In particular, this paper is self-contained.

We may reformulate our result via congruence among abelian varieties introduced in [H15b]. For such reformulation, we recall first the definition of the congruence. An $F$-simple abelian variety (with a polarization) defined over a number field $F$ is called, in this paper, “of GL(2)-type” if we have a subfield $H_A \subset \text{End}^0(A/F) = \text{End}(A/F) \otimes \mathbb{Q}$ of degree $\dim A$ (stable under Rosati-involution).

Then, for the two-dimensional compatible system $\rho_A$ of Galois representations of $A$ with coefficients in $\mathbb{Q}$, $H_A$ is generated by traces $\text{Tr}(\rho_A(Frob_l))$ of Frobenius elements $Frob_l$ for $l$-primes $l$ of good reduction (i.e., the field $H_A$ is uniquely determined by $A$). We always regard $F$ as a subfield of the algebraic closure $\overline{\mathbb{Q}}$. Thus $O_A' := \text{End}(A/F) \cap H_A$ is an order of $H_A$. Write $O_A$ for the integer ring of $H_A$. Replacing $A$ by the abelian variety representing the group functor $R \mapsto A(R) \otimes_{\mathcal{O}_A} O_A$, we may choose $A$ so that $O_A' = O_A$ in the $F$-isogeny class of $A$. Since finiteness of the Tate–Shafarevich group of $A$ (not necessarily its exact size) is determined by the $F$-isogeny class of $A$ for a field extension $K/F$, as we only concern with finiteness property of $\mathfrak{III}_K(A)$, we hereafter assume that $\text{End}(A/F) \cap H_A = O_A$ for any abelian variety of GL(2)-type over $F$. For two abelian varieties $A$ and $B$ of GL(2)-type over $F$, we say that $A$ is congruent to $B$ modulo a prime $p$ over $F$ if we have a prime factor $p_A$ (resp. $p_B$) of $p$ in $O_A$ (resp. $O_B$) and field embeddings $\sigma_A : O_A/p_A \hookrightarrow \mathbb{F}_p$ and $\sigma_B : O_B/p_B \hookrightarrow \mathbb{F}_p$ such that $(A[p_A] \otimes_{\mathcal{O}_A/p_A, \sigma_A} \mathbb{F}_p)^{ss} \cong (B[p_B] \otimes_{\mathcal{O}_B/p_B, \sigma_B} \mathbb{F}_p)^{ss}$ as Gal($\overline{\mathbb{Q}}/F$)-modules, where the superscript “$ss$” indicates the semi-simplification. Hereafter in this article, we always assume the field of definition $F$ is equal to $\mathbb{Q}$, but the coefficient field $K$ is any number field.

Let $E_{/\mathbb{Q}}$ be an elliptic curve. Writing the Hasse–Weil L-function $L(s, E)$ as a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ with $a_n \in \mathbb{Z}$ (i.e., $1 + p - a_p = |E(\mathbb{F}_p)|$ for each prime $p$ of good reduction for $E$), we call $p$ admissible for $E$ if $E$ has good reduction at $p$ and $(a_p \mod p)$ is not in $\Omega_E := \{ \pm 1, 0 \}$. Therefore, the maximal étale quotient of $E[p]$ over $\mathbb{Z}_p$ is not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ up to unramified quadratic twists. By the Hasse bound $|a_p| \leq 2\sqrt{p}$, $p \geq 7$ is not admissible if and only if $a_p \in \Omega_E$ (so, 2 and 3 are not admissible) if $E$ does not have complex multiplication, the Dirichlet density of non-admissible primes is zero by a theorem of Serre as $L(s, E) = L(s, f)$ for a rational Hecke eigenform $f$ (see [H15b, §8]).

**Theorem B.** Let $E_{/\mathbb{Q}}$ be an elliptic curve with $|\mathfrak{III}_K(E)| < \infty$ and $\dim_{\mathbb{Q}} E(K) \otimes \mathbb{Q} \leq 1$. Let $N$ be the conductor of $E$, and pick an admissible prime $p$ for $E$. Consider the set $\mathcal{A}_{E,p}$ made up of all $\mathbb{Q}$-isogeny classes of $\mathbb{Q}$-simple abelian varieties $A_{/\mathbb{Q}}$ of GL(2)-type with prime-to-$p$ conductor $N$.
congruent to $E$ modulo $p$ over $\mathbb{Q}$. Then there exists an explicit (computable) finite set $S_E$ of primes depending on $N$ but independent of $K$ such that if $p \not\in S_E$, almost all members $A \in \mathcal{A}_{E,p}$ have finite $\text{III}_K(A)_{p^A}$ and constant dimension $\dim_{H_A} A(K) \leq 1$. If further $E(K) = \text{III}_K(E) = 0$ (i.e., $\text{Sel}_K(E)p = 0$ in short) and $E$ can be embedded into $J_r$ for some $r > 0$, then as long as $p$ totally splits in $K/\mathbb{Q}$, every $A \in \mathcal{A}_{E,p}$ has finite $\text{III}_K(A)_{p^A}$ and $\text{Sel}_K(A)_{p^A}$ as long as $p \not\in S_E$.

Here for the prime $p_A|p$, we have $(A[p_A] \otimes_{O_A[p_A]} \mathbb{F}_p)^{	ext{ss}} \cong (E[p_A] \otimes_{\mathbb{F}_p} \mathbb{F}_p)^{\text{ss}}$, and $\text{III}_K(A)_{p^A}$ (resp. $\text{Sel}_K(A)_{p^A}$) is the $p_A$-primary part of $\text{III}_K(A)_{p}$ (resp. $\text{Sel}_K(A)_{p}$). The definition of the set $S_E$ will be given in Definition 15.1 and a more general version of this theorem will be given as Theorem 15.2.

Taking $K = \mathbb{Q}$ and applying the above theorem to the modular elliptic curves $E = X_0(N)$ for small $N$, we get the following corollary:

**Corollary C.** Let $N$ be one of 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49 (all the cases when $X_0(N)$ is an elliptic curve with finite $X_0(N)(\mathbb{Q})$). Pick an admissible prime $p$ for $X_0(N)$. Then we have $|\text{III}_Q(A)_{p^A}| < \infty$ and $|\text{Sel}_Q(A)_{p^A}| < \infty$ for almost all $A$ in $\mathcal{A}_{X_0(N),p}$. If further $X_0(N)(\mathbb{Q}) = \text{III}_Q(X_0(N)p = 0$, $\text{Sel}_Q(A)_{p^A}$ and $\text{III}_Q(A)_{p^A}$ are both finite for all $A$ in $\mathcal{A}_{X_0(N),p}$ without exception.

By a celebrated theorem of Kolyvagin [K88] (with modularity of rational elliptic curves [BCDT01]), as long as $\text{rank}_Q E(\mathbb{Q}) \leq 1$, we have $|\text{III}_Q(E)| < \infty$. For the modular elliptic curves $X_0(N)$ listed above, the point of the above corollary is that we have $|X_0(N)(\mathbb{Q})| < \infty$ and that the exceptional set $S_E$ can be taken to be empty (or more precisely, $S_E$ is contained in non-admissible primes); so, the statement becomes more transparent. This corollary produces infinitely many examples of simple abelian varieties of (unbounded) dimension ($> 1$) with finite Tate–Shafarevich group $\text{III}_Q(A)_{p^A}$, as we know $\text{dim}_{\mathbb{F}_p} A_{p^A}$ grows indefinitely in an analytic family [H11].

We have a version of this corollary for some elliptic curve factors $E/\mathbb{Q}$ of $J_0(N)$ (e.g., $N = 37$) of root number $-1$ assuming $\text{rank}_Q E(\mathbb{Q}) = 1$ and $|\text{III}_Q(E)| < \infty$ for the analytic family with constant root number $-1$ containing $E$ (such a family is associated to an exotic tower; see Theorem 15.2).

Instead of starting with the modular elliptic curve as listed above, we can start with a CM elliptic curve $E$ with finite Tate–Shafarevich group over $\mathbb{Q}$ (studied by Rubin [R87]) and then we get a similar result for the CM family of abelian varieties containing the starting elliptic curve (although some CM cases are also covered by Corollary C).

For an extension $X$ of an abelian variety by a finite group scheme defined either over a number field $K$ or a local field $K$ of characteristic 0, we define an fpfp abelian sheaf $\tilde{X}$ by specifying its value as follows:

$$\tilde{X}(R) = \begin{cases} X(R) \otimes \mathbb{Z}_p & \text{if } [K : \mathbb{Q}] < \infty, \\ X[p^\infty](R) & \text{if } [K : \mathbb{Q}] < \infty \text{ (l \neq p) or } [K : \mathbb{R}] < \infty, \\ (X/X(p))(R) & \text{as a sheaf quotient} \quad \text{if } [K : \mathbb{Q}_p] < \infty \end{cases}$$

for fpfp algebras $R/K$, where $X(p)$ is the maximal prime-to-$p$ torsion subgroup of $X$. Throughout the paper, we write $M^\vee$ for the Pontryagin dual module $\text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ for a $\mathbb{Z}_p$-module $M$.

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2. $U(p)$-ISOMORPHISMS

Replacing fppf cohomology we described in [H14b, §3] by étale cohomology, we reproduce the results and proofs in [H14b, §3] as it gives the foundation of our control result, though we need later to adjust technically the method described here to get precise control of the limit Tate–Shafarevich group. Let $S = \text{Spec}(K)$ for a field $K$. Let $X \rightarrow Y \rightarrow S$ be proper morphisms of noetherian schemes. We study

$$H^0_{\text{fppf}}(T, R^1 f_* \mathbb{G}_m) = H^0_{\text{et}}(T, R^1 f_* \mathbb{G}_m) = R^1 f_* \mathcal{O}_X(T) = \text{Pic}_{X/S}(T)$$

for $S$-scheme $T$ and the structure morphism $f : X \rightarrow S$. Write the morphisms as $X \xrightarrow{f} Y \xrightarrow{g} S$ with $f = g \circ \pi$. We note the following general fact:

**Lemma 2.1.** Assume that $\pi$ is finite flat. Then the pull-back of line bundles: $\text{Pic}_{Y/S}(T) \ni \mathcal{L} \mapsto \pi^* \mathcal{L} \in \text{Pic}_{X/S}(T)$ induces the Picard functoriality which is a natural transformation $\pi^* : \text{Pic}_{Y/S} \rightarrow \text{Pic}_{X/S}$ contravariant with respect to $\pi$. Similarly, we have the Albanese functoriality sending $\mathcal{L} \in \text{Pic}_{X/S}(T)$ to $\bigwedge^{\deg(X/Y)} \pi_* \mathcal{L} \in \text{Pic}_{Y/S}(T)$ as long as $X$ has constant degree over $Y$. This map $\pi_* : \text{Pic}_{X/S} \rightarrow \text{Pic}_{Y/S}$ is a natural transformation covariant with respect to $\pi$.

Hereafter we always assume that $\pi$ is finite flat with constant degree.

In [H14b, §3], we assumed that $f$ and $g$ have compatible sections $S \xrightarrow{s_0} Y$ and $S \xrightarrow{s_f} X$ so that $\pi \circ s_f = s_g$. However, in this paper, we do not assume the existence of compatible sections, but we limit ourselves to $T = \text{Spec}(\kappa)$ for an étale extension $\kappa$ of the base field $K$. Then we get (e.g., [NMD, Section 8.1] and [ECH, Chapter 3])

$$\text{Pic}_{X/S}(T) = H^0_{\text{fppf}}(T, R^1 f_* \mathbb{G}_m) \overset{(*)}{=} H^1_{\text{fppf}}(X_T, O^\times_{X_T}) = H^1_{\text{et}}(X_T, O^\times_{X_T})$$

$$\text{Pic}_{Y/S}(T) = H^0_{\text{fppf}}(T, R^1 g_* \mathbb{G}_m) \overset{(*)}{=} H^1_{\text{fppf}}(Y_T, O^\times_{Y_T}) = H^1_{\text{et}}(Y_T, O^\times_{Y_T})$$

for any $S$-scheme $T$. The identity at $(*)$ follows from the fact: $\text{Pic}_T = 0$, since $T$ is a union of points (i.e., $\kappa = k_1 \oplus \cdots \oplus k_m$ for finite separable field extensions $k_j/K$). Here $X_T = X \times_S T$ and $Y_T = Y \times_S T$. We suppose that the functors $\text{Pic}_{X/S}$ and $\text{Pic}_{Y/S}$ are representable by smooth group schemes (for example, if $X, Y$ are smooth proper and geometrically irreducible (and $S = \text{Spec}(K)$ for a field $K$); see [NMD, 8.4.2–3]). We then write $J_f = \text{Pic}^0_{Y/S}$ (? = $X, Y$) for the identity connected component of $\text{Pic}_{Y/S}$. Anyway we suppose hereafter also that $X, Y, S$ are varieties (in the sense of [ALG, II.4]).

For an fppf covering $U \rightarrow Y$ and a presheaf $P = P_U$ on the fppf site over $Y$, we define via Čech cohomology theory an fppf presheaf $U \mapsto \mathcal{H}^q(U, P)$ denoted by $\mathcal{H}^q(P_U)$ (see [ECH, III.2.2 (b)]). The inclusion functor from the category of fppf sheaves over $Y$ into the category of fppf presheaves over $Y$ is left exact. The derived functor of this inclusion of an fppf sheaf $F = F_Y$ is denoted by $\mathcal{H}^*(P_U)$ (see [ECH, III.1.5 (c)]). Thus $\mathcal{H}^*(\mathbb{G}_m/Y)(U) = \mathcal{H}^0_{\text{fppf}}(U, O^\times_U)$ for a $Y$-scheme $U$ as a presheaf (here $U$ varies in the small fppf site over $Y$).

Instead of the Hochschild-Serre spectral sequence used in [H86b, §4] to get a control of modular group cohomology, assuming that $f, g$ and $\pi$ are all faithfully flat of finite presentation, we use the
spectral sequence of Čech cohomology of the flat covering $\pi : X \rightarrow Y$ in the fppf site over $Y$ [ECH, III.2.7]:

$$\hat{H}^n(X_T/Y_T, H^q(Y_T, O^\times_{Y_T})) \Rightarrow H^n_{\text{fppf}}(Y_T, O^\times_{Y_T})$$

for each $S$-scheme $T$. Here $F \mapsto H^n_{\text{fppf}}(Y_T, F)$ (resp. $F \mapsto H^n(Y_T, F)$) is the right derived functor of the global section functor: $F \mapsto F(Y_T)$ from the category of fppf sheaves (resp. Zariski sheaves) over $Y_T$ to the category of abelian groups. The canonical isomorphism $\iota$ is the one given in [ECH, III.4.9].

Write $H^*_{\text{fppf}}$ for $H^*(Y_T, O_U)$ and $H^*(H^0_{\text{fppf}})$ for $H^*(Y_T, H^0_{\text{fppf}})$. Since From this spectral sequence, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\hat{H}^1(H^0_{Y_T}) & \longrightarrow & \text{Pic}_{Y/S}(T) & \longrightarrow & \hat{H}^0(Y_T, H^1(G_{m,Y})) & \longrightarrow & \hat{H}^2(H^0_{Y_T}) \\
\uparrow & & \downarrow i & & \uparrow & & \uparrow & & \\
\hat{H}^1(H^0_{Y_T}) & \longrightarrow & \text{Pic}_{Y/S}(T) & \longrightarrow & \hat{H}^0(Y_T, \text{Pic}_Y(T)) & \longrightarrow & \hat{H}^2(H^0_{Y_T}) \\
\uparrow & & \downarrow j & & \uparrow & & \uparrow & & \\
?_1 & \longrightarrow & J_Y(T) & \longrightarrow & \hat{H}^0(Y_T, J_X(T)) & \longrightarrow & ?_2.
\end{array}$$

Here the horizontal exactness at the top two rows follows from the spectral sequence (2.1) (see [ECH, Appendix B]).

Take a correspondence $U \subset Y \times_S Y$ given by two finite flat projections $\pi_1, \pi_2 : U \rightarrow Y$ of constant degree (i.e., $\pi_j_* O_U$ is locally free of finite rank $\deg(\pi_j)$ over $O_Y$). Consider the pullback $U_X \subset X \times_S X$ given by the Cartesian diagram:

$$\begin{array}{cc}
U_X = U \times_{Y \times_S Y} (X \times_S X) & \longrightarrow & X \times_S X \\
\downarrow & & \downarrow \\
U & \longrightarrow & Y \times_S Y
\end{array}$$

Let $\pi_j, X = \pi_j \times_S \pi : U_X \rightarrow X$ ($j = 1, 2$) be the projections.

Consider a new correspondence $U^{(q)}_X = U_X \times_Y \cdots \times_Y U_X$, whose projections are the iterated product

$$\pi_{j,X}^{(q)} = \pi_j, X \times_Y \cdots \times_Y \pi_j, X : U^{(q)}_X \rightarrow X^{(q)} (j = 1, 2).$$

Here is a first step to get a control result of $\Lambda$-TS groups: for arithmetic cohomology.

**Lemma 2.2.** Let the notation and the assumption be as above. In particular, $\pi : X \rightarrow Y$ is a finite flat morphism of geometrically reduced proper schemes over $S = \text{Spec}(K)$ for a field $K$. Suppose that $X$ and $U_X$ are proper schemes over a field $K$ satisfying one of the following conditions:

1. $U_X$ is geometrically reduced, and for each geometrically connected component $X^o$ of $X$, its pull back to $U_X$ by $\pi_{2,X}$ is also connected; i.e., $\pi^0(X) \pi_{2,X}^0 \sim \pi^0(U_X)$;

2. $(f \circ \pi_2, X)_* O_{U_X} = f_* O_X$.

If $\pi_2 : U \rightarrow Y$ has constant degree $\deg(\pi_2)$, then, for each $q > 0$, the action of $U^{(q)}$ on $H^0(X, O_{X^{(q)}}^\times)$ factors through the multiplication by $\deg(\pi_2)$.

This result is given as [H14b, Lemma 3.1, Corollary 3.2]. Though in [H14b, §3], an extra assumption of requiring the existence of compatible sections to $X \rightarrow Y \rightarrow S$, this assumption is nothing to do with the proof of the above lemma, and hence the proof there is valid without any modification.
To describe the correspondence action of $U$ on $H^0(X, \mathcal{O}_X)$ in down-to-earth terms, let us first recall the Čech cohomology: for a general $S$-scheme $T$,

$$(2.3) \quad \check{H}^q\left(\frac{X_T}{T}, \check{H}^0(G_{m/Y})\right) = \frac{\{(c_{i_0,\ldots,i_q}) \in H^0(X_T^{(q+1)}, \mathcal{O}_T^{X}) \mid \prod_j (c_{i_0,\ldots,i_j \ldots i_{q+1}} \circ p_{i_0,\ldots,i_j \ldots i_{q+1}})^{(-1)^j} = 1\}}{\{db_{i_0,\ldots,i_q} = \prod_j (b_{i_0,\ldots,i_j \ldots i_{q}} \circ p_{i_0,\ldots,i_j \ldots i_{q-1}})^{(-1)^j} \mid b_{i_0,\ldots,i_j \ldots i_{q}} \in H^0(X_T^{(q)}, \mathcal{O}_T^{X(q)})\}}$$

where we agree to put $H^0(X_T^{(0)}, \mathcal{O}_T^{X(0)}) = 0$ as a convention,

$$X^{(q)}_T = \underset{q}{\underbrace{X \times Y \times Y \cdots \times Y}} \times T, \quad O_{X_T^{(q)}} = O_X \times_{O_Y} O_X \times O_Y \cdots \times O_Y O_X \times_{O_T} O_T,$$

the identity $\prod_j (c \circ p_{i_0,\ldots,i_j \ldots i_{q+1}})^{(-1)^j} = 1$ takes place in $O_{X_T^{(q+2)}}$ and $p_{i_0,\ldots,i_j \ldots i_{q+1}} : X^{(q+2)}_T \to X^{(q+1)}_T$ is the projection to the product of $X$ the $j$-th factor removed. Since $T \times T T \cong T$ canonically, we have $X^{(q)}_T \cong \underset{q}{\underbrace{X \times T \times \cdots \times T}} T$ by transitivity of fiber product.

Consider $\alpha \in H^0(X, \mathcal{O}_X)$. Then we lift $\pi^*_1 \mathcal{O}_X = \alpha \circ \pi_1 \mathcal{O}_X \in H^0(U_X, \mathcal{O}_{U_X})$. Put $\alpha U := \pi^*_1 \mathcal{O}_X \alpha$. Note that $\pi_{2 \times X} \mathcal{O}_{U_X}$ is locally free of rank $d = \deg(\pi_2)$ over $\mathcal{O}_X$, the multiplication by $\alpha U$ has its characteristic polynomial $P(T)$ of degree $d$ with coefficients in $\mathcal{O}_X$. We define the norm $N_U(\alpha_U)$ to be the constant term $P(0)$. Since $\alpha$ is a global section, $N_U(\alpha_U)$ is a global section, as it is defined everywhere locally. If $\alpha \in H^0(X, \mathcal{O}_X)$, $N_U(\alpha_U) \in H^0(X, \mathcal{O}_X)$. Then define $U(\alpha) = N_U(\alpha_U)$, and in this way, $U$ acts on $H^0(X, \mathcal{O}_X)$. For a degree $q$ Čech cohomology class $[c] \in \check{H}^q(X/Y, \check{H}^0(G_{m/Y}))$ with a Čech $q$-cocycle $c = (c_{i_0,\ldots,i_q})$, $U([c])$ is given by the cohomology class of the Čech cocycle $U(c) = (U(c_{i_0,\ldots,i_q}))$, where $U(c_{i_0,\ldots,i_q})$ is the image of the global section $c_{i_0,\ldots,i_q}$ under $U$. Indeed, $(\pi^*_1 \mathcal{O}_X \mathcal{O}_X \cdots \mathcal{O}_X)$ plainly satisfies the cocycle condition, and $(N_U(\pi^*_1 \mathcal{O}_X \mathcal{O}_X \cdots \mathcal{O}_X))$ is again a Čech cocycle as $N_U$ is a multiplicative homomorphism. By the same token, this operation sends coboundaries to coboundaries, and define the action of $U$ on the cohomology group. Thus we get the following vanishing result:

**Proposition 2.3.** Suppose that $S = \text{Spec}(K)$ for a field $K$. Let $\pi : X \to Y$ be a finite flat covering of (constant) degree $d$ of geometrically reduced proper varieties over $K$, and let $Y \overset{\pi_1}{\to} U \overset{\pi_2}{\to} Y$ be two finite flat coverings (of constant degree) identifying the correspondence $U$ with a closed subscheme $U \overset{\pi_1 \times \pi_2}{\to} Y \times_S Y$. Write $\pi_1 \times \pi_2 : U_X = U \times_Y X \to X$ be the base-change to $X$. Suppose one of the conditions (1) and (2) of Lemma 2.2 for $(X, U)$. Then

1. The correspondence $U \subset Y \times_S Y$ sends $\check{H}^q(H_Y^0)$ into $\deg(\pi_2)(\check{H}^q(H_Y^0))$ for all $q > 0$.
2. If $d$ is a $p$-power and $\deg(\pi_2)$ is divisible by $p$, $\check{H}^q(H_Y^0)$ for $q > 0$ is killed by $U^M$ if $p^M > d$.
3. The cohomology $\check{H}^q(H_Y^0)$ with $q > 0$ is killed by $d$.

This follows from Lemma 2.2, because on each Čech $q$-cocycle (whose value is a global section of iterated product $X^{(q+1)}_T$), the action of $U$ is given by $U^{(q+1)}$ by (2.3). See [H14b, Proposition 3.3] for a detailed proof.

Assume that a finite group $G$ acts on $X/\mathcal{O}_X$ faithfully. Then we have a natural morphism $\phi : X \times G \to X \times_Y X$ given by $\phi(x, \sigma) = (x, \sigma(x))$. In other words, we have a commutative diagram

$$\begin{array}{ccc}
X \times G & \xrightarrow{(x,\sigma)\mapsto \phi(x)} & X \\
(x,\sigma)\mapsto \sigma & \downarrow & \downarrow \\
X & \longrightarrow & Y,
\end{array}$$

which induces $\phi : X \times G \to X \times_Y X$ by the universality of the fiber product. Suppose that $\phi$ is surjective; for example, if $Y$ is a geometric quotient of $X$ by $G$; see [GME, §I.8.3]). Under this
map, for any fppf abelian sheaf \( F \), we have a natural map \( H^0(X/Y, F) \to H^0(G, F(X)) \) sending a \( Čech \) 0-cocycle \( c \in H^0(X, F) = F(X) \) (with \( p_1^*c = p_2^*c \) to \( c \in H^0(G, F(X)) \)). Obviously, by the surjectivity of \( \phi \), the map \( H^0(X/Y, F) \to H^0(G, F(X)) \) is an isomorphism (e.g., [ECH, Example III.2.6, page 100]). Thus we get

**Lemma 2.4.** Let the notation be as above, and suppose that \( \phi \) is surjective. For any scheme \( T \) fppf over \( S \), we have a canonical isomorphism: \( H^0(X_T/Y_T, F) \cong H^0(G, F(X_T)) \).

We now assume \( S = \text{Spec}(K) \) for a field \( K \) and that \( X \) and \( Y \) are proper reduced connected curves. Then we have from the diagram (2.2) with the exact middle two columns and exact horizontal rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\uparrow & & \text{deg} \uparrow \text{onto} & & \text{deg} \uparrow \text{onto} & & \uparrow \\
\hat{H}^1(H_0^0) & \longrightarrow & \text{Pic}_{Y/S}(T) & \longrightarrow & \hat{H}^1(Y, \text{Pic}_{Y/S}(T)) & \longrightarrow & \hat{H}^2(H_0^0) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
?_1 & \longrightarrow & J_Y(T) & \longrightarrow & \hat{H}^0(Y, J_X(T)) & \longrightarrow & ?_2,
\end{array}
\]

Thus we have \( ?_j = \hat{H}^j(H^0_Y) \) (\( j = 1, 2 \)).

By Proposition 2.3, if \( q > 0 \) and \( X/Y \) is of degree \( p \)-power and \( p \mid \text{deg}(\pi_2) \), \( \hat{H}^q(H^0_Y) \) is a \( p \)-group, killed by \( U^M \) for \( M \gg 0 \).

3. **Exotic modular curves**

We study a more general modular tower \( \{X_r\}_r \) than the standard one \( \{X_1(Np^r)\}_r \), considered in the introduction (thus hereafter, \( X_r \) could be no longer \( X_1(Np^r) \)). We introduce open compact subgroups of \( \text{GL}_2(\mathbb{A}^{(\infty)}) \) giving rise to the general tower \( \{X_r\}_r \).

Let \( \Gamma := 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times \), where \( \epsilon = 2 \) if \( p = 2 \) and \( \epsilon = 1 \) otherwise. The group \( \Gamma \) is a maximal torsion-free subgroup of \( \mathbb{Z}_p^\times \). Fix an exact sequence of profinite groups \( 1 \to H_p \to \Gamma \times \Gamma \to \Gamma \to 1 \), and regard \( H_p \) as a subgroup of \( \Gamma \times \Gamma \). This implies \( \pi_1(a, d) = a^\epsilon d^{-\epsilon} \) for a pair \( (a, d) \in \mathbb{Z}_p^2 \) with \( aZ_p + dZ_p = Z_p \) and hence \( H_p = \{(a, d) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times : a^\epsilon d^{-\epsilon} = 1\} \). Writing \( \mu \) for the maximal torsion subgroup of \( \mathbb{Z}_p^\times \), we pick a character \( \xi : \mu \times \mu \to \mathbb{Z}_p^\times \) and define \( H = H_\xi = H_{\alpha, \delta, \xi} := H_p \times \ker(\xi) \) in \( \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times = \Gamma \times \Gamma \times \mu \times \mu \). We can take \( \xi(\zeta, \zeta') = \zeta^\alpha \zeta'^{-\delta} \) for \((\alpha, \delta) \in \mathbb{Z}^2\). Write \( \pi := \pi_1 \times \xi : \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \) and the image of \( H \) in \( \mathbb{Z}_p^\times / (\Gamma^{p\text{-power}})^2 \) as \( H_r \). Then define, for \( \mathbb{Z} = \prod_l\text{primes } \mathbb{Z}_l \),

\[
\begin{align*}
\hat{\Gamma}_0(M) & := \left\{ \left( \begin{array}{cc} a & d \\ c & d \end{array} \right) \in \text{GL}_2(\overline{\mathbb{Z}}) \mid c \in \mathbb{M}\overline{\mathbb{Z}} \right\}, \\
\hat{\Gamma}_1(M) & := \left\{ \left( \begin{array}{cc} a & d \\ c & d \end{array} \right) \in \hat{\Gamma}_0(M) \mid d - 1 \in \mathbb{M}\overline{\mathbb{Z}} \right\}, \\
\hat{\Gamma}_1(M) & := \left\{ \left( \begin{array}{cc} a & d \\ c & d \end{array} \right) \in \hat{\Gamma}_1(M) \mid a - 1 \in \mathbb{M}\overline{\mathbb{Z}} \right\}, \\
\hat{\Gamma}_s = \hat{\Gamma}_{H,s} & := \left\{ \left( \begin{array}{cc} a & d \\ c & d \end{array} \right) \in \hat{\Gamma}_0(p^s) \cap \hat{\Gamma}_1(N) \mid (a_p, d_p) \in H_s \right\}, \\
\hat{\Gamma}_s = \hat{\Gamma}_{H,s} & := \hat{\Gamma}_0(p^s) \cap \hat{\Gamma}_r, \quad (s \geq r).
\end{align*}
\]

By definition, \( \hat{\Gamma}_r \cap SL_2(\mathbb{Q}) = \Gamma_1(Np^r) \) as in the introduction if \( H_p = \Gamma \times \{1\} \) (i.e., \( \langle \alpha, \delta \rangle = (0, 1) \)) and \( \xi(a, d) = \omega(d) \) for \( \omega(a) = \lim_{m \to \infty} a^{p^m} \) if \( p \) is odd and otherwise \( \omega(a) = \left( \frac{q_{\nu} \nu - \tau}{\alpha} \right) \) (the quadratic residue symbol). We write this \( \xi \) as \( \omega_d \).

Consider the moduli problem over \( \mathbb{Q} \) of classifying the following triples

\[
(E, \mu_N \xrightarrow{\phi_N} E, \mu_{p^r} \xrightarrow{\phi_{p^r}} E[p^r] \xrightarrow{[\phi_{p^r}]} \mathbb{Z}/p^r\mathbb{Z})/R,
\]

where \( E \) is an elliptic curve defined over a \( \mathbb{Q} \)-algebra \( R \) and the sequence \( \mu_{p^r} \hookrightarrow E[p^r] \to \mathbb{Z}/p^r\mathbb{Z} \) is meant to be exact in the category of finite flat group schemes. As is well known (e.g., [AME]),
the triples are classified by a modular curve $U_r/Q$, and we write $Z_r$ for the compactification of $U_r$ smooth at cusps. In Shimura’s terminology, writing $Z_r'$ for the canonical model attached to $U_r := \Gamma_1'(p') \cap \Gamma_1(N)$, the curve $Z_r'$ is defined over $Q(\mu_{p'})$ and is geometrically irreducible, while we have $Z_r = \Res_{Q(\mu_{p'})/Q} Z_r'$ (when $N \geq 4$) which is not geometrically irreducible. We have the identity of the complex points $Z_r(C) = \GL_2(Q) \setminus \GL_2(k)/U_r \hat{\times} \SO_2^o(R)$.

Each element $(u, a, d)$ the group $G := (\mathbb{Z}/N\mathbb{Z})^\times \times Z_{p'}^\times \times Z_{p}^\times$ acts on $Z_r$ by sending $(E, u, a, d)$ to $(E, \mu_{p'} \circ \varphi_r = E[p'] \varphi_{r,p'} = Z/p'Z)$ to

\[(3.2) \quad (E, \mu_{p'} \circ \varphi_r = E[p'] \varphi_{r,p'} = Z/p'Z),\]

where $a \circ \varphi_{r,p'}(x) = a \varphi_{r,p'}(x)$ and the action on $\varphi_{r,p'}$ and $\mu_N$ is the one we have described in the introduction. For $z = (z_N, z_p) \in (\mathbb{Z}/N\mathbb{Z})^\times \times Z_{p}^\times$, we write the action of $(u, a, d) = (z_N, z_p, z_p)$ as $(z)$. Via the inclusion $\Gamma \times \Gamma \subset G$, the two variable Iwasawa algebra acts $\Lambda := \mathbb{Z}[\Gamma \times \Gamma]$ and $\mathbb{Z}[\Gamma]$ by canonical duality $\langle \cdot, \cdot \rangle$.

By (3.2) we have $E = \Res_r \ker_{\Gamma(\mathbb{Q})} \phi \subset \hat{\varphi}_{r,p'}(z) \subset \varphi_{r,p'}(z_a)$, where $\varphi_{r,p'}$ is determined by $(x, \varphi_{r,p'}(z_a)) = \hat{\varphi}_{r,p'}(x)$ for $x \in E'[p']$. We define an operator $w_r = w_{(a, u, d)}$ acting on $Z_r$ by sending $(E, u, a, d) \mapsto E[p'] \varphi_{r,p'} = Z/p'Z$ to the above $(E', \phi_{r,N}, \varphi_{r,p'})$. We have the following fact from the definition:

**Lemma 3.1.** The tower $\{X_r/Q\}$ with respect to $(-\delta, -\alpha, \xi)$ is isomorphically sent by $w_r$ defined over $Q$ to the tower over $Q$ with respect to $(-\delta, -\alpha, \xi)$ for $\xi(c, d) = \xi(d, a)$. In other words, $H$ defining the tower $\{X_r\}$ is sent to $H'$ defining the other by the involution $(a, d) \mapsto (d, a)$. Regarding $w_r$ as an involution of $X_r$ defined over $Q(\mu_{p'})$, if $s \in \Gal(Q(\mu_{p'}))/Q$ for $z = (\mathbb{Z}/N\mathbb{Z})^\times \times Z_{p}^\times$ is given by $s(z_{p'}) = z'_{p'}$, we have $w_{s_{p'}} = (z) \circ w_r = w_r \circ (z)^{-1}$.

The last assertion of the lemma follows from $w_{s_{p'}} = w_{s_{p'}}(z_{p'}) = w_{s_{p'}}(z) = (z) \circ w_{s_{p'}}$ and $w_r^2 = \id$.

We consider the quotient curves $X_r := Z_r/H$. The complex points of $X_r$ removed cusps is given by $Y_r(C) = \GL_2(Q) \setminus \GL_2(k)/\Gamma_r \hat{\times} \SO_2^o(R)$. Indeed, the action of $(a_p, d_p) \in H$ regarded as an element $\begin{pmatrix} a_p & 0 \\ 0 & d_p \end{pmatrix}$ in $GL_2(Z_p) \subset GL_2(\hat{Z})$ is given by $(\varphi_{r,p'}, \varphi_{r,p'}) \mapsto (\varphi_{r,p'} \circ d_p, \varphi_{r,p'} \circ a_p)$. If $\det(\Gamma_r) = \hat{Z}^\times$, by [IAT, Chapter 6], $X_r$ is a geometrically connected curve canonically defined over $Q$. We have an adelic expression of their complex points.

\[X_r(C) = \GL_2(Q) \setminus \GL_2(k)/\Gamma_r \hat{\times} \SO_2^o(R) = \Gamma_r \setminus \mathfrak{X} \text{ and } X_r(C) = \Gamma_r \setminus \mathfrak{X},\]

where $\Gamma_r^* = \hat{\Gamma}_r \cap \text{SL}_2(Q)$ and $\Gamma_r = \hat{\Gamma}_r \cap \text{SL}_2(Q)$. If $\det(\Gamma_r) \supseteq \hat{Z}^\times$, our curve $X_r^* = \Res_{F_r/Q} V_{\Gamma_r}$ and $X_r = \Res_{F_r/Q} V_{\Gamma_r}$ for Shimura’s geometrically irreducible canonical model $V_{S}$ defined over $F_r$ for $S = \hat{\Gamma}_r$ and $\hat{\Gamma}_r$ (see [IAT, Chapter 6]). In any case, these curves are geometrically reduced curves defined over $Q$ with equal number of geometrically connected components (i.e., it is $[F_r : Q]$ for Shimura’s field of definition $F_r$ fixed by $\det(\Gamma_r) \subset \hat{Z}^\times \cong \Gal(Q(ab)/Q)$).

The group $\Gamma_r^*$ $(s > r)$ normalizes $\Gamma_s$, and we have $\Gamma_r^/\Gamma_s = \Gamma_r^0/\Gamma_s$ is canonically isomorphic to $H \mod p^s$ by sending coset $(c, c') \Gamma_s$ to $(a_p, d_p) \mod p^s \in (H \mod p^s)$, and the moduli theoretic action of $H$ coincides with the action of $\Gal(X_r/X_r^0) = (H \mod p^s)$. Through $\Gamma \cong (\Gamma \times \Gamma)/H_p$ (resp. $Z_{p'} \times Z_{p}^\times \mod p^s$), the one variable Iwasawa algebra $\Lambda$ (resp. $Z_p[[Z_{p'}^\times]] = \Lambda[[\mu]]$) acts on the tower $\{X_r\}$. 

If \( \det(\hat{\Gamma}_{H,r}) = \widehat{\mathbb{Z}}^\times \), as explained in [IAT, Chapter 6], \( X_{r/Q} \) and \( X_{r/Q}^r \) is geometrically irreducible. Though we do not need geometric irreducibility, we indicate here an easy criterion when geometric irreducibility holds. We note that \( \det(\hat{\Gamma}_r) \supset (\widehat{\mathbb{Z}}(p))^{\times} \), where \( \widehat{\mathbb{Z}}(p) = \prod_{t \not| p} \mathbb{Z}_t \cong \widehat{\mathbb{Z}}/\mathbb{Z}_p \). Thus the problem is reduced to the study of the determinant map at \( p \). By \( \alpha Z_p + \delta Z_p = Z_p \), it is easy to see by definition, embedding diagonally \( H \) into \( GL_2(\mathbb{Z}_p) \), that

\[
(3.3) \quad \det : H_p \to \Gamma \text{ is an isomorphism if and only if } p \nmid (\alpha + \delta) \text{ or } \alpha \cdot \delta = 0.
\]

If \( (\alpha', \delta') \in \mathbb{Z}^2 \) with \( \alpha'Z + \delta'Z = Z \) and \( \xi(a, d) = \omega(a)^\alpha \omega(d)^{-\delta} \).

\[
(3.4) \quad \det : (H \cap \mu \times \mu) \to \mu \text{ is an isomorphism if } \alpha' + \delta' \text{ is prime to } 2 \cdot (p - 1) \text{ or } \alpha' \cdot \delta' = 0.
\]

The second condition becomes also a necessary condition if we replace \( \alpha' \cdot \delta' = 0 \) by \( \alpha' \cdot \delta' \equiv 0 \mod p - 1 \) if \( p \) is odd and by \( \alpha' \cdot \delta' \equiv 0 \mod 2 \) if \( p = 2 \). If \( \alpha' = \delta' = i \), then \( \text{Ker}(\xi) \supset (\zeta, \zeta) \), and hence \( \det(H) \supset \mu^2 \). Thus to have a non-trivial element in \( \text{Ker}(\omega^i) \) in \( \mu \setminus \mu^2 \), \( \omega^i \) has to have odd order.

\[
(3.5) \quad \det : (H \cap \mu \times \mu) \cong \mu \text{ if } \alpha' = \delta' = i \text{ and } \omega^i \text{ has odd order}.
\]

The image \( \det(H) \) can be a proper subgroup in \( \mathbb{Z}_p^2 \), and the curve \( X_r \) and \( X_r^r \) becomes reducible over the subfield \( F = F_{\xi/Q} \) fixed by \( \det(H) \) identifying \( \text{Gal}(Q(\mu_{p^\infty})/Q) \) with \( \mathbb{Z}_p^2 \) by the \( p \)-adic cyclotomic character.

The most interesting case is when \( \xi(a, d) = \omega^i(a)^{-\epsilon}(d) \) \( (i = 0, 1, \ldots, p - 2) \) and \( \alpha = \delta = 1 \). Suppose \( \alpha' = \delta' = i \) for \( 0 \leq i < p \) (so, \( \alpha'Z + \delta'Z = i\mathbb{Z} \)). In this case, the L-function \( L(s, F_p) \) can have root number \( \pm 1 \) (so, Birch–Swinnerton Dyer conjecture would force the non-triviality of the Mordell–Weil group of \( A_F \) if the root number is \( -1 \)). By (3.5), \( \det : \text{Ker}(\xi) \to \mu \) is onto if and only if \( \omega^i \) has odd order (including the case where \( i = 0 \)), and hence \( \det(\hat{\Gamma}_{H,r}) = \widehat{\mathbb{Z}}^\times \) if \( p > 2 \) and \( \omega^i \) has odd order. Otherwise, if \( p > 2 \), \( F_\xi \) is a unique quadratic extension of \( Q \) inside \( Q(\mu_p) \). If \( p = 2 \), if \( \alpha = \delta = 1 \) and \( \alpha' = \delta' = 0 \), \( F_\xi = Q(\sqrt{2}) \); and if \( \alpha = \delta = 1 \) and \( \alpha' = \delta' = 1 \), then \( F_\xi = Q(\sqrt{-1}, \sqrt{2}) \).

Taking \( (X, Y, U)/\mathbb{S} \) to be \( (X_{s/Q}, X_{s/Q}^r, U(p))/\mathbb{Q} \) for \( s > r \geq 1 \), we get for the projection \( \pi : X_s \to X_r^r \). The result of the previous section is plainly applicable if \( X_{s/Q}^r \) is geometrically irreducible (as discussed above), since \( U(p) \) is also geometrically irreducible as it is the image of \( X_{s+1}^r := \delta_f/(\Gamma_s \cap \Gamma_0(p^{s+1})) \) by the diagonal product of two degeneration maps from \( X_{s+1}^r \) in \( X_s^r \times X_r^r \). If not, writing \( X_{s/Q} \) for geometrically irreducible components \( X_{s+i}^r \), then \( U(p) \) restricted in each \( X_{s+i}^r \times X_r^r \) is geometrically irreducible by the same argument above and its degree is a \( p \)-power independent of the components; so, we can apply the argument in Section 2 in this geometrically reducible cases even over \( \mathbb{Q} \).

**Corollary 3.2.** Let \( F \) be a number field or a finite extension of \( Q_l \) for a prime \( l \). Then we have, for integers \( r, s \) with \( s \geq r \geq \epsilon \),

\[
(\text{u}) \quad \pi^* : J_{s/Q}^r(F) \to \tilde{H}^0(X_s/X_r^r, J_{s/Q}(F)) \xrightarrow{(\ast)} J_{s/Q}(F)[\gamma^p] - 1 \text{ is a } U(p)\text{-isomorphism,}
\]

where \( J_{s/Q}(F)[\gamma^p] - 1 = \text{Ker}(\gamma^p - 1 : J_s(F) \to J_s(F)) \) and \( \epsilon = 1 \) if \( p > 2 \) and \( \epsilon = 2 \) if \( p = 2 \).

Here the identity at (\ast) follows from Lemma 2.4. The kernel \( A \to \text{Ker}(\gamma^p - 1 : J_s(A) \to J_s(A)) \) is an abelian fpf sheaf (as the category of abelian fpf sheaves is abelian and regarding a sheaf as a presheaf is a left exact functor), and it is represented by the scheme theoretic kernel \( J_{s/Q}[\gamma^p] - 1 \) of the endomorphism \( \gamma^p - 1 \) of \( J_{s/Q} \). From the exact sequence \( 0 \to J_s[\gamma^p - 1] \to J_s[\gamma^p - 1] \to J_s \), we get another exact sequence

\[
0 \to J_s[\gamma^p - 1](F) \to J_s(F) \xrightarrow{\gamma^p - 1} J_s(F).
\]

Thus

\[
J_{s/Q}(F)[\gamma^p] - 1 = J_{s/Q}[\gamma^p] - 1(F).
\]

The above (u) combined with this implies the sheaf identity (u2) below for integers \( r, s \) with \( s \geq r \geq \epsilon \):

(u2) \( \pi^* : J_r/Q \to J_s/Q[\gamma^{p^{r-s}} - 1] = \text{Ker}(\gamma^{p^{r-s}} - 1 : J_s/Q \to J_s/Q) \) is a \( U(p) \)-isomorphism.

As long as

\[(3.6) \quad X_\epsilon(K) \neq \emptyset, \]

\( J_r \) for \( r < s \) is self-dual, and we have Albanese functoriality (cf. [ARG, VII.6]). Even if \( X(K) = \emptyset \), assuming \( X \xrightarrow{\pi} Y \) is finite flat with constant degree, as explained in Lemma 2.1, we still have covariant Albanese map \( \text{Pic}_{X/S} \xrightarrow{\pi} \text{Pic}_{Y/S} \). Thus to have the map \( \pi_* \), the condition like (3.6) is not necessary.

Let \( \{X'_r/Q\} \) be the (dual) tower out of the data \((-\delta, -\alpha, \xi')\) as in Lemma 3.1. Write \( J_r = \text{Pic}_{X_r/Q}^0 \) and \( J'_r = \text{Pic}_{X'_r/Q}^0 \) (see [NMD, §9.3]). Then we define Hecke equivariant projection \( \pi'_* : J_r \to J_s \) \((s \geq r)\) by the following commutative diagram:

\[
\begin{array}{ccc}
J_s & \xrightarrow{\pi'_*} & J_s' \\
\downarrow{w_r} & & \downarrow{w_r} \\
J'_s & \xrightarrow{\pi''_*} & J'_s',
\end{array}
\]

where \( \pi'_{s,r,*} \) is induced by the projection \( \pi'_{s,r} : X'_r \to X'_r \) by Albanese functoriality. The Albanese projection \( \pi'_{s,r,*} \) commutes with \( T^*(n) \) (the dual Hecke operator). Since \( T^*(n) \circ w_s = w_s \circ T(n) \) \((e.g., [MFM, Theorem 4.5.5]), \( \pi'_* \) commutes with \( T(n) \).

**Lemma 3.3.** For integers \( r, s \) with \( s \geq r \geq \epsilon \), we have morphisms

\[ \iota'_s : J_{s/Q}[\gamma^{p^{r-s}} - 1] \to J'_{s/Q} \quad \text{and} \quad \iota''_s : J'_{s/Q} \to J_{s/Q}/(\gamma^{p^{r-s}} - 1)(J_{s/Q}) \]

satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
J'_{s/Q} & \xrightarrow{\pi'} & J_{s/Q}[\gamma^{p^{r-s}} - 1] \\
\downarrow{u} & & \downarrow{u''} \\
J'_{s/Q} & \xrightarrow{\iota'_s} & J_{s/Q}[\gamma^{p^{r-s}} - 1],
\end{array}
\]

and

\[
\begin{array}{ccc}
J'_{s/Q} & \xrightarrow{\pi''} & J_{s/Q}/(\gamma^{p^{r-s}} - 1)(J_{s/Q}) \\
\downarrow{u^*} & & \downarrow{u'^*} \\
J'_{s/Q} & \xrightarrow{\iota''_s} & J_{s/Q}/(\gamma^{p^{r-s}} - 1)(J_{s/Q}),
\end{array}
\]

where \( u \) and \( u'' \) are \( U(p^{s-r}) = U(p)^{s-r} \) and \( u^* \) and \( u'^* \) are \( U^*(p^{s-r}) = U^*(p)^{s-r} \). In particular, for an fpfp extension \( T_{/Q} \), the evaluated map at \( T : (J_{s/Q}/(\gamma^{p^{r-s}} - 1)(J_{s/Q}))(T) \xrightarrow{\pi''} J'_s(T) \) (resp. \( J'_s(T) \xrightarrow{\pi'} J_{s/Q}/(\gamma^{p^{r-s}} - 1)(J_{s/Q}) \)) is a \( U^*(p) \)-isomorphism (resp. \( U(p) \)-isomorphism).

**Proof.** We first prove the assertion for \( \pi^* \). We note that the category of groups schemes fpfp over a base \( S \) is a full subcategory of the category of abelian fpfp sheaves. We may regard \( J'_{s/Q} \) and \( J_{s/Q}[\gamma^{p^{r-s}} - 1]/Q \) as abelian fpfp sheaves over \( Q \) in this proof. Since these sheaves are represented by (reduced) algebraic groups over \( Q \), we can check being \( U(p) \)-isomorphism by evaluating the sheaf at a field \( K \) of characteristic 0 \((e.g., [EAI, Lemma 4.18])\). By Proposition 2.3 (2) applied to \( X = X_{s/Q} \) and \( Y = X'_{s/Q} \) \((with \ S = \text{Spec}(Q) \) and \( s \geq r)\),

\[ \mathcal{K} := \text{Ker}(J'_{s/Q} \to J_{s/Q}[\gamma^{p^{r-s}} - 1]) \]

is killed by \( U(p)^{s-r} \) as \( d = p^{s-r} = \text{deg}(X_s/X'_s) \). Thus we get

\[ \mathcal{K} \subset \text{Ker}(U(p)^{s-r} : J'_{s/Q} \to J'_{s/Q}). \]

Since the category of fpfp abelian sheaves is an abelian category \( (because \ of \ the \ existence \ of \ the \ sheafification \ functor \ from \ presheaves \ to \ sheaves \ under \ fpfp \ topology \ described \ in \ [ECH, \ §II.2]) \), the above inclusion implies the existence of \( \iota'_s \) with \( \pi^* \circ \iota'_s = U(p)^{s-r} \) as a morphism of abelian fpfp
sheaves. Since the category of group schemes fppf over a base $S$ is a full subcategory of the category of abelian fppf sheaves, all morphisms appearing in the identity $\pi^* \circ \iota_s = U(p)^{s-r}$ are morphism of group schemes. This proves the assertion for $\pi^*$.

Take a number field so that $X_s(K) \neq \emptyset$ (for example, the infinity cusp of $X_s$ is rational over $\mathbb{Q}(\mu_p)$). Then $\text{Pic}^0_{J_s/K} \cong J_s^*$ for any $s \geq r \geq 0$ by the self-duality of the jacobian variety. Note that the second assertion is the dual of the first under this self-duality; so, over $K$, it can be proven reversing all the arrows and replacing $J_s[\gamma^{p^{-\epsilon}} - 1]/K$ (resp. $\pi^*, U(p)$) by the quotient $J_s/(\gamma^{p^{-\epsilon}} - 1)J_s$ as fppf abelian sheaves (resp. $\pi^*, U^*(p)$). By Lemma 2.1, every morphism and abelian variety of the diagram in question are all well defined over $\mathbb{Q}$. In particular $J_s/(\gamma^{p^{-\epsilon}} - 1)(J_s)$ is an abelian variety quotient over $\mathbb{Q}$ (cf., [NMD, Theorem 8.2.12] combined with [ARG, §V.7]). Then by Galois descent for projective varieties (e.g., [GME, §1.11]), the diagram descends to $\mathbb{Q}$. Since being $U^*(p)$-isomorphism or $U(p)$-isomorphism is insensitive to the descent process, we get the final assertion. 

\[\square\]

4. Hecke algebras for exotic towers

Hereafter, whenever the tower $\{X_r\}_r$ is dealt with, the data $(\alpha, \delta, \xi)$ which defines the exotic tower is fixed. We introduce the Hecke algebra $h_{\alpha,\delta,\xi}$ for the exotic tower $\{X_r\}_r$ defined for $(\alpha, \delta, \xi)$. Fix a prime $p$. We assume in the rest of the paper the following condition:

(F) The Hecke algebra $h_{\alpha,\delta,\xi}$ is $\Lambda$-free.

If $(\alpha, \delta) = (0, 1)$ and $\xi(a, d)$ only depends on $d$, this is always true, and as we see in this section, the freeness (F) holds for $p \geq 5$ without any other assumptions, and even for $p = 3$, for most of $(\alpha, \delta)$ including the interesting case of $\alpha = \delta = 1$, (F) holds still true (see Proposition 18.2).

Let $\{X_r/\mathbb{Q}\}_r$ be the exotic tower as in Section 3. As described in (3.2), $z \in \mathbb{Z}_{p^\infty}$ acts on $X_r$. Recall that $J_s/\mathbb{Q}$ (resp. $J_s/\mathbb{Q}$) is the Jacobian of $X_r$ (resp. $X_r^*$). We regard $J_s$ as the degree 0 component of the Picard scheme of $X_r$. For an extension $K/\mathbb{Q}$, we consider the group of $K$-rational points $J_s(K)$.

For each prime $l$, we consider $\varpi_l := \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_l)$, and regard $\varpi_l \in \text{GL}_2(\mathbb{A})$ so that its component at each place $v | l$ is trivial. Then $\Delta := \varpi_l^{-1} \Gamma_s \varpi_l \cap \Gamma_s$ gives rise to a modular curve $X(\Delta)$ whose $\mathbb{C}$-points is given by $GL_2(\mathbb{Q}) \backslash (GL_2(\mathbb{A}^{\infty}) \times (\mathfrak{h} \cup \mathfrak{f})) / \Delta$. We have a projection $\pi_l : X(\Delta) \to X_s^\circ$ given by $\mathfrak{h} \ni z \mapsto z/l \in \mathfrak{f}$ in addition to the natural one $\pi_l : X(\Delta) \to X_s$ coming from the inclusion $\Delta \subset \Gamma_s^\circ$. Then embedding $X(\Delta)$ into $X_s^\circ \times X_s^\circ$ by these two projections, we get the modular correspondence written by $T(l)$ if $l \nmid Np$ and $U(l)$ if $l \mid Np$. We can extend this definition to $T(n)$ for all $n > 0$ prime to $Np$ via Picard/Albanese functoriality (see Lemma 2.1). We use the same symbol $T(n)$ and $U(l)$ to indicates the endomorphism (called the Hecke operator) given by the corresponding correspondence $T(n)$ and $U(l)$. The Hecke operator $U(p)$ acts on $J_s(K)$ and the $p$-adic limit $\epsilon = \lim_{n \to \infty} U(p)^{n\epsilon}$ is well defined on the Barsotti–Tate group $J_s[p^{\infty}]$ and the completed Mordell–Weil group $\hat{J}_s(K)$ as defined in (S) above.

Let $\Gamma$ be the maximal torsion-free subgroup of $\mathbb{Z}_{p^\infty}$ given by $1 + p^r \mathbb{Z}_p$ for $\epsilon = 1$ if $p > 2$ and $\epsilon = 2$ if $p = 2$. Writing $\gamma = 1 + p^r \in \Gamma$, $\gamma$ is a topological generator of the multiplicative group $\Gamma = \mathbb{Z}_{p^r}$. Throughout this paper $\epsilon = 1$ if $p > 2$ and $\epsilon = 2$ if $p = 2$.

Let $h_r(\mathbb{Z}) = \mathbb{Z}[T(n), U(l) : l \nmid Np, (n, Np) = 1] \subset \text{End}(J_r)$, and put $h_r(R) = h_r(\mathbb{Z}) \otimes_\mathbb{Z} R$ for any commutative ring $R$. Then we define $h_r = h_{r,\alpha,\delta,\xi} := e(h_r(\mathbb{Z}_p))$. The restriction morphism $h_{r,s}(\mathbb{Z}) \ni h \mapsto h|_{J_s} \in h_r(\mathbb{Z})$ for $s > r$ induces a projective system $\{h_r\}_r$ whose limit gives rise to the big ordinary Hecke algebra

$h = h_{\alpha,\delta,\xi}(\mathbb{N}) := \lim_{\rightarrow} h_r$.

Writing $l(l)$ (the diamond operator) for the action of $l \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$ identified with $\text{Gal}(\mathbb{X}_r/\mathbb{X}_0(Np^r))$, we have an identity $l(l) = T(l)^2 - T(l^2) \in h_r(\mathbb{Z}_p)$ for all primes $l \nmid Np$. Thus we have a canonical $\Lambda$-algebra structure $\Lambda = \mathbb{Z}[l(l)] \hookrightarrow h$. If $(\alpha, \delta) = (0, 1) = (\alpha', \delta')$, it is now well known that $h$ is a free of finite rank over $\Lambda$ and $h_r = h \otimes_\Lambda (\gamma^{p^{-\epsilon}} - 1)$ (cf. [H86a], [GK13] or [GME, §3.2.6]).
More generally, by [PAF, Corollary 4.31], assuming \( p \geq 5 \), the same facts hold (and we expect this to be true without any assumption on primes). Anyway, if \( p = 2, 3 \), the specialization map \( \mathbf{h} \otimes_{\Lambda} \Lambda/\langle \gamma^p - 1 \rangle \rightarrow \mathbf{h}_r \) is onto with finite kernel, and \( \mathbf{h} \) is a torsion-free \( \Lambda \)-module of finite type. We will prove the \( \Lambda \)-freeness of \( \mathbf{h}_{\alpha, \delta, \xi}(N) \) and isomorphisms \( \mathbf{h} \otimes_{\Lambda} \Lambda/\langle \gamma^p - 1 \rangle \cong \mathbf{h}_r \) for most cases of \( p = 3 \) in Section 18 for the sake of completeness.

A prime \( P \) in \( \Omega_r := \bigcup_{r > 0} \text{Spec}(h_r)(\overline{\mathbb{Q}}_p) \subset \text{Spec}(h)(\overline{\mathbb{Q}}_p) \) is called an arithmetic point of weight 2. In this paper, we only deal with arithmetic point of weight 2; so, we often omit the word “weight 2” and just call them arithmetic points/primes. Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism \( \lambda : h \rightarrow \overline{\mathbb{Q}}_p \) killing \( \gamma^r - 1 \) for \( r \geq 0 \) to a classical Hecke eigenform, we need to fix (once and for all) an embedding \( \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p \) of the algebraic closure \( \overline{\mathbb{Q}} \) in \( \mathbb{C} \) into a fixed algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \). We write \( i_{\infty} \) for the inclusion \( \overline{\mathbb{Q}} \subset \mathbb{C} \).

More generally, we consider the jacobian variety \( J(Z_r) \) of the curve \( Z_r \) defined above (3.2), and define \( h^\text{ord}_{r} \) to be the maximal \( \Lambda \)-algebra direct summand of \( \text{End}(J(Z_r)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \) in which \( U(p) \) is invertible. Then as before we define \( h^\text{ord}_r = h(N)^{n, \text{ord}} := \operatorname{lim}_{\leftarrow r} h^r_{\text{odd}} \), which is a \( \Lambda \)-algebra. Take a semi-simple modular residual representation \( \overline{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{F}) \) of prime-to-\( p \) conductor \( N \).

We consider

\[
h^{\text{ord}, \varphi} = h^{\text{ord}} \otimes_{\mathbb{Z}_p} W/\mathfrak{a}_r h^{\text{ord}} \otimes_{\mathbb{Z}_p} W
\]

where \( \mathfrak{a}_r \) is the kernel of the algebra homomorphism \( W[[Z_p^\times \times Z_p^\times]] \rightarrow W[[Z_p^\times]]^\times \) induced by the character \( (a, d) \mapsto \varphi(a, d) \xi(a, d) : Z_p^\times \times Z_p^\times \rightarrow Z_p^\times \). If we take \( \varphi(a, d) = a^d d^{-\delta} \), for \( (a, d) \in \Gamma \times \Gamma \) and \( W = \mathbb{Z}_p \), we have \( h^{\text{ord}, \varphi} = h_{\alpha, \delta, \xi}(N) \) under present notation. Then by [PAF, Corollary 4.31], \( h^{\text{ord}, \varphi} \) is \( \Lambda \)-free of finite rank for \( \Lambda = \mathbb{Z}_p[[\Gamma^2/H_\Gamma]] \).

**Proposition 4.1.** Assume \( p \geq 5 \) or \( (\alpha, \delta, \xi) = (0, 1, \omega_3) \), where \( \omega_3(a, d) = \omega(d) \). Then \( h_{\alpha, \delta, \xi}(N) \) is \( \Lambda \)-free of finite rank for \( \Lambda = \mathbb{Z}_p[[\Gamma^2/H_\Gamma]] \).

**Remark 4.2.** For \( p \leq 3 \), we will prove in Proposition 18.2 freeness over \( \Lambda \) of \( h_{0,1,\omega_3}(N) \) if it is obtained by systematic twists of \( h_{0,1,\omega_3}(N) \). This covers the interesting cases of analytic families of abelian variety, including some corresponding to the \( p \)-adic \( L \)-function \( k \mapsto L(2k, k) \) as in the introduction.

Picard functoriality gives injective limits \( J_{\infty}(K) = \lim_{\leftarrow \gamma} J_{\gamma}(K) \) and \( J_{\infty}[p^\infty](K) = \lim_{\leftarrow \gamma} J_{\gamma}[p^\infty](K) \), on which \( e \) acts. Write \( \mathcal{G} = \mathcal{G}_{\alpha, \delta, \xi} := e(J_{\infty}[p^\infty]) \), which is called the \( \Lambda \)-adic Barsotti–Tate group in [H14a] and whose arithmetic property was scrutinized there. Adding superscript or subscript “ord”, we indicate the image of \( e \).

The compact cyclic group \( \Gamma \) acts on these modules by the diamond operators. In other words, we identify canonically \( \text{Gal}(X_r/\text{Spec}(\mathbb{Z}/(Np^r))) \) for modular curves \( X_r \) and \( X_0(Np^r) \) with \( (\mathbb{Z}/Np^r)^\times \), and the group \( \Gamma \) acts on \( J_{\gamma} \) through its image in \( \text{Gal}(X_r/\text{Spec}(\mathbb{Z}/(Np^r))) \). Thus \( J_{\infty}(K)^{\text{ord}} \) is a module over \( \Lambda := \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]] \) by \( \gamma \mapsto t = 1 + T \) for a fixed topological generator \( \gamma \) of \( \Gamma = \mathbb{Z}_p^\times \). The big ordinary Hecke algebra \( h \) acts on \( J_{\infty}^{\text{ord}} \) as endomorphisms of functors.

Let \( \text{Spec}(\mathcal{T}) \) be a connected component of \( \text{Spec}(h) \) and \( \text{Spec}(\mathcal{L}) \) be a primitive irreducible component of \( \text{Spec}(\mathcal{T}) \). We write \( \Omega_\ell = \bigcup_{\ell > 0} \text{Spec}(\mathcal{L}/(\mathbb{Z}/(p^\ell - 1))) \) (which is the set of all arithmetic points of weight 2). For \( P \in \Omega_\ell \) with \( P \in \text{Spec}(\mathcal{L}/(\mathbb{Z}/(p^\ell - 1))) \), we write \( r(P) \) for the minimal \( r > 0 \) with this property. Then the corresponding Hecke eigenform \( f_P \) belongs to \( S_2(\Gamma_0(Np^r), \varepsilon_r \chi_r) \) for a character \( \varepsilon_r : \mathbb{Z}_p^\times \rightarrow \mu_{p^r} \) and a character \( \chi_r : \mu \times (\mathbb{Z}/N^\times) \rightarrow \mathbb{C}^\times \). Here \( f \) in this space satisfies

\[
f(aba^{-d}) = e_P(a^{-d}d^\delta \chi_p(\xi(a, d)))\chi(d) f \]

for \( (a, b, c) \in \Gamma_0(Np^r) \). Here \( \chi_p = \chi_\mu \) and \( \chi_N = \chi_{(\mathbb{Z}/N^\times)} \). The corresponding adelic form \( f \) satisfies

\[
f(axu) = e_P(a^{-d}d^\delta \chi_p(\xi^{-1}(a, p))\chi(d_N) f \]

for all \( u = (a, b, c) \in \Gamma_0(Np^r) \). Here \( \chi(d) = \chi_N(d_N^{-1}) = \chi(d_N) \) regarding \( d \in \mathbb{A}_\Gamma^\times \).

For each \( h \)-module \( M \), we put \( M_{\mathcal{T}} := M \otimes h_{\mathcal{T}} \); in particular, \( J_{\infty, \mathcal{T}}^{\text{ord}} := J_{\infty}^{\text{ord}} \otimes h_{\mathcal{T}}^\times \) as an fppf sheaf.
5. Abelian factors of \( J_r \).

We give a description of abelian factors \( A_s \) and \( B_s \) (we study in this paper) of the modular jacobian varieties \( \{ J_{r,s} \} \) of the exotic modular tower introduced in the previous section which behaves coherently in the limit process under the Hecke operator action. Let \( \text{Spec}(T) \) be a connected component of \( \text{Spec}(h(N)) \). Write \( m_\Sigma \) for the maximal ideal of \( T \) and \( 1_\Sigma \) for the idempotent of \( T \) in \( h(N) \).

We assume the following condition

(A) We have \( \varpi \in m_\Sigma \) such that \( (\varpi) \cap \Lambda \) is a factor of \((\gamma^p - 1)\) in \( \Lambda \) and that \( T/(\varpi) \) is free of finite rank over \( \mathbb{Z}_p \).

Write \( \varpi_s \) for the image of \( \varpi \oplus (1 - 1_\Sigma) \) in \( h_s \) (\( s \geq r \)) and define an \( h_s(\mathbb{Z}) \)-ideal by

\[
a_s = (\varpi_s h_s + (1 - e)h_s(\mathbb{Z}_p)) \cap h_s(\mathbb{Z}).
\]

Write \( A_s \) for the identity connected component of \( J_s[a_s] = \bigcap_{a \in a_s} J_s[a] \), and put \( B_s = J_s/a_s J_s \), where \( a_s J_s \) is a rational abelian subvariety of \( J_s \) given by \( a_s J_s(\mathbb{Q}) = \sum_{a \in a_s} a(J_s(\mathbb{Q})) \subset J_s(\mathbb{Q}) \).

Taking a finite set \( G \) of generators of \( a_s \), \( \{ a_s J_s \} \) is the image of \( a : \bigoplus_{g \in G} J_s \xrightarrow{\sum g(x)} J_s \). The kernel \( J_s[a] = \text{Ker}(a) \) is a well defined fppf sheaves, which is represented by an extension of the abelian variety \( A_s \) by a finite étale group scheme both over \( \mathbb{Q} \). Then by [NMD, Theorem 8.2.12], the quotient \( A_s/J_s \cong (\bigoplus_{g \in G} J_s)/K \) is well defined as an abelian scheme and is the sheaf fppf quotient.

Then again \( B_s := J_s/a_s J_s \) is the fppf sheaf quotient and also abelian variety quotient again by [NMD, Theorem 8.2.12]. By definition, \( A_s \) is stable under \( h_s(\mathbb{Z}) \) and \( h_s(\mathbb{Z})/a_s \to \text{End}(A_s) \).

**Lemma 5.1.** Assume (F) and (A). Then we have \( \hat{A}_s^{\text{ord}} = \hat{J}_s^{\text{ord}}[\varpi_s] \) and \( \hat{J}_s[a_s] = \hat{A}_s \). The abelian variety \( A_s \) (\( s > r \)) is the image of \( A_s \) in \( J_s \) under the morphism \( \pi^s = \pi^{s,r} : J_s \to J_s \) induced by Picard functoriality from the projection \( \pi = \pi_{s,r} : X_s \to X_r \) and is \( \mathbb{Q} \)-isogenous to \( B_s \). The morphism \( J_s \to B_s \) factors through \( J_s \xrightarrow{\pi^s} J_s \to B_s \). In addition, the sequence

\[
0 \to \hat{A}_s^{\text{ord}} \to \hat{J}_s^{\text{ord}} \xrightarrow{\varpi} \hat{J}_s^{\text{ord}} \to \hat{B}_s^{\text{ord}} \to 0
\]

is an exact sequence of fppf sheaves.

Passing to the limit, we get the following exact sequence of fppf sheaves:

\[
0 \to \hat{A}_s^{\text{ord}} \to \hat{J}_s^{\text{ord}} \xrightarrow{\varpi} \hat{J}_s^{\text{ord}} \to \hat{B}_s^{\text{ord}} \to 0,
\]

where \( J_s^{\text{ord}} = \lim_{s \to s} \hat{J}_s^{\text{ord}} \) and \( \hat{X}^{\text{ord}} = \lim_{s \to s} \hat{X}_s^{\text{ord}} \) for \( X = A, B \).

**Proof.** Taking a finite set \( G \) of generators of \( a_s \) containing \( \varpi_s \), we get an exact sequence \( 0 \to J_s[a_s] \to J_s \xrightarrow{\sum g(x) \in G} \bigoplus_{g \in G} J_s \). Since \( X \to \hat{X} \) as in (S) is left exact, we have \( \hat{A}_s \subset \bigcap_{a \in a_s} \hat{J}_s[a] \) with finite quotient. Applying further the idempotent, since \( a_s = ((\varpi_s) \oplus (1 - e)h_s(\mathbb{Z}_p)) \cap h_s(\mathbb{Z}) \), we find

\[
\hat{J}_s[a_s]^{\text{ord}} = \bigcap_{a \in a_s} \hat{J}_s[a]^{\text{ord}} = \hat{J}_s^{\text{ord}}[\varpi_s].
\]

We have an exact sequence

\[
0 \to J_s[a_s][p^\infty]^{\text{ord}} \to J_s[p^\infty]^{\text{ord}} \xrightarrow{\varpi} J_s[p^\infty]^{\text{ord}} \to \text{Coker}(\varpi_s) \to 0,
\]

and \( \text{Coker}(\varpi_s) \) is \( p \)-divisible and is dual to \( J_s[a_s][p^\infty]^{\text{ord}} \) under the \( w_s \)-twisted self Cartier duality of \( J_s[p^\infty]^{\text{ord}} \) (over \( \mathbb{Q} \); see [H14a, §4]). This shows \( \hat{J}_s[a_s][p^\infty]^{\text{ord}} \) is \( p \)-divisible (so, \( (J_s[a_s]/A_s)^{\text{ord}} \) has order prime to \( p \)), and hence \( \hat{A}_s^{\text{ord}} = \hat{J}_s[a_s]^{\text{ord}} \).

Plainly by definition, \( \pi^s(J_s[a_s]) \subset J_s[a_s] \). Since we have the following commutative diagram:

\[
\begin{array}{ccc}
\hat{h}_s(\mathbb{Z}) & \longrightarrow & h_r(\mathbb{Z}) \\
\downarrow & & \downarrow \\
\hat{h}_s(\mathbb{Z}_p)/(\varpi_s h_s(\mathbb{Z}) + (1 - e)h_s(\mathbb{Z}_p)) & \longrightarrow & h_r(\mathbb{Z}_p)/(\varpi h_r + (1 - e)h_r(\mathbb{Z}_p)),
\end{array}
\]

with
Thus we have \( \dim A_s = \text{rank}_\mathbb{Z} h_s(\mathbb{Z})/a_s = \text{rank}_\mathbb{Z} h_t(\mathbb{Z})/a_t = \dim A_t \); so, \( A_s = \pi^*(A_t) \).

The above commutative diagram also tells us that \( a_s \supset b_s := \text{Ker}(h_s(\mathbb{Z}) \to h_r(\mathbb{Z})) \) in \( h_s(\mathbb{Z})_p \). Thus the projection \( J_s \to J_s/a_sJ_s = B_s \) factors through \( J_r = J_s/b_sJ_s \). Indeed, the natural projection: \( J_s/b_sJ_s/Q \to J_r/Q \) has to be a finite morphism (as the tangent space at the origin of the two are isomorphic), and we conclude \( J_s/b_sJ_s = J_r \) by the universality of the categorical quotient \( J_s/a_sJ_s \) (cf., [NMD, page 219]).

Assuming \( J_s(K) \neq \emptyset \) for a finite extension \( K/Q \), the dual sequence (over \( K \)) of the exact sequence of fppf sheaves: \( 0 \to J_s[a_s] \to J_s \xrightarrow{x-\gamma(x)+x} \bigoplus_{g \in G} J_s \to B_s \to 0 \).

Thus \( A_s \) is isogenous to \( B_s \) over \( K \), and by Galois descent, \( A_s \) is \( \mathbb{Q} \)-isogenous to \( B_s \). Indeed, for the complementary abelian subvariety \( A_s^\perp \) in \( J_s \) of \( A_s \), we have \( J_s/A_s^\perp = B_s \), and the \( \mathbb{Q} \)-isogeny follows without taking duality. Here note that the quotient \( J_s/A_s^\perp \) exists as an abelian variety and also as an fppf sheaves by [NMD, Theorem 8.2.12] (and [ARG, V.7]).

Since the morphism \( h \) is verified. Thus \( J_s \to J_s[a_sJ_s] \) is \( \text{fppf sheaves} \) by \([NMD, \text{Theorem } 8.2.12]\) (and \([ARG, \text{V.7}]\)).

As explained just below (A), we have \( \text{Im}(\bigoplus_{g \in G} J_s \xrightarrow{x-\gamma(x)+x} J_s) = a_sJ_s \) is fppf sheaves. Then applying the argument of [H14b, Section 1] to the exact sequence

\[
0 \to J_s[a_s] \to \bigoplus_{g \in G} J_s \to aJ_s \to 0
\]

of fppf sheaves, we confirm the exactness of

\[
0 \to \hat{J}_s[a_s] \to \bigoplus_{g \in G} \hat{J}_s \to \bar{a}\hat{J}_s \to 0
\]
as fppf sheaves. Thus applying the idempotent \( e \), we confirm

\[
\text{Im}(\bigoplus_{g \in G} \hat{J}_s^{\text{ord}} \xrightarrow{x-\gamma(x)+x} \hat{J}_s^{\text{ord}}) = a_s\hat{J}_s^{\text{ord}}.
\]

Since the morphism \( \bigoplus_{g \in G} \hat{J}_s^{\text{ord}} \xrightarrow{x-\gamma(x)+x} \hat{J}_s^{\text{ord}} \) factors through \( \varpi_s(\hat{J}_s^{\text{ord}}) \) as all \( g = \varpi_s x \) with \( x \in h_s \), noting \( \varpi \in G \), \( \varpi_s(\hat{J}_s^{\text{ord}}) \hookrightarrow a_s\hat{J}_s^{\text{ord}} \). Thus \( a_s\hat{J}_s^{\text{ord}} = \varpi_s(\hat{J}_s^{\text{ord}}) \) as fppf sheaves. This shows the exactness:

\[
0 \to \hat{A}_s^{\text{ord}} \to \hat{J}_s^{\text{ord}} \xrightarrow{\varpi_s} a_s\hat{J}_s = \varpi_s(\hat{J}_s^{\text{ord}}) \to 0.
\]

Since \( B_s = J_s/a_sJ_s \) as fppf sheaves, we see the exactness of

\[
0 \to \varpi_s(\hat{J}_s^{\text{ord}}) \to \hat{J}_s^{\text{ord}} \to \hat{B}_s^{\text{ord}} \to 0
\]
as fppf sheaves. Combining the two exact sequence, we obtain the exactness of the last sequence in the lemma.

Assuming \( X_s(K) \neq \emptyset \), \( J_s \cong \text{Pic}_X^0/K \) via the polarization of the canonical divisor (e.g., [ARG, VII.6]). The Rosati involution \( h \mapsto h^* \) and \( T(n) \mapsto T^*(n) \) brings \( h_r(\mathbb{Z}) \) to \( h_r^*(\mathbb{Z}) \subset \text{End}(J_r/K) \).

At the level of double coset operator \([\Gamma\alpha\Gamma']^* = [\Gamma^*\alpha^*\Gamma'] \). Because of this fact, the involution \( h \mapsto h^* \) itself is well defined giving \( h_r(\mathbb{Z}) \cong h_r^*(\mathbb{Z}) \) in \( \text{End}(J_r/K) \) (even if \( X_r(Q) = \emptyset \)). Note that \( X_1(Np^r)(Q) \) contains the infinity cusp; so, for the standard tower, we have \( X_r(\mathbb{Q}) \neq \emptyset \).

The Weil involution \( w_s = [\Gamma_s \left( \begin{array}{cc} 0 & -1 \\ Np & 0 \end{array} \right) \Gamma_s] \) has the effect that \( w_s[\Gamma_s\alpha\Gamma_s] = [\Gamma_s\alpha\Gamma_s]w_s \) as easily verified. Thus \( w_s \circ T^*(n) = T(n) \circ w_s \) for all \( n \) including \( T(l) = U(l) \) for \( l|Np \). We write \( \{A_s^\perp\}_{s>0} \) for the tower corresponding to \( \{(\hat{\Gamma}_s^\perp)^t = w_s\hat{\Gamma}_s^\perp w_s^{-1}\}_{s>0} \) with the main involution \( \iota \) given by \( x^\iota = \det(x) \).

Thus \( \{X_s^\iota\}_{s>0} \) corresponds to the triple \((-\alpha, -\delta, \xi') \) for \( \xi'(a, d) = \xi(d, a) \), and the \( b \)-component of \( (\hat{\Gamma}_s^\perp)^t \) for \( l|N \) is given by

\[
\{(a, b, c) \in \text{GL}_2(\mathbb{Z}_l) | c \in N \mathbb{Z}_l, a - 1 \in N \mathbb{Z}_l \}.
\]
Then $w_s$ gives an isomorphism $w_s : X^r_s \to X^r_s$ defined over $\mathbb{Q}$. Note that the fixed isomorphism $\mu_{p^r} \cong \mathbb{Z}/p^r \mathbb{Z}$ induces an isomorphism $X^r_s \cong X^r_s$ over $\mathbb{Q}(\mu_{p^r})$. As an automorphism of $X^r_s/\mathbb{Q}(\mu_{p^r})$, $w_s$ satisfies $w_s(z) = (z) \circ w_s = w_s \circ (z)^{-1}$ for $z \in \mathbb{Z}^*$. (see Lemma 3.1). Write $J_f^r = \text{Pic}^0_{\mathbb{C}^\omega}/\mathbb{Q}$.

Take a connected (resp. irreducible) component $\text{Spec}(\mathbb{T})$ (resp. $\text{Spec}(\mathbb{I})$) of $\text{Spec}(\mathfrak{h})$ and assume that $I$ is primitive in the sense of [H86a, Section 3]. For each arithmetic $P \in \text{Spec}(\mathbb{I})(\mathbb{C}_p)$, the corresponding cusp form $f_P$ is a $p$-stabilized Hecke eigenform of weight 2 new at each prime $l|N$ if and only if $I$ is primitive.

We get directly from Lemma 5.1 the following proposition giving sufficient conditions for the validity of (A) for $A_{f,s}$ when $f = f_P$ is in a $p$-adic analytic family indexed by $P \in \text{Spec}(\mathbb{I})$.

**Proposition 5.2.** Let Spec($\mathbb{T}$) be a connected component of Spec($\mathfrak{h}$) and Spec($\mathbb{I}$) be a primitive irreducible component of Spec($\mathbb{T}$). Assume the condition (F) (the $\Lambda$-freeness of $h_{\alpha,\delta,\xi}$). Then the condition (A) holds for the following choices of $(\varpi, A_s, B_s)$:

1. Suppose that an eigen cusp form $f = f_P$ new at each prime $l|N$ belongs to Spec($\mathbb{T}$) and that $\mathbb{T} = \mathbb{I}$ is regular (or more generally a unique factorization domain). Then writing the level of $f_P$ as $Np^r$, the algebra homomorphism $\lambda : \mathbb{T} \to \mathbb{C}_p$ given by $f(T(l)) = \lambda(T(l))f$ gives rise to the prime ideal $P = \text{Ker}(\lambda)$. Since $P$ is of height 1, it is principal generated by $\varpi \in \mathbb{T}$. This $\varpi$ has its image $a_s \in T_s = T \otimes_{\Lambda} A_s$ for $A_s = \Lambda/(\gamma p^{r'} - 1)$. Write $h_s = h \otimes_{\Lambda} A_s = T_s + 1, h_s$ as an algebra direct sum for an idempotent $1_s$. Then, the element $\varpi_s = a_s + 1_s \in h_s$ for the identity $1_s$ of $X_s$ satisfies (A).

2. Fix $r > 0$. Then $\varpi \in \mathfrak{m}_l^\omega$ for a factor $\varpi (\gamma p^{r'} - 1)$ in $\Lambda$, satisfies (A).

Here is a criterion from [F02, Theorem 3.1] for regularity of $\mathbb{T}$:

**Theorem 5.3.** Assuming (F) for $h_{\alpha,\delta,\xi}$. Let $f$ be a Hecke eigenform of conductor $N$, of weight 2 and with Neben character $\chi$, and define $a_p \in \mathbb{C}_p$ by $f(T(p)) = a_p f$. Let $p$ be a prime outside $6D_{\chi}(N)$ (for $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^*|$). Suppose that for the prime ideal $P$ of $\mathbb{Z}[a_p]$ induced by $i_p : \mathbb{Q} \hookrightarrow \mathbb{C}_p$, $(a_p \mod p)$ is different from 0 and $\pm \sqrt{\chi(p)}$. Then for the connected component Spec($\mathbb{T}$) of Spec($h_{\alpha,\delta,\xi}$) $\text{acting non-trivially}$ on the $p$-stabilized Hecke eigenform corresponding to $f$ in $S_2(\Gamma_0(Np), \chi)$, $\mathbb{T}$ is a regular integral domain isomorphic to $W \otimes_{\mathbb{Z}_p} \Lambda = W[[T]]$ for a complete discrete valuation ring $W$ unramified at $p$.

The result is valid always for $p \geq 5$ and for $p = 3$ under (F) (see Propositions 4.1 and 18.2. Here is a proof of this fact since [F02, Theorem 3.1] is slightly different from the above theorem.

**Proof.** Let $\psi = \lim_{n \to \infty} T(p)(p)^n \in h_2(\Gamma_0(N), \chi; A)$ for $\mathbb{Z}_p[\chi]$-algebra $A$. Put $h_2^{\text{ord}}(\Gamma_0(N), \chi; A) := e^\psi h_2(\Gamma_0(N), \chi; A)$. Since $U(p) \equiv T(p) \mod p$ on $A[[q]]$, the natural algebra homomorphism:

$$h_2^{\text{ord}}(\Gamma_0(Np), \chi; A) \to h_2^{\text{ord}}(\Gamma_0(N), \chi; A)$$

sending $U(p)$ to the unit root of $X^2 - \chi(X) + \chi(p) \in h_2^{\text{ord}}(\Gamma_0(N), \chi; A)[X]$ and $T(l)$ to $T(l)$ for all primes $l \neq p$ is a well defined surjective $A$-algebra homomorphism.

Since $p \nmid 12D_{\chi}(N \varphi(N)$, we have $p > 3$ and $p \nmid \varphi(Np)$. Write $\mathfrak{h}$ for $\mathfrak{h}_{\alpha,\delta,\xi}(N)$. Then $\mathfrak{h}$ is $\Lambda$-free by (F) and an exact control is valid (see Propositions 4.1 and 18.2). By the diamond operators (z) for $\mathfrak{h}$ in $(\mathbb{Z}/Np\mathbb{Z})^\times$, $\mathfrak{h}$ is an algebra over $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times]$. We can decompose $\psi = \otimes_{\chi} \psi(\chi)$ so that the diamond operator $(z)$ acts by $\psi(\chi)$ on $\mathfrak{h}(\psi)$, where $\psi$ runs over all even characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$. From the exact control $\mathfrak{h}(\chi)/\mathfrak{h}(\psi) \cong h_1(\chi; A) = \mathfrak{h}_1(\chi; A) = h_1(\chi; A)$ for the character $\chi$ of $(\mathbb{Z}/Np\mathbb{Z})^\times$, where

$$h_2(\Gamma_0(Np), \chi; \mathbb{Z}_p[\chi]) = h_2(\Gamma_1(Np), \chi; \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}/Np\mathbb{Z}]^\times, \chi} \mathbb{Z}_p[\chi]$$

and $\mathbb{Z}_p[\chi]$ is the $\mathbb{Z}_p$-subalgebra of $\mathbb{C}_p$ generated by the values of $\chi$. Here the tensor product is with respect to the algebra homomorphism $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \to \mathbb{Z}_p[\chi]$ induced by $\chi$. Writing $\Sigma = \text{Hom}_{\text{alg}}(\mathfrak{h}(\chi), \mathfrak{h}(\psi))$, for each $\lambda \in \Sigma$, $\Sigma := \{\text{ker}(\lambda) | \lambda \in \Sigma\}$ is the set of all maximal ideals of
of $T$ we have a unique algebra direct factor reduction map modulo $h$ that $\text{Im}(\pi)$ is reduced by the theory of new forms ([H86a, §3] and [MFM, §4.6]) and unramified over $\mathbb{Z}_p$ by $p \nmid D_X$, the reduction map modulo $p$: $\text{Hom}_{\text{alg}}(h, \mathbb{F}_p) \to \Sigma$ is a bijection. In particular, we have $h = h^{\text{new}} \oplus h^{\text{old}}$ where $h^{\text{new}}$ is the direct sum of $h_{\text{max}}$ for $\lambda$ coming from the eigenvalues of $N$-primitive forms. Again by Hensel’s lemma, we have the algebra decomposition $h_N = h^{\text{new}} \oplus h^{\text{old}}$ with $h'/Th' = h''$ for $? = \text{new, old}$. Since $h^{\text{new}}$ is reduced by the theory of new forms ([H14a, §6] and [MFM, §4.6]) and unramified over $\mathbb{Z}_p$ by $p \nmid D_X(N)$, we conclude $h^{\text{new}} \cong \bigoplus W$ for discrete valuation rings $W$ finite unramified over $\mathbb{Z}_p$ (one of the direct summand $W$ acts on $f$ non-trivially; i.e., $W$ given by $\mathbb{Z}_p[f] = \mathbb{Z}_p[a_n|n = 1, 2, \ldots| \subset \mathbb{Q}_p$ for $T(n)$-eigenvalues $a_n$ of $f$). Thus again by Hensel’s lemma, we have a unique algebraic direct factor $T$ of $h^{\text{new}}$ such that $T/\mathbb{T} = \mathbb{Z}_p[f] = W$. Since $W$ is unramified over $\mathbb{Z}_p$, by the theory of Witt vectors [BCM, IX.1], we have a unique section $W \to \mathbb{T}$ of $\mathbb{T} \to \mathbb{Z}_p[f] = W$. Then $W[[T]] \subset \mathbb{T}$ which induces a surjection after reducing modulo $T$. Then by Nakayama’s lemma, we have $\mathbb{T} = W[[T]] = W \otimes_{\mathbb{Z}_p} A$ as desired. 

\section{Limit Abelian Factors}

We recall some elementary but useful results (e.g. [H14b, §6, after (6.6)]) with its proof. Let $\pi_{s,r,s} : J_s \to J_r$ for $s > r$ be the morphism induced by the covering map $X_s \to X_r$ through Albanese functoriality. Then we define $\pi_{r,s} = w_r \circ \pi_{s,r,s} \circ w_s$. Note that $\pi_{r,s}$ is well defined over $\mathbb{Q}$, and satisfies $T(n) \circ \pi_{r,s} = \pi_{r,s} \circ T(n)$ for all $n$ prime to $Np$ and $U(q) \circ \pi_{r,s} = \pi_{r,s} \circ U(q)$ for all $q|Np$ (as $w_r \circ h \circ w_s = h^*$ for $h \in h^*(\mathbb{Z})$ ($? = s, r$) by [MFM, Theorem 4.5.5].

Let $\iota : C_{r/q} \subset J_{r/q}$ (resp. $\pi : J_{r/q} \to D_{r/q}$) be an abelian subvariety (resp. an abelian variety quotient) stable under Hecke operators (including $U(l)$ for $l|Np$) and $w_r$. Here the stability means that $\text{Im}(\iota)$ and $\text{Ker}(\pi)$ are stable under Hecke operators. Then $\iota$ and $\pi$ are Hecke equivariant. Let $\iota_s : C_s := \pi_{s,r,s}(C) \subset J_s$ for $s > r$ and $D_s$ is the quotient algebraic variety of $\pi_{s,r,s} : J_s \circlearrowright J_r \to D_r$, where $\pi_{s,r,s} = w_r \circ \pi_{s,r,s} \circ w_s$. The twisted projection $\pi_{s,r,s}^* \circ \iota_{s,r,s}$ is rational over $\mathbb{Q}$ as $w_s[z] = \langle z \rangle w_s = w_s \circ \langle z \rangle^{-1}$ for $z \in \mathbb{Q}_p^\times$.

Since the two morphisms $J_s \to J'_s$ and $J'_s \to J_s[\gamma^{p-r-1}]$ (Picard functoriality) are $U(p)$-isomorphism of fpf abelian sheaves by (u1) and Corollary 3.2, we get the following two isomorphisms of fpf abelian sheaves for $s > r > 0$:

\begin{align}
(6.1) \quad & C_r[p^{\infty}]^{\text{ord}} \overset{\sim}{\longrightarrow} C_s[p^{\infty}]^{\text{ord}} \quad \text{and} \quad C_r^{\text{ord}} \overset{\sim}{\longrightarrow} C_s^{\text{ord}},
\end{align}

since $C_s^{\text{ord}}$ is the isomorphic image of $C_r^{\text{ord}} \subset J_r$ in $J_s[\gamma^{p-r-1}]$. By $w$-twisted Cartier duality [H14a, §4], we have

\begin{align}
(6.2) \quad & D_s[p^{\infty}]^{\text{ord}} \overset{\sim}{\longrightarrow} D_r[p^{\infty}]^{\text{ord}}.
\end{align}
Thus by Kummer sequence, we have the following commutative diagram
\[
\begin{align*}
\hat{D}_s^{\text{ord}}(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z} &= (D_s(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z})^{\text{ord}} \quad \rightarrow \quad H^1(D_s[p^m]^{\text{ord}}) \\
\hat{C}_s^{\text{ord}}(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z} &= (D_s(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z})^{\text{ord}} \quad \rightarrow \quad H^1(D_s[p^m]^{\text{ord}})
\end{align*}
\]

This shows
\[
\hat{D}_s^{\text{ord}}(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z} \cong \hat{D}_r^{\text{ord}}(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z}.
\]

Passing to the limit, we get
\[
(6.3) \quad \hat{D}_s^{\text{ord}} \xrightarrow{\pi_s^*} \hat{D}_r^{\text{ord}} \quad \text{and} \quad (D_s \otimes \mathbb{Z}/p\mathbb{Z})^{\text{ord}} \xrightarrow{\pi_r^*} (D_r \otimes \mathbb{Z}/p\mathbb{Z})^{\text{ord}}
\]
as fpf abelian sheaves. In short, we get

**Lemma 6.1.** Suppose that \( \kappa \) is a field extension of finite type of either a number field or a finite extension of \( \mathbb{Q}_p \). Then we have the following isomorphism
\[
\hat{C}_r(\kappa)^{\text{ord}} \xrightarrow{\sim} \hat{C}_s(\kappa)^{\text{ord}} \quad \text{and} \quad \hat{D}_s(\kappa)^{\text{ord}} \xrightarrow{\sim} \hat{D}_r(\kappa)^{\text{ord}}
\]
for all \( s > r \) including \( s = \infty \).

Taking \( C_r \) to be \( A_r \) (and hence \( D_s = B_s \) by Lemma 5.1) and applying this lemma to the exact sequence (5.1), we get a new exact sequence (for \( \varpi \) in (A)):
\[
(6.4) \quad 0 \to \hat{A}_s^{\text{ord}} \to \hat{J}_s^{\text{ord}} \xrightarrow{\pi_s^*} \hat{J}_r^{\text{ord}} \to \hat{B}_s^{\text{ord}} \to 0,
\]
since \( \hat{A}_s^{\text{ord}} = \lim_s \hat{A}_s^{\text{ord}} \cong \hat{A}_r^{\text{ord}} \) by the lemma.

We make \( \hat{B}_s^{\text{ord}} \) explicit. By computation, \( \pi_r^* \circ \pi_s^* = p^{s-r}U(p^{s-r}) \). To see this, as Hecke operators, \( \pi_{r,s}^* = [\Gamma^r_s], \pi_{r,s,s}^* = [\Gamma_r] \). Thus we have
\[
(6.5) \quad \pi_r^* \circ \pi_{r,s}^* = [\Gamma^r_s] \cdot [w_s] \cdot [\Gamma_r] \cdot [w_r] = [\Gamma_s] \cdot [w_s w_r] \cdot [\Gamma_r] = [\Gamma_s^r \cdot \Gamma_r]^r \Gamma_r = p^{s-r}U(p^{s-r})\).

**Lemma 6.2.** We have the following two commutative diagram for \( s > s' \)
\[
\begin{align*}
\hat{C}_s' & \xrightarrow{\sim} \hat{C}_s \\
\hat{D}_s' & \xrightarrow{\sim} \hat{D}_s \\
\pi_{s,s'}^* & \downarrow \quad p^{s'-s} U(p^{s-s})
\end{align*}
\]
and
\[
\begin{align*}
\hat{C}_s & \xrightarrow{\sim} \hat{C}_s' \\
\hat{D}_s & \xrightarrow{\sim} \hat{D}_s' \\
\pi_{s,s'}^* & \downarrow \quad p^{s'-s} U(p^{s-s})
\end{align*}
\]

In particular, we get \( \hat{D}_s^{\text{ord}} := \lim_s \hat{D}_s^{\text{ord}} = \hat{D}_s^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \).

**Proof.** By \( \pi_{r,s}^* \) (resp. \( \pi_r^* \)), we identify \( \hat{C}_s' \) with \( \hat{C}_r^{\text{ord}} \) (resp. \( \hat{D}_s^{\text{ord}} \) with \( \hat{D}_r^{\text{ord}} \)) as in Lemma 6.1. Then the above two diagrams follow from (6.5).

For a free \( \mathbb{Z}_p \)-module \( F \) of finite rank, we suppose to have a commutative diagram:
\[
\begin{align*}
F \xrightarrow{p^n} F \\
\downarrow \quad \downarrow \\
F \xrightarrow{p^{-n}} F
\end{align*}
\]
Thus we have \( \lim_{n,x \to p^n x} F = \lim_{n,x \to p^n x} p^{-n} F \cong F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). If \( T \) is a torsion \( \mathbb{Z}_p \)-module with \( p^B T = 0 \) for \( B \gg 0 \), we have \( \lim_{n,x \to p^n x} T = 0 \). Thus for general \( M = F \oplus T \), we have \( \lim_{n,x \to p^n x} M \cong M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \).

Identifying \( \widehat{D}_s^\text{ord} \) with \( \widehat{D}_r^\text{ord} \) by \( \pi_s^r \) for all \( s \geq r \), the transition map of the inductive limit \( \lim_{s \to r} \widehat{D}_s^\text{ord} \) is given by

\[
\begin{array}{ccc}
\widehat{D}_s^\text{ord} & \longrightarrow & \widehat{D}_s^\text{ord} \\
\downarrow & & \downarrow \\
\widehat{D}_r^\text{ord} & \longrightarrow & \widehat{D}_r^\text{ord}
\end{array}
\]

Thus applying the above result for \( M = \widehat{D}_s^\text{ord}(K) \), we find \( \lim_{s \to r} \widehat{D}_s^\text{ord}(K) = \widehat{D}_r^\text{ord}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). \( \square \)

Applying this lemma to \( D_s = B_s \), we get from (6.4), the following exact sequence:

**Corollary 6.3.** Assume (F) for \( h, \delta, \xi \). Let \( K \) be either a number field or a finite extension of \( \mathbb{Q}_l \) for a prime \( l \). For \( (\pi, A_r, B_r) \) satisfying (A), we get the following natural short exact sequence of étale sheaves over \( \text{Spec}(K) \):

\[
0 \to A_r^\text{ord} \to J_r^\text{ord} \to J_r^\text{ord} \to B_r^\text{ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.
\]

In particular, for \( K' = K_S \) if \( K \) is a number field and \( K' = \overline{K} \) if \( K \) is local, we have the following exact sequence of Galois modules:

\[
0 \to A_r^\text{ord}(K') \to J_r^\text{ord}(K') \to J_r^\text{ord}(K') \to B_r^\text{ord}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.
\]

**Proof.** Since a finite étale extension \( R \) of \( K \) is a product of finite field extensions of \( K \), we may assume that \( R \) is a field extension of \( K \). Then by (S), \( B_s(R)^{\text{ord}} \cong B_s(R)^{\text{ord}} \) is a \( \mathbb{Z}_p \)-module of finite type.

Then by the above lemma Lemma 6.2, taking \( D_s \) to be \( B_s \), we find that \( \lim_{s \to r} B_s(R)^{\text{ord}} = B_r^\text{ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). Since passing to injective limit is an exact functor, this proves the first exact sequence:

\[
0 \to A_r^\text{ord} \to J_r^\text{ord} \to J_r^\text{ord} \to B_r^\text{ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.
\]

Since \( \widehat{X}^\text{ord}(K') = \lim_{F/K} \widehat{X}^\text{ord}(F) \) for \( F \) running a finite extension of \( K \), we get the exactness of

\[
0 \to \widehat{A}_r^\text{ord}(K') \to J_r^\text{ord}(K') \to J_r^\text{ord}(K') \to B_r^\text{ord}(K') \to 0.
\]

Since

\[
0 \to A_r^\text{ord}(K') \to J_r^\text{ord}(K') \to J_r^\text{ord}(K') \to B_r^\text{ord}(K') \to 0
\]

is an exact sequence of Galois modules, passing to the limit, we still have the exactness of

\[
0 \to \widehat{A}_r^\text{ord}(K') \to J_r^\text{ord}(K') \to J_r^\text{ord}(K') \to B_r^\text{ord}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.
\]

Note here that \( B_r^\text{ord}(K') \) is \( p \)-divisible, and hence

\[
B_r^\text{ord}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = B_r^\text{ord}(K') / B_r^\text{ord}[p^\infty](K').
\]

\( \square \)

Consider the composite \( f_s \) of the inclusion \( \widehat{A}_s^\text{ord} \to \widehat{J}_r^\text{ord} \) and the projection: \( \widehat{J}_r^\text{ord} \to \widehat{B}_s^\text{ord} \). Since \( \widehat{A}_s^\text{ord} \cong \mathbb{Q}_l(\widehat{J}_r^\text{ord}) \cong \text{Ker}(\widehat{A}_s^\text{ord} \to \widehat{B}_s^\text{ord}) \), we may think \( \text{Ker}(f_s) \) also as a group subscheme of \( \widehat{J}_r^\text{ord} \). Since we have a commutative diagram:

\[
\begin{array}{ccc}
\widehat{A}_s^\text{ord} & \xrightarrow{f_s} & \widehat{B}_s^\text{ord} \\
\downarrow_{\pi_s^r} & & \downarrow_{\pi_s^r} \\
\widehat{A}_r^\text{ord} & \xrightarrow{f_r} & \widehat{B}_r^\text{ord}
\end{array}
\]
we have \( \text{Ker}(f_s) \hookrightarrow \hat{A}^\text{ord}_{\infty}[p^{s-r}] \cong \hat{A}^\text{ord}_{r}[p^{s-r}] \) (from \( \pi^r_s \circ \pi^*_s, r = p^{s-r}U(p)^{s-r} \)) whose cokernel is bounded by \( \text{Ker}(f_r) \). Passing to the limit, we find
\[
\hat{A}^\text{ord}_{\infty}[p^{\infty}] = \lim_{s} \text{Ker}(f_s) = \hat{A}^\text{ord}_{\infty} \cap \varpi(J^\text{ord}_{\infty})
\]
inside \( J^\text{ord}_{\infty} \). This shows

**Corollary 6.4.** We have an isomorphism \( J^\text{ord}_{\infty} \cong (\hat{A}^\text{ord}_{\infty} \oplus \varpi(J^\text{ord}_{\infty})) / \hat{A}^\text{ord}_{\infty}[p^{\infty}] \) with \( \hat{A}^\text{ord}_{\infty} \cong \hat{A}^\text{ord}_{r} \).

### 7. Generality of Galois cohomology

We prove some general result on Galois cohomology for our later use. Let \( S \) be a set of places of a number field \( K \). Suppose that \( S \) contains all archimedean places and \( p \)-adic places of \( K \) (and primes for bad reduction of the abelian varieties when we deal with abelian varieties). Let \( K^S \) be the maximal extension unramified outside \( S \).

**Lemma 7.1.** Let \( \{ M_n \}_n \) be a projective system of finite \( \mathbb{Z}_p[\text{Gal}(K^S/K)] \)-modules \( M_n \). Write \( M_{\infty} := \text{lim}_n M_n \) and \( M_{\infty}' := \text{lim}_{n \to \infty} M'_n \) for the Pontryagin dual \( M'_n \) of \( M_n \). Write \( G \) (resp. \( G_v \), for a place \( v \) of \( K \)) for the (point by point) stabilizer of \( M_{\infty}' \) in \( \text{Gal}(K^S/K) \) (resp. \( \text{Gal}(K_v/K_v) \)) and \( \hat{G} = \text{Gal}(K^S/K)/G \) (resp. \( \hat{G}_v := \text{Gal}(K^S_v/K_v)/G_v \)). Then, we have

1. \( \text{III}^1(K^S/K, M_{\infty}) = \text{lim}_{n} \text{III}^1(K^S/K, M'_n), \text{III}^1(K^S/K, M_{\infty}) = \text{lim}_{n} \text{III}^1(K^S/K, M_n) \).

2. \( \text{II}^2(K^S/K, M_{\infty}) = \text{lim}_{n} \text{II}^2(K^S/K, M'_n), \text{II}^2(K^S/K, M_{\infty}) = \text{lim}_{n} \text{II}^2(K^S/K, M_n) \), and if \( S \) is a finite set, we have

\[
\text{H}^2(K^S/K, M_{\infty}) = \text{lim}_{n} \text{H}^2(K^S/K, M_n) \quad \text{and} \quad \text{II}^2(K^S/K, M_{\infty}) = \text{lim}_{n} \text{II}^2(K^S/K, M_n).
\]

**Proof.** Since the proof two identities of (1) is dual each other, we only prove the result for projective limit. Since \( H^0(? , M_n) \) (resp. \( K^S/K \) and \( K_v \)) is finite for all \( n \), we have \( \text{lim}_{n} H^1(? , M_n) = H^1(? , M_{\infty}) \) for \( ? = K^S/K \) and \( K_v \) by [CNF, Corollary 2.7.6]. By definition, we have an exact sequence:

\[
0 \to \text{III}^1(K^S/K, M_n) \to H^1(K^S/K, M_n) \to \prod_{v \in S} H^1(K_v, M_n).
\]

Since any continuous cochain with values in \( \text{lim}_{n} M_n \) is a projective limit of continuous cochains with values in \( M_n \), we have a natural map \( H^1(? , \text{lim}_{n} M_n) \to \text{lim}_{n} H^1(? , M_n) \) for \( ? = K^S/K \) and \( K_v \). Passing to the limit, we get the following commutative diagram with exact rows

\[
\begin{array}{c}
\text{III}^1(K^S/K, \text{lim}_{n} M_n) \xrightarrow{\sim} H^1(K^S/K, \text{lim}_{n} M_n) \xrightarrow{i} \prod_{v \in S} H^1(K_v, \text{lim}_{n} M_n) \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{lim}_{n} \text{III}^1(K^S/K, M_n) \xrightarrow{\sim} \text{lim}_{n} H^1(K^S/K, M_n) \xrightarrow{i} \prod_{v \in S} \text{lim}_{n} H^1(K_v, M_n).
\end{array}
\]

This shows

\[
\text{III}^1(K^S/K, \text{lim}_{n} M_n) = \text{lim}_{n} \text{III}^1(K^S/K, M_n)
\]
as desired.

As for (2), since cohomology functor commutes with injective limit, the assertion (3) for injective limits follows from the same argument as in the case of (1). If \( S \) is finite, \( H^1(K^S/K, M_{\infty}) \) is finite (e.g., [ADT, I.5.1]). Thus by [CNF, Corollary 2.7.6], we have \( \text{lim}_{n} H^2(? , M_n) = H^2(? , \text{lim}_{n} M_n) \) for \( ? = K^S/K \) and \( K_v \), and hence once again the same argument works (replacing \( H^1 \) by \( H^2 \)). \( \square \)

Let \( A \) be an abelian variety over a field \( K \). Since the Galois group \( \text{Gal}(K/K) \) and \( \text{Gal}(K^S/K) \) is profinite and \( \hat{A}(K) \) and \( A(K^S) \) are discrete modules, for \( q > 0 \), the continuous cohomology group \( H^q(K^S/K, A) \) for a number field \( K \) and \( H^q(K, A) \) for a local field \( K \) are torsion discrete modules (see [MFG, Corollary 4.26]).
Lemma 7.2. If $K$ is either a number field or a local field of characteristic 0, we have a canonical isomorphism for $0 < q \in \mathbb{Z}$:

\[(7.1) \quad H^q(\widehat{A}) \cong H^q(A)_p = H^q(A)[p^\infty],\]

where $H^q(\cdot)$ is the cohomology of $\cdot$ with coefficients in $\mathbb{Q}$ if $K$ is a number field, and $H^q(\cdot)$ is the cohomology of $\cdot$ with coefficients in $\mathbb{Q}$ if $K$ is local.

Proof. By (S), if $K$ is a number field, we have

$$H^q(K^S/K, \hat{A}) \cong H^q(K^S/K, A \otimes \mathbb{Z}) \cong H^q(K^S/K, A) \otimes \mathbb{Z}_p = H^q(K^S/K, A)_p,$$

as $H^q(K^S/K, A)$ is a torsion module. Here the identity \(*\) follows from [CNF, 2.3.4].

Now suppose that $K$ is an $l$-adic or archimedean local field with $l \neq p$. Then $\hat{A} = A[p^\infty]$, and we have a natural inclusion $0 \to \hat{A} \to A(K) \to Q \to 0$ for the quotient Galois module $Q$. Thus $Q$ is $p$-primary and $p$-divisible; i.e., the multiplication by $p$ is invertible on $Q$. Therefore the $p$-primary part $H^q(K, Q)_p$ is a $Q_p$-vector space for $q \geq 0$. By the exact sequence $H^{q-1}(K, Q)_p \to H^q(K, \hat{A})_p \to H^q(K, A)_p \to H^q(K, Q)_p$, we conclude $H^q(K, \hat{A})_p \cong H^q(K, A)_p$ as the two modules are $p$-torsion.

If $l = p$, we have $A(K) = A(K) = A[p^\infty] = A[p^\infty]$. Since $A(K)$ is a union of $p$-profinite group, we have $H^q(\hat{A}) = H^q(A)$. Since $A[p^\infty]$ is prime-to-$p$ torsion, we have $H^q(K, A[p^\infty]) = 0$ as desired. \hfill $\square$

8. Diagrams of Selmer groups and Tate–Shafarevich groups

We describe a commutative diagram involving different cohomology groups and Tate–Shafarevich groups, which lays a base of the proof of the control result in later sections. We assume $p > 2$ for simplicity.

Recall the definition of the $p$-part of the Selmer group and the Tate–Shafarevich group for an abelian variety $A$ defined over a number field $K$:

\[
\begin{align*}
\text{III}_K(A)_p &= \ker(H^1(K^S/K, A)_p) \xrightarrow{\text{Res}} \prod_{v \in S} H^1(K_v, A)_p, \\
\text{Sel}_K(A)_p &= \ker(H^1(K^S/K, A[p^\infty]) \xrightarrow{\text{Res}} \prod_{v \in S} H^1(K_v, A)_p).
\end{align*}
\]

As long as $S$ is sufficiently large containing all bad places for $A$ in addition to all archimedean and $p$-adic places, these groups are independent of $S$ (see [ADT, I.6.6]) and are $p$-torsion modules. By Lemma 7.2, we can replace $A$ in the above definition by $\hat{A}$, and we get

\[(8.1) \quad \begin{align*}
\text{III}_K(A)_p &= \text{III}_K(\hat{A}) = \ker(H^1(K^S/K, \hat{A}) \xrightarrow{\text{Res}} \bigoplus_{v \in S} H^1(K_v, \hat{A})), \\
\text{Sel}_K(A)_p &= \text{Sel}_K(\hat{A}) = \ker(H^1(K^S/K, A[p^\infty]) \xrightarrow{\text{Res}} \bigoplus_{v \in S} H^1(K_v, \hat{A})).
\end{align*}\]

It is known that image of global cohomology classes lands in the direct sum $\bigoplus_{v \in S} H^1(K_v, \hat{A})$ in the product $\prod_{v \in S} H^1(K_v, \hat{A})$ (see [ADT, I.6.3]).

Since $A \to \text{III}_K(\hat{A})$ is a covariant functor from abelian varieties defined over a number field $K$ to an abelian groups, from Lemma 3.3, we get the commutative diagram for $X = \text{III}$ and Sel:

\[
\begin{align*}
X_K(\hat{J}_r) & \xrightarrow{\pi^*} X_K(\hat{J}_r) \\
\downarrow u & \swarrow u' \downarrow u'' \\
X_K(\hat{J}_r) & \xrightarrow{\pi^*} X_K(\hat{J}_r),
\end{align*}\]

\[(8.2)\]
Similarly to the diagram as above, from Corollary 3.2, we get the following commutative diagram:

\[
\begin{array}{ccc}
X_K(\tilde{J}_n) & \xrightarrow{\pi^n} & X_K(\tilde{J}_s[\gamma^{p^n-1} - 1]) \\
\downarrow u & \nearrow u'' & \downarrow \pi''
\end{array}
\]

Equation (8.3)

These diagrams provide us the following canonical isomorphisms

\[
X_K(\tilde{J}_s)^{\text{ord}} \cong X_K(\tilde{J}_s[\gamma^{p^n-1} - 1])^{\text{ord}} \quad \text{for } X = \text{III and Sel.}
\]

Equation (8.4)

For any group subvariety \( A/\mathbb{Q} \) of \( J_s \) proper over \( \mathbb{Q} \) or any abelian variety quotient \( A/\mathbb{Q} \) of \( J_s \) stable under \( U(p) \), we have \( A = \tilde{A}^{\text{ord}} \oplus (1-e)\tilde{A} \), and hence \( H^q(\mathcal{A}, \tilde{A}^{\text{ord}}) = H^q(\mathcal{A}, \tilde{A})^{\text{ord}} \) for \( \mathcal{A} = K \) and \( K^S \). This shows \( H^q(\mathcal{A}, \tilde{A}^{\text{ord}}) = H^q(\mathcal{A}, \tilde{A})^{\text{ord}} \), and hence \( X^q_{\mathcal{A}}(\mathcal{A}^{\text{ord}}) = X^q_{\mathcal{A}}(\mathcal{A})^{\text{ord}} = X^q_{\mathcal{A}}(\mathcal{A})^{\text{ord}} \) for \( X = \text{III and Sel.} \) Thus hereafter, we attach the superscript “ord” inside the cohomology/Tate-Shafarevich group if the coefficient is \( p \)-adically completed in the sense of (S).

We define the ind \( \Lambda \)-TS group and the ind \( \Lambda \)-Selmer group by

\[
\begin{align*}
\text{III}(J_{\infty})^{\text{ord}} := \text{III}(J_{\infty}^{\text{ord}}) &= \lim_{r} \text{III}_{K}(\tilde{J}_{r}^{\text{ord}}) = \lim_{r} \text{III}_{K}(J_{r}^{\text{ord}}), \\
\text{Sel}(J_{\infty})^{\text{ord}} := \text{Sel}(J_{\infty}^{\text{ord}}) &= \lim_{r} \text{Sel}_{K}(\tilde{J}_{r}^{\text{ord}}) = \lim_{r} \text{Sel}_{K}(J_{r}^{\text{ord}})
\end{align*}
\]

which are naturally \( \mathbf{h} \)-modules.

Write \( H_{S}^{2}(M) = \bigoplus_{s \in S} H^{2}(K_{s}/M, M) \) and \( H^{q}(M) = H^{q}(K^{S}/K, M) \) for a Gal\((K^{S}/K)\)-module \( M \). By [ADT, I.6.6], \( \text{III}(K^{S}/K, A)_{p} = \text{III}_{K}(A)_{p} \) for an abelian variety \( A_{/\mathbb{K}} \) as long as \( S \) contains all bad places of \( A \) and all archimedean and \( p \)-adic places. Consider a triple \((\mathcal{A}, A_{s} = J_{s}[a_{s}], B_{s} = J_{s}/a_{s}J_{s})\) satisfying the condition (A) of Section 5 and (F) in Section 4. Note that \( J_{\infty}^{\text{ord}}(\mathcal{A}) = A_{s}^{\text{ord}} \) (see Lemma 5.1), we have \( H^{q}(\mathcal{A}, J_{\infty}^{\text{ord}}(\mathcal{A})) = H^{q}(\mathcal{A}, J_{\infty}^{\text{ord}}(\mathcal{A})) \). This implies \( \text{III}_{S}(J_{\infty}^{\text{ord}}(\mathcal{A})) = \text{III}_{S}(A_{s}^{\text{ord}}) = \text{III}_{K}(\tilde{A}_{r}^{\text{ord}}), \) where the last identity follows from [ADT, I.6.6]. Recall the following exact sequence from Corollary 6.3:

\[
0 \rightarrow \tilde{A}_{r}^{\text{ord}}(K') \rightarrow J_{\infty}^{\text{ord}}(K') \xrightarrow{\varpi} J_{\infty}^{\text{ord}}(K') \rightarrow \widehat{B}_{r}^{\text{ord}}(K') \otimes_{\mathbb{Z}_{p}} \mathbb{Q}_{p} \rightarrow 0,
\]

Equation (8.6)

where \( J_{\infty}^{\text{ord}} = \lim_{\rightarrow_{r}} J_{r}^{\text{ord}} \) and \( K' = K^{S} \) and \( K_{r} . \) We separate it into two short exact sequences:

\[
\begin{align*}
0 & \rightarrow \tilde{A}_{r}^{\text{ord}}(K') \rightarrow J_{\infty}^{\text{ord}}(K') \xrightarrow{\varpi} \varpi(J_{\infty}^{\text{ord}}(K')) \rightarrow 0, \\
0 & \rightarrow \varpi(J_{\infty}^{\text{ord}}(K')) \rightarrow J_{\infty}^{\text{ord}}(K') \rightarrow \widehat{B}_{r}^{\text{ord}}(K') \otimes_{\mathbb{Z}_{p}} \mathbb{Q}_{p} \rightarrow 0.
\end{align*}
\]

Equation (8.7)

Look into the following commutative diagram of sheaves with exact rows:

\[
\begin{array}{ccccccc}
A_{r}[p^{\infty}]^{\text{ord}} & \xrightarrow{\iota} & J_{\infty}^{\text{ord}}[p^{\infty}] & \xrightarrow{\varpi[p^{\infty}]} & J_{\infty}^{\text{ord}}[p^{\infty}] & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\tilde{A}_{r}^{\text{ord}} & \xrightarrow{\iota} & J_{\infty}^{\text{ord}} & \xrightarrow{\varpi} & J_{\infty}^{\text{ord}} & \rightarrow & \widehat{B}_{r}^{\text{ord}} \otimes \mathbb{Q}_{p}.
\end{array}
\]

Equation (8.8)

Since \( \widehat{B}_{r}^{\text{ord}} \otimes \mathbb{Q}_{p} \) is a sheaf of \( \mathbb{Q}_{p} \)-vector spaces and \( J_{\infty}^{\text{ord}}[p^{\infty}] \) is \( p \)-torsion, the inclusion map \( \iota \) factors through the image \( \text{Im}(\varpi) = \varpi(J_{\infty}^{\text{ord}}) \), so,

\[
\varpi(J_{\infty}^{\text{ord}})[p^{\infty}] = J_{\infty}^{\text{ord}}[p^{\infty}].
\]

Equation (8.9)

From the exact sequence, \( \varpi(J_{\infty}^{\text{ord}}) \hookrightarrow J_{\infty}^{\text{ord}} \rightarrow \widehat{B}_{r}^{\text{ord}} \otimes_{\mathbb{Z}_{p}} \mathbb{Q}_{p} \), taking its cohomology sequence, we get the bottom sequence of the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \rightarrow & H^{1}([\varpi(J_{\infty}^{\text{ord}})[p^{\infty}]) & \rightarrow & H^{1}(J_{\infty}^{\text{ord}}[p^{\infty}]) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod_{s \in S} \text{Coker}(J_{\infty}^{\text{ord}}(K_{s})) \rightarrow \widehat{B}_{r}^{\text{ord}}(K_{s}) \otimes_{\mathbb{Z}_{p}} \mathbb{Q}_{p} & \xrightarrow{\iota} & H^{1}_{S}(\varpi(J_{\infty}^{\text{ord}})) & \rightarrow & H^{1}_{S}(J_{\infty}^{\text{ord}}).
\end{array}
\]
Here we have written $H_S^1(X) := \prod_{v \in S} H^1(K_v, X)$. define
\[ \text{Sel}_K(\varpi(J^\text{ord})) := \ker (i : H^1(K^S/K, \varpi(J^\text{ord})[\mathbb{P}^\infty]) \rightarrow H^1_S(\varpi(J^\text{ord}))). \]
(8.10)
\[ \text{III}_K(\varpi(J^\text{ord})) := \ker (i : H^1(K^S/K, \varpi(J^\text{ord})) \rightarrow H^1_S(\varpi(J^\text{ord}))), \]
\[ E_{\text{Sel}}^*(K_v) := \text{Coker}(J^\text{ord}(K_v) \rightarrow \widehat{B}_r(J^\text{ord}(K_v)) \otimes \mathbb{Q}_p). \]

By the snake lemma, we get an exact sequence
(8.11)
\[ 0 \rightarrow \text{Sel}_K(\varpi(J^\text{ord})) \rightarrow \text{Sel}_K(J^\text{ord}) \rightarrow \prod_{v \nmid p} E_{\text{Sel}}^*(K_v), \]

since $\widehat{B}_r(J^\text{ord}(K_v)) \otimes \mathbb{Q}_p = E_{\text{Sel}}^*(K_v) = 0$ if $v \mid p$ by (S) in the introduction.

Define error terms
\[ E_{\text{Sel}}(F) := \frac{J^\text{ord}_1[p^\infty](F)}{\varpi(J^\text{ord}_2[p^\infty](F))} = \lim_{\longrightarrow} \frac{\varpi(J^\text{ord}_3[p^\infty](F))}{\varpi(J^\text{ord}_3[p^\infty](F))} \quad \text{and} \quad E^*(F) := \frac{\varpi(J^\text{ord}_3[p^\infty](F))}{\varpi(J^\text{ord}_3[p^\infty](F))} = \lim_{\longrightarrow} \frac{\varpi(J^\text{ord}_3[p^\infty](F))}{\varpi(J^\text{ord}_3[p^\infty](F))} \]
for $F = K, K_v$, and put $E_S^*(K) = \prod_{v \in S} E_{\text{Sel}}^*(K_v)$. Noting $G \xrightarrow{\varpi} G$ is epimorphism of sheaves for $G = J^\text{ord}_3[p^\infty]$ (see [H14a, §3 (DV)]), we get the following commutative diagram with two bottom exact rows and columns:
\[
\begin{array}{ccc}
\text{Sel}_K(\widehat{A}_{\text{r}, \varpi}) & \xrightarrow{\text{Sel}_K(J^\text{ord})} & \text{Sel}_K(\varpi(J^\text{ord})) \\
\cap & & \cap \\
\text{Sel}_K(J^\text{ord}) & \xrightarrow{\text{Sel}_K(J^\text{ord})} & \text{Sel}_K(\varpi(J^\text{ord})) \\
\downarrow & & \downarrow \\
H^1(\widehat{A}_{\text{r}, \varpi}) & \xrightarrow{\text{Sel}_K(J^\text{ord})} & H^1(\varpi(J^\text{ord})) \\
\downarrow & & \downarrow \\
E_S^*(K) & \xrightarrow{\text{Sel}_K(J^\text{ord})} & H^1_S(\varpi(J^\text{ord})).
\end{array}
\]

Here the last map $\varpi_{S, \ast}$ could have 2-torsion finite cokernel if $p = 2$.

We look into $\Lambda$-TS groups. Let $\varpi \in \mathfrak{h}$ coming from $\varpi_r \in \text{End}(J_{r/\mathbb{Q}})$ and suppose that $(\varpi) = \varpi \circ (\gamma^{p^{r-1}} - 1)$. The long exact sequence obtained from (8.7) produces the following commutative diagram with exact columns and bottom two exact rows:
\[
\begin{array}{ccc}
\text{Sel}_K(\widehat{A}_{\text{r}, \varpi}) & \xrightarrow{\text{Sel}_K(J^\text{ord})} & \text{Sel}_K(\varpi(J^\text{ord})) \\
\cap & & \cap \\
\text{Sel}_K(J^\text{ord}) & \xrightarrow{\text{Sel}_K(J^\text{ord})} & \text{Sel}_K(\varpi(J^\text{ord})) \\
\downarrow & & \downarrow \\
H^1(\widehat{A}_{\text{r}, \varpi}) & \xrightarrow{\text{Sel}_K(J^\text{ord})} & H^1(\varpi(J^\text{ord})) \\
\downarrow & & \downarrow \\
E_S^*(K) & \xrightarrow{\text{Sel}_K(J^\text{ord})} & H^1_S(\varpi(J^\text{ord})).
\end{array}
\]

By the vanishing of $H^1_S(\widehat{A}_{\text{r}})$ ([ADT, Theorem I.3.2] and Lemma 7.2), $\varpi_{S, \ast}$ are surjective. In each term of the diagram (8.13), we can bring the superscript “ord” inside the functor $\text{III}$ and $H^1$ to outside the functor as the ordinary projector acts on $\widehat{J}_{S}$, $J_{\infty}$ and $\widehat{A}_{r}$ and gives direct factor of the sheaf. The diagram “ord” inside is the one obtained directly from the short exact sequence of Corollary 6.3.


In this section, we prove vanishing of the error term $E^\infty(K)$ for local fields of residual characteristic $l \neq p$, which will be put together in the next section to prove the control result up to finite error of the limit Selmer group and the limit Tate–Shafarevich group.

Starting more generally, for the moment, we denote by $K$ either a number field or an $l$-adic field (the prime $l$ can be $p$ unless we mention that $l \neq p$).
Lemma 9.1. Let $K$ either a number field or an $l$-adic field. Then the Pontryagin dual $E^\infty(K)^{\vee}$ of $E^\infty(K)$ is a $\mathbb{Z}_p$-module of finite type (i.e., $E^\infty(K)$ is $p$-torsion of finite corank).

Proof. Let $K' = \overline{K}$ if $K$ is local and $K = K^S$ if $K$ is global. We have an exact sequence

$$0 \to E^\infty(K) \to H^1(K'/K, \widehat{A}_r^{\text{ord}}) \to H^1(K'/K, \widehat{J}_s^{\text{ord}}).$$

By [ADT, I.3.4], if $K$ is local, $H^1(K'/K, \widehat{A}_r^{\text{ord}}) \cong \text{Pic}_{A/K}(K)^{\vee}$; so, we get the desired result. If $K$ is global, $\widehat{A}_r^{\text{ord}}(K) \otimes_{\mathbb{Z}} \mathbb{F} \to H^1(K'/K, \widehat{A}_r^{\text{ord}}[p]) \to H^1(K'/K, \widehat{A}_r^{\text{ord}})[p]$ is exact, and the middle term is finite by Tate’s computation of the global cohomology (taking $S$ to be finite); so, $H^1(K'/K, \widehat{A}_r^{\text{ord}})$ has Pontryagin dual finite type over $\mathbb{Z}_p$. This finishes the proof. □

As before, we write $H^q(M)$ for $H^q(K, M)$ (resp. $H^q(K^S/K, M)$) if $K$ is local (resp. global). For each $\mathbb{Z}_p$-module $M$, we write $T_p M := \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, M) = \varprojlim_n M[p^n]$. For any abelian variety $A/K$, we have an exact sequence $A(K) \otimes_{\mathbb{Z}} \mathbb{F} \to H^1(A[p^n]) \to H^1(\widehat{A})[p^n]$ by Kummer theory. If $K$ is a number field, by Mordell–Weil theorem, $\text{Pic}_{A/K}(K)^{\vee}$ is finite by Tate’s computation of the global cohomology (taking $S$ to be finite); so, $H^1(K'/K, \widehat{A}_r^{\text{ord}})$ has Pontryagin dual finite type over $\mathbb{Z}_p$. This finishes the proof.

By the same reasoning as above, passing to the limit with respect to $n$, by [H14b, Lemma 2.1], we find the exactness of $0 \to \varpi(\widehat{J}_s^{\text{ord}})[p^n](K') \to \varpi(\widehat{J}_s^{\text{ord}})(K') \xrightarrow{\varpi} \varpi(\widehat{J}_s^{\text{ord}})[p^n] \to 0$. Via the associated long exact sequence, we get another exact sequence:

$$0 \to \varpi(\widehat{J}_s^{\text{ord}})(K) \otimes_{\mathbb{Z}_p} \mathbb{F}/p^n \mathbb{F} \to H^1(\varpi(\widehat{J}_s^{\text{ord}})[p^n]) \to H^1(\varpi(\widehat{J}_s^{\text{ord}}))[p^n] \to 0.$$
Note $\text{Ext}^1_{\mathbb{F}_p}(\mathbb{Q}_p/\mathbb{Z}_p, E^s(K)) \cong E^s(K)$ (as $E^s(K)$ is finite $p$-torsion) and
\[
\text{Ext}^1_{\mathbb{F}_p}(\mathbb{Q}_p/\mathbb{Z}_p, H^1(\hat{A}_v^\text{ord})) \cong H^1(\hat{A}_v^\text{ord})^{\vee} \cong [p^\infty]
\]
which is finite. Here $M \to M^\vee$ is the Pontryagin duality. Since the natural map $i : T_p \text{Im}(b) \to T_p H^1(\hat{A}_v^\text{ord})$ is injective with $\text{Im}(i) = \hat{H}(h_s)$ (as $M \to T_p M$ is left exact), we have an exact sequence $0 \to \text{Ker}(h_s) \to \text{Im}(b_s) \to E^s(K) \to H^1(\hat{A}_v^\text{ord})^{\vee} [p^\infty]$ whose extreme right term has finite order independent of $s$. Since $T_p E^s(K) = 0$ as $E^s(K)$ is finite, we find $b_s$ is injective from the left exactness of the functor $T_p$.

To see the existence of the map $\hat{e}_s$, we suppose that $x = \varpi_s(y) \in \varpi_s(\hat{J}_s^\text{ord}(K)))$. Then we have
\[
\varpi_s(\beta(x)) = \varpi_s(\beta(\varpi_s(y))) = \varpi_s(j(f(y))) = 0.
\]
If $b \equiv b' \mod \varpi_s(\hat{J}_s^\text{ord}(K)))$ for $b, b' \in \hat{B}^\text{ord}(K)$, we have $\varpi_s(\beta(b)) = \varpi_s(\beta(b'))$. In other words, $\pi(b) \mapsto \varpi_s(\beta(\varpi_s(y)))$ is a well-defined homomorphism from $E^s(K) \cong \varpi_s(\hat{J}_s^\text{ord}(K)/\varpi_s(\hat{J}_s^\text{ord}(K)))$ into $\text{Im}(\varpi_s) \cong \text{Coker}(j) \subset H^2(T_p A^\text{ord})$, which we have written as $e_s$.

We have the following fact.

**Lemma 9.3.** Let $K$ be either a number field or a local field over $\mathbb{Q}_l$ for a prime $l$. If $A/K$ is an abelian variety defined over $K$ with its Tate module $T_p A$. Then we have $H^0(K, T_p A) = 0$. In particular, the map $a$ in (9.1) is injective.

This lemma applied to $T_p \varpi_s(\hat{J}_s^\text{ord}) \subset T_p J_s$ tells us $H^0(K, T_p \varpi_s(\hat{J}_s^\text{ord})) \subset H^0(K, T_p J_s) = 0$, and therefore the map $a$ in (9.1) is injective.

**Proof.** Note that $H^0(K, T_p A^\text{ord}) = 0$ (from $H^0(K, T_p A) = 0$) which surjects to $\text{Ker}(a)$, and hence $a$ is injective.

If $K$ is a number field, by [B81], the image of Galois group in $\text{Aut}(T_p A)$ contains an open subgroup of $\mathbb{Z}_p^\times$ as scalars, we conclude $H^0(K, T_p A) = 0$.

Assume now that $K$ is local. After extending scalars, we may assume that $A$ extends to a semi-abelian scheme over the integer ring $W$ of $K$ (see [NMD, §7.4]). Since $T_p A$ can be embedded into the product of the Tate modules of simple factors of $A$, we may assume that $A$ is absolutely simple. By extending scalars, we may assume that $A$ has semi-stable reduction over $W$. Changing $A$ by an isogeny, we may assume that the abelian factor of its reduction modulo $mm$ has complex multiplication by the maximal order of a CM semi-simple algebra. Writing $\hat{A} := \text{Pic}^0(A/K)$ and $\hat{A}$ for the formal group of $A$ with its toric part $\hat{A}_m$. Then we have a unique dual étale quotient $A[p^\infty]^{\text{ét}}_m$ of $\hat{A}_m[p^\infty]$ by Cartier duality, which is actually a direct factor of the Barsotti–Tate group $A[p^\infty]^{\text{ét}}$. This gives rise to an extension
\[
0 \to T_p A_m \to T_p A_m \to T_p A[p^\infty]^{\text{ét}}_m \to 0.
\]
Then we conclude from the theory of degeneration of abelian varieties [DAV] this extension of the toric part
\[
0 \to T_p A_m \to T_p A[p^\infty]_m \to T_p A[p^\infty]^{\text{ét}}_m \to 0
\]
does not have any split factor as the module of the inertia subgroup $I_l$ of $\text{Gal}(\mathbb{K}/K)$. Thus we conclude $H^0(K, T_p A_m) = 0$. On the other hand, the quotient $A[p^\infty]/A[p^\infty]_m$ is a Barsotti–Tate group over $W$, which is étale if $l \neq p$. On $A[p^\infty]/A[p^\infty]_m$, the Frobenius action $\phi$ has eigenvalues given by Weil $l$-number of weight $f > 0$ if the residue field of $W$ has order $U$. This shows that $H^0(K, T_p A/p^\infty]_m) = 0$. Combined with the vanishing of the toric part, we conclude $H^0(K, T_p A) = 0$ if $l \neq p$.

If $l = p$, $A[p^\infty]/A[p^\infty]_m$ is an extension of a product $A^{LT}$ of Lubin–Tate formal groups over $W$ associated with the $p$-Frobenius eigenvalues by the étale quotient on which the Frobenius has eigenvalues given by Weil $p$-number of weight $f$. Then we see $H^0(K, T_p (A[p^\infty]/A[p^\infty]_m)) = 0$, and again we conclude $H^0(K, T_p A) = 0$. □
Remark 9.4. Just to know $H^0(K, T_p\omega(\tilde{s}_s^{\text{ord}})) = 0$ which we really need, this follows from [H14b, Corollary 4.4] directly without using the result in Lemma 9.3 for a general abelian variety.

We have a similar lemma for $H^2$.

Lemma 9.5. Let $K$ be a local field over $\mathbb{Q}_l$ for a prime $l$. If $A_K$ is an abelian variety defined over $K$ with its Tate module $T_pA$. Then we have $H^2(K, T_pA) \cong A[p\infty]$ which is a finite module. Here $A[p\infty](-1) = A[p\infty] \otimes_{Z_p} Z_p(-1)$ for the Galois module $Z_p(-1) := \text{Hom}_{Z_p}(\mu_{p\infty}(\overline{\mathbb{Q}}), Q_p/Z_p)$.

Proof. By Lemma 7.1 (2), we have $H^2(K, T_pA) = \lim_n H^2(K, A[p^n])$. By Tate duality (e.g., [MFG, Theorem 4.43]), we have $H^2(K, A[p^n]) \cong A'[p^n](K)$. Thus we have

$$H^2(K, T_pA) = \lim_n A'[p^n](K)^\vee \cong (\lim_n A'[p^n](K))^\vee = A'[p\infty](K)^\vee.$$ 

By Lemma 9.3, $A'[p\infty](K)$ is a finite module. Since we have a canonical pairing $A[p^n] \times A'[p^n] \to \mu_{p^n}$, we have $A'[p\infty](K)^\vee \cong A[p\infty](-1)(K)$. Thus we get the desired assertion.

Lemma 9.6. Assume (A) and (F). Let $K$ be a finite extension of $\mathbb{Q}_l$ for a prime $l \neq p$. Then we have $\text{Ker}(h_s) = \text{Im}(b_s) = 0$.

Proof. By [ADT, I.3.4], if $K$ is local, for an abelian variety $A$ over $K$, we have $H^1(K, A) = A'(K)^\vee$ for $A' = \text{Pic}^0_{A/K}$. Then if $l \neq p$, we find $T_pH^1(K, A) = 0$ as $A'(K) \cong W^{\dim A} \times \Delta$ for a finite group $\Delta$ and the $l$-adic integer ring $W$ of $K$. Thus $b_s$ and $h_s$ are zero maps.

Proposition 9.7. Assume (A) and (F). Let $K$ be a finite extension of $\mathbb{Q}_l$ for a prime $l$. Suppose that the complex (9.2) is exact. Then we have $E^*(K) = 0$. In particular, $E^*(K) = 0$ if $l \neq p$. If $l = p$, writing $X' = \text{Pic}^0_{X/K}$ for an abelian variety $X/K$, we have

$$E^*(K) \cong \text{Coker}(\tilde{J}^s_t(K)^{\text{co-ord}} \to \tilde{A}'_s(K)^{\text{co-ord}})^\vee.$$ 

We will prove the finiteness and boundedness of $E^*(K)$ when $l = p$ later in Section 17 under some extra assumptions (see Theorem 17.3).

Proof. By the exactness of (9.2), we may apply the snake lemma to the middle two exact rows of (9.1), and we find an exact sequence

$$(9.3) \quad 0 \to E^*(K) \xrightarrow{\epsilon_s} \text{Im}(\omega_s) \to \text{Coker}(h_s) \to 0.$$ 

This implies $E^*(K) \hookrightarrow \text{Im}(\omega_s) \subset H^2(T_rA^{\text{ord}})$. By Lemma 9.5, we have

$$H^2(K, T_rA^{\text{ord}}) \cong A_r[p\infty]^{\text{ord}}(1) = J_r[p\infty]^{\text{ord}}(1) = H^2(K, T_rJ_r^{\text{ord}}),$$

which is injective as $A_r \subset J_r$. We have an exact sequence

$$H^1(K, T_r\omega(\tilde{J}^{\text{ord}}_s)) \xrightarrow{\epsilon_s} H^2(K, T_rA^{\text{ord}}) \xrightarrow{a_2} H^2(K, T_rJ_r^{\text{ord}}).$$

Since $a_2$ is injective, we find $\text{Im}(\omega_s) = 0$; so, $E^*(K) = 0$. If $l \neq p$, (9.2) is exact by Lemmas 9.2 and 9.6, and hence $E^*(K) = 0$.

Suppose $l = p$. We have an exact sequence

$$0 \to E^*(K) := \text{Coker}(\omega_s : \tilde{J}_s^{\text{ord}}(K) \to \omega(\tilde{J}^{\text{ord}}_s)(K)) \xrightarrow{\epsilon_s} H^1(K, \tilde{A}'_s^{\text{ord}}) \xrightarrow{a_1} H^1(K, \tilde{J}_s^{\text{ord}}).$$

By [ADT, I.3.4], if $K$ is local, for an abelian variety $A$ over $K$, we have $H^1(K, A) = A'(K)^\vee$ for $A' = \text{Pic}^0_{A/K}$. This shows

$$E^*(K) = \text{Ker}(\tilde{A}'_s^{\text{co-ord}}(K)^\vee \to \tilde{J}_s^{\text{co-ord}}(K)^\vee) = \text{Coker}(\tilde{J}_s^{\text{co-ord}}(K) \to \tilde{A}'_s^{\text{co-ord}}(K))^\vee$$

as desired.

□
10. Control of \( A \)-Selmer groups

We start with a lemma.

**Lemma 10.1.** For a number field or an \( l \)-adic field \( K \) and \( \mathcal{G} = J^\infty[p^\infty] \), the Pontryagin dual \( \mathcal{G}(K)^\vee \) is a \( \Lambda \)-torsion module of finite type. For any arithmetic prime \( P \), \( \mathcal{G}(K)^\vee \otimes_{h} h/P^n \), \( \mathcal{G}(K) \otimes_{h} h/P^n \) and \( \mathcal{G}(K)[P^n] \) are all finite for any positive integer \( n \).

**Proof.** We give a detailed argument when \( K \) is a number field and briefly touch an \( l \)-adic field as the argument is essentially the same. Let \( P \in \Omega_h \), and suppose \( K \) is a number field. Suppose that the Galois representation \( \rho_P \) associated with \( P \) contains an open subgroup \( G \) of \( SL_2(\mathbb{Z}_p) \). Let \( L \) be the Pontryagin dual module of \( \mathcal{G}(\mathbb{Q}) \). If the cusp form \( f_P \) associated to \( P \) has conductor divisible by \( N \), the localization \( L_P \) is free of rank 2 over the valuation ring \( V = h_P \) finite over \( \Lambda_p \) (e.g., [HMI, Proposition 3.78]). If not, by the theory of new form (e.g. [H86a, §3.3]), \( L_P \) is free of rank 2 over a local ring of the form \( V[X_1, \ldots, X_m]/(X_1^{a_1}, \ldots, X_m^{a_m}) = h \) with nilradical coming from forms (e.g., [H13a, Corollary 1.2]). The contragredient \( \tilde{\rho}_P = \rho_P^{-1} \) of \( \rho_P \) is realized by \( L_P/PL_P \). Then \( G \) is also contained in \( \text{Im}(\tilde{\rho}_P) \), and \( H_0(K, L_P/PL_P) \cong H(K, L_P)/PH_0(K, L_P) \) is a surjective image of \( H_0(G, L_P/PL_P) \), which vanishes. Thus \( H_0(K, L_P/PL_P) = 0 \), which implies \( H_0(K, L_P) = 0 \) by Nakayama’s lemma. In particular, \( H_0(K, L) \) is a \( \Lambda \)-torsion module whose support is outside \( P \).

If \( \rho_P \) does not contain an open subgroup of \( SL_2(\mathbb{Z}_p) \), by Ribet [R85], there exists an imaginary quadratic field \( M \) such that \( \tilde{\rho}_P = \text{Ind}_M^Q \phi \) for an finite Hecke character \( \phi \) of \( \text{Gal}(\overline{\mathbb{Q}} \mid M) \). Then it is easy to show that \( H_0(K, L_P/PL_P) = 0 \), and in the same way as above, we find \( H_0(K, L_P) = 0 \) and that \( H_0(K, L) \otimes h/P^n \) is finite for all \( n \). Thus for any arithmetic prime \( P \in \Omega_h \), \( L_P = 0 \) and hence \( \mathcal{G}(K)^\vee = H_0(K, L) \) is \( \Lambda \)-torsion with support outside \( P \). Thus in any case, \( \mathcal{G}(K)^\vee \otimes_{h} h/P^n \) is finite for all \( n \) as \( h \) is a semi-local ring of dimension 2 finite torsion-free over \( \Lambda \). The module \( \mathcal{G}(K)[P^n] \) is just the dual of \( H_0(K, L) \otimes h/P^n \) and hence is finite. Then \( (\mathcal{G}(K) \otimes_{h} h/P^n)^\vee = H_0(K, L)[P^n] \), which is finite by the above fact that \( H_0(K, L) \) is \( \Lambda \)-torsion with support outside \( P \).

If \( K \) is \( l \)-adic, replacing \( K \) by its finite extension, we may assume that \( A_P \) has split semi-stable reduction. Write \( F \) for the residu field of \( P \). Then either \( \tilde{\rho}_P(\text{Frob}) \) for a Frobenius element \( \text{Frob} \) of \( \text{Gal}(\overline{\mathbb{Q}} \mid K) \) has infinite order without eigenvalue 1 or the space \( V(\tilde{\rho}_P) \) fits into a non-split extension \( F \hookrightarrow V \twoheadrightarrow F(-1) \) for the Tate twist \( F(-1) \). Because of this description \( H_0(K, L_P/PL_P) = 0 \), and by the same argument above, the results follows.

Since \((\varpi)\) is supported by finitely many arithmetic primes, \( E_{\text{Sel}}(K)^\vee := (\mathcal{G}(K) \otimes_{h} h/(\varpi))^{\vee} \cong \mathcal{G}(K)^\vee[\varpi] \) is finite by the above lemma; so, we get

**Corollary 10.2.** Assume (A). If \( K \) is a number field, then \( E_{\text{Sel}}(K) \) is finite.

Let \( T \) be the local ring such that \( \varpi \in m_T \). We see \( E_{\text{Sel}}(K)_T = \text{Coker}(\varpi : \mathcal{G}(K)_T \to \mathcal{G}(K)^\vee_T) \), where \( M_T = M \otimes h_T \) for an \( h \)-module \( M \). Thus for the Galois representation \( \rho_T \) acting on \( T\mathcal{G}_T = \lim \bigwedge T_P \mathcal{G}_p[p^\infty - 1] \), if \( \rho_T \) modulo \( m_T \) is absolutely irreducible over \( \text{Gal}(\overline{\mathbb{Q}} \mid K) \), we conclude \( E_{\text{Sel}}(K) = 0 \). Here \( \overline{\text{Tr}}_T = (\rho_T \mod m_T) \) is the semi-simple two dimensional representation whose trace is given by \( \text{Tr}(\rho_T) \mod m_T \). Indeed, the Galois module \( \mathcal{G}(m_T) \) has Jordan-Hölder sequence whose sub-quotients are all isomorphic to \( \overline{\text{Tr}}_T \); so, by Nakayama’s lemma, \( \mathcal{G}(K) = 0 \). Write \( \overline{\text{Tr}}_T \mid \text{Gal}(\overline{\mathbb{Q}}_p/Q_p) \cong \left( \begin{array}{cc} \overline{\varpi} - s & 0 \\ 0 & \overline{\varpi} \end{array} \right) \) mod \( m_T \) with the nearly ordinary character \( \overline{\varpi} \) (i.e., \( \overline{\varpi}([p, Q_p]) \) is equal to the image modulo \( m_T \) of \( U(p) \)). Here \( \overline{\varpi}_p = \nu_p \mod p \). Then it is plain that \( \mathcal{G}(K_{ne}) = 0 \) for all place \( v \mid p \) of \( K \) if \( \overline{\varpi}_p \overline{\psi} \) and \( \overline{\varpi} \) are both non-trivial over \( \text{Gal}(\overline{\mathbb{Q}}_p/K_v) \) for all \( p \)-adic places \( v \mid p \) of \( K \).

**Corollary 10.3.** Let \( p > 2 \), and suppose one of the following two conditions:

1. \( \overline{\varpi}_p \) is irreducible over \( \text{Gal}(\overline{\mathbb{Q}}_p/K_v) \);
2. \( \overline{\varpi}_p \overline{\psi} \) and \( \overline{\varpi} \) are both non-trivial over \( \text{Gal}(\overline{\mathbb{Q}}_p/K_v) \) for all \( p \)-adic places \( v \mid p \) of \( K \).

Then we have \( E_{\text{Sel}}(K) = 0 \).
If $\xi(a, d) = \alpha(a)\beta(d)$, we have $\overline{\varphi}(z, Q_p) = \alpha(z) \mod p$ and $\overline{\varphi}(z, Q_p) = \beta(z) \mod p$ for $z \in Q_p^\times$, where $[z, Q_p]$ is the local Artin symbol. Let $\rho_A$ be the Galois representation realized on $T_p\widehat{\mathcal{A}}_{\text{ord}}$ into $GL_2(W)$ for a finite flat extension $W$ of $\mathbb{Z}_p$. Write $\rho_A|_{\text{Gal}(\overline{\mathbb{Q}}_p/Q_p)} \simeq \left( \begin{smallmatrix} \nu & 0 \\ 0 & \phi \end{smallmatrix} \right)$. Supposing $p > 2$, for each $p$-adic place $v|p$ of $K$, let $\sigma_v$ (resp. $\text{Frob}_v$) be a topological generator of $\text{Gal}(K_v[\mu_{p^n}]/K_v)$ (resp. $\text{Gal}(K_v[\mu_{p^n}]/K_v)$) inducing on $K_v[\mu_{p^n}]$ a power of the local Artin symbol $[p, Q_p]$, where $K_v^{ur}$ is the maximal unramified extension of $K_v$ in $\overline{K}_p$. Recall $M_T = M \otimes_{\mathbb{Q}} T$ for an $h$-module $M$.

**Theorem 10.4.** Suppose $p > 2$, (A) and (F). Let $K$ be a number field and $\text{Spec}(T)$ be the connected component such that $\varpi \in \pi_T$.

1. (e1) $E^\infty(K_v)_T$ is finite for all $v|p$.
2. (e2) $A_v$ does not have split multiplicative reduction modulo $p$ at all primes $p|p$ of $K$.

Then the following sequence

$$0 \rightarrow \text{Sel}_K(A_v^{\text{ord}}) \rightarrow \text{Sel}_K(J_{\text{ord}}^{\infty, T}) \xrightarrow{\varpi} \text{Sel}_K(J_{\text{ord}}^{\infty, T})$$

is exact up to finite error.

2. Assume one of the following two conditions:
   (E1) $E_{\text{Sel}}(K_v)^T = E^\infty(K_v)_T = 0$ for all $v|p$,
   (E2) $K/Q$ is unramified at $p$ and for a generator $\sigma$ of $\text{Gal}(\mathbb{Q}[\mu_{p^n}]/\mathbb{Q})$

$$|\nu_p\psi(\sigma) - 1|_p = |\psi(\sigma) - 1|_p = |\phi(\text{Frob}_v) - 1|_p = 1$$

for all $v|p$. Here $\text{Frob}_v$ is a Frobenius element in $\text{Gal}(\overline{K}_v/K_v)$ acting trivially on $K[\mu_{p^n}]$.

Then the sequence $(10.1)$ is exact, and if in addition $E_{\text{Sel}}^*(K_v)_T = 0$ for all $v|p$, we have $\text{Sel}_K(\varpi(J_{\text{ord}}^{\infty, T})) \cong \text{Sel}_K(J_{\text{ord}}^{\infty, T})$.

By $(8.11)$, we have an exact sequence: $0 \rightarrow \text{Sel}_K(\varpi(J_{\text{ord}}^{\infty})) \rightarrow \text{Sel}_K(J_{\text{ord}}^{\infty}) \rightarrow \prod_{v|p} E_{\text{Sel}}^*(K_v)_T$ and for most cases, $\prod_{v|p} E_{\text{Sel}}^*(K_v)_T$ is finite (see Theorem 17.3 for a sufficient condition for the finiteness of $E_{\text{Sel}}^*(K_v)_T$). By our choice of $T$, we have $\widehat{A}_v^{\text{ord}} = A_v^{\text{ord}}$ and $\widehat{B}_v^{\text{ord}} = B_v^{\text{ord}}$. By Corollary 10.3, (E2) implies $E_{\text{Sel}}(K) = 0$. Since we will later prove in Theorem 17.3 that (e2) (resp. (E2)) implies (e1) (resp. (E1)), we prove the theorem under (E1) or (e1).

**Proof.** Recall the following commutative diagram with two bottom exact rows and three right exact columns from $(8.12)$ (tensored with $T$ over $h$):

$$
\begin{array}{ccccccc}
\text{Ker}(i_{\text{Sel},*}) & \xrightarrow{i} & \text{Sel}_K(\widehat{A}_v^{\text{ord}}) & \xrightarrow{i_{\text{Sel},*}} & \text{Sel}_K(J_{\text{ord}}^{\infty, T}) & \xrightarrow{\varpi_{\text{Sel},*}} & \text{Sel}_K(\varpi(J_{\text{ord}}^{\infty, T})) \\
\downarrow & & \downarrow i & & \downarrow \cap & & \downarrow \cap \\
E_{\text{Sel}}(K) & \xrightarrow{\epsilon} & H^1(\widehat{A}_v^{\text{ord}}[p^\infty]) & \xrightarrow{i} & H^1(J_{\text{ord}}^{\infty, T}[p^\infty]) & \xrightarrow{\varpi} & H^1(J_{\text{ord}}^{\infty, T}[p^\infty]) \\
\downarrow \epsilon & & \downarrow & & \downarrow & & \downarrow \\
\prod_{v|p} E^\infty(K_v) & \xrightarrow{\epsilon_0} & H^1_S(\widehat{A}_v^{\text{ord}}) & \xrightarrow{\epsilon_{S, *}} & H^1_S(J_{\text{ord}}^{\infty, T}) & \xrightarrow{\varpi_{S, *}} & H^1_S(\varpi(J_{\text{ord}}^{\infty, T}))
\end{array}
$$

Since the middle two columns are exact, the left column is exact with injection $i$ (e.g., [BCM, I.1.4.2 (1)]). Since the bottom row is exact with injection $\epsilon_0$, the map $\epsilon_0$ is injective and $\text{Im}((\epsilon_0)) = \text{Ker}(i_{\text{Sel},*})$. Suppose (E1). Then all the terms of the left column vanish. Then we get $\text{Ker}(i_{\text{Sel},*}) = 0$ and the following exact sequence:

$$0 \rightarrow \text{Sel}_K(A_v^{\text{ord}}) \rightarrow \text{Sel}_K(J_{\text{ord}}^{\infty}) \rightarrow \text{Sel}_K(\varpi(J_{\text{ord}}^{\infty})) \xrightarrow{(8.11)} \text{Sel}_K(J_{\text{ord}}^{\infty})$$

is exact. The cokernel $\text{Coker}(\text{Sel}_K(J_{\text{ord}}^{\infty}) \xrightarrow{\varpi} \text{Sel}_K(\varpi(J_{\text{ord}}^{\infty})))$ is global in nature and seems difficult to determine, although $\text{Coker}(\text{Sel}_K(\varpi(J_{\text{ord}}^{\infty})) \xrightarrow{\varpi} \text{Sel}_K(J_{\text{ord}}^{\infty}))$ is local as in $(8.11)$, and if $E_{\text{Sel}}(K_v) = 0$ for all $v|p$, it vanishes.
Now we assume \((e1)\). We need to prove the sequence \((10.3)\) is exact up to finite error. By Corollary 10.2, \(E_{\text{Sel}}(K)\) is finite. Since we know \(E_{\infty}^\wedge(K_v) = 0\) for \(v\) prime to \(p\) by Proposition 9.7, we conclude from \((e1)\) that \(E_{\infty}^\wedge(K)\) is finite. Then the diagram \((8.12)\) has two bottom rows exact up to finite error. Since the Pontryagin dual of all the modules in the above diagram are \(\Lambda\)-modules of finite type, we can work with the category of \(\Lambda\)-modules of finite type up to finite error [e.g., [BCM, VII.4.5]]. Then in this new category, the bottom two rows are exact and the extreme left terms a pseudo-null. Thus the dual sequence of the theorem is exact up to finite error, and by taking dual back, the sequence in the theorem is exact up to finite error. \(\Box\)

Here is an obvious corollary:

**Corollary 10.5.** Assume \((F)\) and \(p > 2\). Then we have

1. The Pontryagin dual Sel\(_K(J_{\infty}^\text{ord})^\vee\) of Sel\(_K(J_{\infty}^\text{ord})\) is a \(\Lambda\)-module of finite type.
2. If further Sel\(_K(J_{\infty}^\text{ord})^\vee\) for a single element \(\varpi \in m_\tau\) satisfying \((A)\) and \((e1)\), then Sel\(_K(J_{\infty}^\text{ord}) = 0\) and Sel\(_K(\hat{A}_r^\text{ord}) = 0\) for every \(\varpi \in m_\tau\) satisfying \((A)\) and \((e1)\).
3. Suppose that \(\mathbb{T}\) is an integral domain. If Sel\(_K(\hat{A}_r^\text{ord})\) is finite for some \(\varpi\) satisfying \((A)\) and \((e1)\), then Sel\(_K(J_{\infty}^\text{ord})^\vee\) is a torsion \(\mathbb{T}\)-module of finite type. Thus if \(\mathbb{T}\) is a unique factorization domain, for almost all \(P \in \Omega_\mathbb{T}\), Sel\(_K(\hat{A}_P^\text{ord})\) is finite.

**Proof.** The condition \((A)\) and \((e2)\) is satisfied by any non-trivial factor \(\varpi\) of \((\gamma^p - 1)/(\gamma - 1)\). Thus Sel\(_K(J_{\infty}^\text{ord})^\vee/\varpi\cdot\text{Sel}_K(J_{\infty}^\text{ord})^\vee\) pseudo isomorphic to Sel\(_K(\hat{A}_r^\text{ord})^\vee\) which is \(\mathbb{Z}_p\)-module of finite type; so, by the topological Nakayama’s lemma, we conclude that Sel\(_K(J_{\infty}^\text{ord})\) is a \(\Lambda\)-module of finite type.

The last two assertions can be proven similarly. \(\Box\)

Suppose that every prime factor of \(p\) in \(K/\mathbb{Q}\) has residual degree 1 and that \(\mathbb{T}\) is a unique factorization domain (so, \((A)\) holds for every \(P \in \Omega_\mathbb{T}\)). Then if \(A_P\) for every \(P \in \Omega_\mathbb{T}\) has potential good reduction and \(A_P\) for some \(P\) has good reduction at \(p\), writing \(f_p|U(p) = a_pf_P\), we have \(|a_p - 1|_p = 1\) as otherwise \(f_P\) has level raising congruence, and therefore \(A_Q\) for some \(Q \in \Omega_\mathbb{T}\) is potentially multiplicative, contradicting to \(A_P\) having potentially good reduction for every \(P \in \Omega_\mathbb{T}\).

Therefore, \((e1)\) is satisfied by Theorem 17.3 \((a)\), and taking \(K = \mathbb{Q}\), the assertion \((4)\) of Theorem A follows from Corollary 10.5.

From the exact sequence \((J_{\infty, \mathbb{T}}^\text{ord}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p/\mathbb{Z}_p) \hookrightarrow \text{Sel}_K(J_{\infty, \mathbb{T}}^\text{ord}) \rightarrow \text{III}_K(J_{\infty, \mathbb{T}}^\text{ord})\), we get

**Corollary 10.6.** Assume \((F)\). The limit Mordell–Weil group group \((J_{\infty, \mathbb{T}}^\text{ord}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) and the limit Tate–Shafarevich group \(\text{III}_K(J_{\infty, \mathbb{T}}^\text{ord})^\vee\) are \(\Lambda\)-module of finite type.

11. CONTROL OF IND \(\Lambda\)-MW GROUPS

**Theorem 11.1.** Assume \((A)\) and \((F)\). Let \(p > 2\) and \(K\) be either a number field or an \(l\)-adic field. Put \(E_{\text{MW}} := \text{Coker}(\varpi(J_{\infty}^\text{ord})(K) \hookrightarrow J_{\infty}^\text{ord}(K))\). The Pontryagin dual of the following sequence

\[
(11.1) \quad 0 \to \hat{A}_r^\text{ord}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J_{\infty}^\text{ord}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} J_{\infty}^\text{ord}(K) \otimes_{\mathbb{Q}_p} \mathbb{Z}_p/\mathbb{Z}_p \to E_{\text{MW}}(K) \otimes_{\mathbb{Q}_p} \mathbb{Z}_p/\mathbb{Z}_p \to 0
\]

is exact up to \(\Lambda\)-torsion error of finite type. More precisely, we have

1. Suppose \(K\) is a number field. Then except possibly for \(\text{Ker}(\varpi)/\text{Im}(\iota)\) which is pseudo isomorphic to \(E_{\infty}^\wedge(K)\), \((11.1)\) is exact up to finite error; so, if \(E_{\infty}^\wedge(K)\) is finite (in particular, if \(\text{III}_K(\hat{A}_r^\text{ord})\) is finite), the entire sequence \((11.1)\) is exact up to finite error.

2. If \(K\) is \(l\)-adic and either \(l \neq p\) or \(A_r\) does not have split multiplicative reduction over \(W\), the sequence \((11.1)\) is exact up to finite error.

**Proof.** Since \(\varpi(J_{\infty}^\text{ord}(K)) = J_{\infty}^\text{ord}(K)/\hat{A}_r^\text{ord}(K)\), we have the following three exact sequences:

\[
(11.2) \quad 0 \to \varpi(J_{\infty}^\text{ord}(K)) \to \varpi(J_{\infty}^\text{ord}(K)) \to E_{\infty}(K) \to 0,
\]

\[
(11.2) \quad 0 \to \hat{A}_r^\text{ord}(K) \to J_{\infty}^\text{ord}(K) \to \varpi(J_{\infty}^\text{ord}(K)) \to 0,
\]

\[
(11.2) \quad 0 \to \varpi(J_{\infty}^\text{ord}(K)) \to J_{\infty}^\text{ord}(K) \to E_{\text{MW}}(K) \to 0.
\]
Tensoring with \( \mathbb{Q}_p/\mathbb{Z}_p \) over \( \mathbb{Z}_p \), we get the following exact sequences

\[
\varpi(J^\infty_\infty)\mathbb{Q}_p/\mathbb{Z}_p(K) \rightarrow E^\infty(K) \rightarrow \varpi(J^\infty_\infty(K)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong \varpi(J^\infty_\infty(K)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0,
\]

(11.3) \( \varpi(J^\infty_\infty(K)) \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \hat{A}^\ord_r(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \varpi(J^\infty_\infty(K)) \otimes \mathbb{Q}_p/\mathbb{Z}_p, \)

\[
0 \rightarrow \varpi(J^\infty_\infty(K)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J^\ord_\infty(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow E^\infty_{MW} \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0,
\]

The last sequence is exact since, by Corollary 6.3, \( E^\infty_{MW} \) is torsion-free \( \mathbb{Z}_p \)-module. From the exact sequence \( \hat{A}^\ord \rightarrow J^\infty \rightarrow \varpi(J^\infty) \) of sheaves, we get \( J^\ord(\hat{A}^\ord_r(K)) \hookrightarrow \varpi(J^\ord_r(K)) \hookrightarrow J^\infty_r(K) \). Then taking its \( p \)-torsion part, we have \( \varpi(J^\ord_r(K)) \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \varpi(J^\ord_r(K)) \mathbb{Q}_p/\mathbb{Z}_p \). On the other hand, the image \( \text{Im}((\varpi(J^\ord_r(K)))) \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \hat{A}^\ord_r(K) \mathbb{Q}_p/\mathbb{Z}_p \) is killed by \( \varpi \). Since \( \varpi \) is arithmetic, by Lemma 10.1, this image is finite. Thus for the sequence (11.1):

\[
0 \rightarrow \hat{A}^\ord_r(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J^\infty_r(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J^\ord_r(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow E^\infty_r(\mathbb{Z}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0
\]

\( \iota \) has finite kernel, \( \text{Ker}(\varpi) / \text{Im}(\iota) \) is the image of \( E^\infty \) with finite cokernel and the last three right terms are exact. By Lemma 9.1, \( E^\infty(K) \) has \( p \)-torsion with finite corank over \( \mathbb{Z}_p \), which is finite if \( \text{I}^p(K, \hat{A}^\ord_r) \) is finite by (8.13) when \( K \) is a number field. If \( K \) is \( l \)-adic with \( l \neq p \), \( E^\infty(K) = 0 \) by Proposition 9.7, \( E^\infty(K) = 0 \). If \( K \) is \( p \)-adic, by Theorem 17.3, \( E^\infty(K) \) is finite if \( A_r \) does not have split multiplicative reduction over \( W \).

12. Control of A-BT groups and its cohomology

Let \( G := G_{\alpha, \delta, \xi} = J^\ord_r[p^\infty] \) which is a A-BT group in the sense of [H14a]. Here the set \( S \) is supposed to be finite. We study the control of the Tate–Shafarevich group of \( G \).

**Theorem 12.1.** Let \( K \) be a number field. Suppose \( |S| < \infty \), (F) and (A) for \( \varpi \). Then the sequence

\[
0 \rightarrow \text{III}(K^S/K, \hat{A}^\ord_r[p^\infty]) \rightarrow \text{III}(K^S/K, G) \rightarrow 0
\]

is exact up to finite error.

**Proof.** From the exact sequence 0 \( \rightarrow \hat{A}^\ord_r[p^\infty] \rightarrow G \xrightarrow{\varpi} G \rightarrow 0 \) [H14a, §3 (DV)], we get a commutative diagram with exact rows and exact columns:

\[
\begin{array}{cccc}
\text{Ker}(\iota) & \rightarrow & \text{III}(K^S/K, \hat{A}^\ord_r[p^\infty]) & \rightarrow & \text{III}(K^S/K, G) \\
\cap & \cap & \cap & \cap & \\
E^\infty_{BT}(K) & \rightarrow & H^1(\hat{A}^\ord_r[p^\infty]) & \rightarrow & H^1(G) \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\prod_{v \in S} E^\infty_{BT}(K_v) & \rightarrow & H^1(\hat{A}^\ord_r[p^\infty]) & \rightarrow & H^1(G) \\
\end{array}
\]

(12.1)

where \( E^\infty_{BT}(K) = \text{Coker}(\varpi : G(\text{K}) \rightarrow G(K)) \).

By Lemma 10.1, \( E^\infty_{BT}(K) \) and \( E^\infty_{BT}(K_v) \) are finite. Thus as long as \( S \) is finite, \( \prod_{v \in S} E^\infty_{BT}(K_v) \) is finite. Then the above diagram proves the desired exactness. \( \square \)

Here is an obvious corollary:

**Corollary 12.2.** Let the notation and the assumption be as in the theorem. Assume that \( T \) is an integral domain. If \( (\varpi) \) is a prime of \( T \) and \( \text{III}(K^S/K, \hat{A}^\ord_r[p^\infty]) \) is finite, then \( \text{III}(K^S/K, G_T) \) is a torsion \( T \)-module of finite type. If in addition \( T \) is a unique factorization domain, for almost all arithmetic point \( P \in \Omega \), \( \text{III}(K^S/K, \hat{A}^\ord_r[p^\infty]) \) is finite.

13. Control of A-TS groups

We now study control of the limit Tate–Shafarevich group from which Theorems A and B in the introduction follow directly. We start with a lemma.
Lemma 13.1. Consider the following commutative diagrams of discrete $\mathbb{Z}_p$-modules of co-finite type:

$$\begin{array}{ccc}
\text{Ker} & \rightarrow & D_1 \xrightarrow{\pi} D_2 \\
\cap & \downarrow & \cap \\
Y & \rightarrow & Y_1 \xrightarrow{1} Y_2,
\end{array}$$

(13.1)

and

$$\begin{array}{ccc}
D_1 & \xrightarrow{\pi} & D_2 \rightarrow \text{Coker} \\
\cap & \downarrow & \cap \\
Y_1 & \xrightarrow{1} & Y_2 \rightarrow Y,
\end{array}$$

(13.2)

Here the word “co-finite type” means the Pontryagin dual is of finite type as $\mathbb{Z}_p$-modules. Suppose that all the vertical arrows are injections and that all the rows are exact. If $D_1$ and $D_2$ are of finite corank and $Y$ is finite, we can decompose $Y_1 = \iota_1(D_1) \oplus D_1^\perp$ and $Y_2 = \iota_2(D_2) \oplus D_2^\perp$ up to finite error so that $I$ sends $\iota_1(D_1)$ (resp. $D_1^\perp$) to $\iota_2(D_2)$ (resp. $D_2^\perp$) and $I = I_{\iota_1(D_1)} \oplus I_{D_1^\perp}$.

Since the proof is basically the same for the two cases, we give a proof for the diagram (13.1).

Proof. We prove the Pontryagin dual version. Consider the following diagram of $\mathbb{Z}_p$-modules:

$$\begin{array}{ccc}
F_2 & \xrightarrow{\pi} & F_1 \rightarrow C \\
\text{onto} & \uparrow & \text{onto} \\
V_2 & \xrightarrow{\Pi} & V_1 \rightarrow V.
\end{array}$$

Since all the modules of this diagram are $\mathbb{Z}_p$-modules of finite type, to get the result up to finite error, we may tensor $\mathbb{Q}_p$ over $\mathbb{Z}_p$ and prove the result for the diagram of above type of $\mathbb{Q}_p$-vector spaces. Thus we assume that the above diagram is made of $\mathbb{Q}_p$-vector spaces with $V = C = 0$. Then just splitting $V_j$ as a direct sum $\tilde{F}_j \oplus F_j^{\perp}$ for a subspace $\tilde{F}_j \subset V_j$ with $\tilde{F}_j \cong F_j$ by $\rho_j$ so that the diagram

$$\begin{array}{ccc}
F_2 & \xrightarrow{\pi} & F_1 \\
\uparrow & \rho_2 & \uparrow \rho_1 \\
\tilde{F}_2 & \xrightarrow{\Pi} & \tilde{F}_1,
\end{array}$$

is commutative. Then we want to diagonalize the linear map $\Pi : V_2 \rightarrow V_1$ by modifying the splitting. Choosing basis of each space and writing $n = \dim F_2 \geq m = \dim F_1$ with $n' = \dim F_2^{\perp} \geq m' = \dim F_1^{\perp}$, in down-to-earth terms, we multiply the $(m + m') \times (n + n')$ matrix $P := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\Pi$ with $m \times n \text{ block } A$ by a $(n + n') \times (n + n')$ unipotent matrix $U := \begin{pmatrix} 1_n & X \\ 0 & 1_{n'} \end{pmatrix}$ from the right to achieve a “diagonal” form $PU = \begin{pmatrix} A & 0 \\ 0 & C' \end{pmatrix}$. In other words, we need to solve $AX = -B$. Since $\Pi|_{\tilde{F}_2}$ is onto, for the $j$-th entry $b_j \in \mathbb{Q}_p^n = \tilde{F}_1$ of $B$ ($j = 1, \ldots, n'$), we can find $x_j \in \mathbb{Q}_p^n = F_2$ such that $Ax_j = -b_j$; so, we get $AX = -B$ for $X = (x_1, \ldots, x_{n'})$. Thus this is possible by choosing basis of $V_j$ well. \hfill \Box

Proposition 13.2. Let $K$ be a number field. Assume (F) and (A) for $(\varpi, A_r)$. Let $\text{Ker}_{MW}$ be the kernel of the diagonal map: $\hat{A}_r^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \prod_{v \mid p} \hat{A}_r^{\text{ord}}(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Then we have the following exact sequence

$$0 \rightarrow \text{Ker}_{MW} \rightarrow \prod(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty]) \rightarrow \prod_K(\hat{A}_r^{\text{ord}}).$$

In particular, if $\prod(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty])$ vanishes (resp. is finite), the error term $\text{Ker}_{MW}$ vanishes (resp. is finite). Further, if $\prod(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty])$ is finite, we have

1. the map: $\prod_K(\varpi(J_\infty)) \rightarrow \prod_K(\hat{J}_\infty)$ induced from the inclusion $\varpi(J_\infty) \rightarrow J_\infty$ has finite kernel.
(2) the Tate-Shafarevich group $\Pi_K(\hat{A}_r)$ is finite.

Recall $E_{MW} \cong \text{Im}(\psi_{ord}(K) \to \hat{B}_r^\infty) \cong \text{Im}(\psi_{ord}(K)/M(k))$ for any field $k$ finite over $\mathbb{Q}_p$ or $\mathbb{Q}$, which is $p$-torsion-free isomorphic to $\mathbb{Q}_p^n \oplus \mathbb{Z}_p^m$ with $m+n = \dim_{\mathbb{Q}_p} \hat{B}_r^\infty/\hat{B}_r^\infty \to \mathbb{Z}_p$.

Proof. Recall $H_\mathbb{S}^1(?) = \bigoplus_{v \in S} H_1(K_v, ?)$ and $H_1(?) = H_1(K^\mathbb{S}/K, ?)$. Recall also from Corollary 6.4 the following exact sequence for any field $k$ finite over $\mathbb{Q}_l$ or $\mathbb{Q}$:

$$0 \to \hat{A}_r^\infty \to \hat{A}_r(k) \oplus M(k) \xrightarrow{\pi_k} J_\mathbb{S}(k).$$

Since the right factor $\hat{A}_r^\infty$ injects into $E_{MW}$ and $A_r$ is isogenous to $B_r$, we conclude $\text{Coker}(\pi_k)$ is a $p$-torsion quotient of $E_{MW}$. Since $\hat{B}_r^\infty$ is compact onto its image in $\hat{B}_r^\infty/\hat{B}_r^\infty \to \mathbb{Z}_p$, $\text{Coker}(\pi_k)$ is co-finite of corank $\leq \dim_{\mathbb{Q}_p} \hat{B}_r^\infty/\hat{B}_r^\infty \to \mathbb{Z}_p$ if $k = K$ and $k = K_v$ with $v | p$. If $k$ is $l$-adic with $l \neq p$, it vanishes as $\hat{B}_r^\infty$ is $l$-torsion-free. From this we get the following commutative diagram with bottom two exact rows:

\begin{equation}
\begin{array}{ccc}
\text{Ker}(\imath_{\star, \star}) & \xrightarrow{\imath_{\star, \star}} & \Pi(K^\mathbb{S}/K, \hat{A}_r^\infty) \\
\downarrow & \downarrow & \downarrow \\
\text{Coker}(\pi_K) & \xrightarrow{\iota} & H^1(\hat{A}_r^\infty) \\
\downarrow & \downarrow & \downarrow \\
\Pi_{v \in \mathbb{S}} \text{Coker}(\pi_{K_v}) & \xrightarrow{\iota_{\star, \star}} & H_\mathbb{S}^1(\hat{A}_r^\infty) \\
\end{array}
\end{equation}

Note that the left column is exact by [BCM, I.1.4.2 (1)]. Suppose finiteness of $\Pi(K^\mathbb{S}/K, \hat{A}_r^\infty)$; so, $\text{Ker}(\imath_{\star, \star})$ is finite as it injects into $\Pi(K^\mathbb{S}/K, \hat{A}_r^\infty)$. Then apply Lemma 3.1 to the left block of (13.3) taking $D_1 = \text{Coker}(\pi_K)$ and $D_2 = \Pi_{v \in \mathbb{S}} \text{Coker}(\pi_{K_v})$ in (13.1), we get the following commutative diagram ("\xrightarrow{\imath_{\star, \star}}" indicating having finite kernel):

\begin{equation}
\begin{array}{c}
\text{Ker}(\text{MW}) \\
\downarrow \\
\text{Coker}(\pi_K) \\
\downarrow \\
\Pi_{v \in \mathbb{S}} \text{Coker}(\pi_{K_v}) \\
\end{array}
\end{equation}

The bottom two rows and three columns are exact up to finite error, and $\text{Ker}(\imath_{\star, \star})$ is finite. Thus the top sequence is exact up to finite error, and this shows the assertion (1) by [BCM, I.1.4.2 (1)].

From the short exact sequence $A_r^\infty \to A_r \to A_r^\infty \to \mathbb{Q}_p$, we get the following commutative diagram with the bottom two exact rows:

\begin{equation}
\begin{array}{c}
\text{Ker}(\text{MW}) \\
\downarrow \\
\text{Coker}(\pi_K) \\
\downarrow \\
\Pi_{v \in \mathbb{S}} \text{Coker}(\pi_{K_v}) \\
\end{array}
\end{equation}

Thus, we have $H_\mathbb{S}^1(\hat{A}_r^\infty) = 0$. This completes the proof. 

\[\text{(13.4)}\]
The injectivity of $I_S$ and exactness of the bottom row proves the exact sequence in the proposition again by [BCM, I.1.4.2 (1)].

To prove the last assertion (2), we apply Lemma 13.1 to $D_1 = \hat{A}_r^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ and $D_2 = \prod_{v \mid p}(\hat{A}_r^{\text{ord}}(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ in the left block of (13.4), which produces the following commutative diagram with two bottom rows and the left column exact up to finite error:

\[
\begin{array}{cccccc}
\text{Ker}_{MW} & \to & \text{III}(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty]) & \to & \text{III}_K(\hat{A}_r) \\
\downarrow & & \downarrow & & \uparrow \\
0 & \to & D^+_1 & \to & H^1(\hat{A}_r^{\text{ord}}) & \to & 0 \\
\delta & & \Res[p^\infty] & & \Res & & \\
0 & \to & D^+_2 & \to & H^1_S(\hat{A}_r^{\text{ord}}) & \to & 0.
\end{array}
\]

By the finiteness of $\text{III}(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty])$, the middle column is exact up to finite error. By the snake lemma up to finite error, we find that $\text{Coker}(\iota_{\text{III,}S})$ is finite; in particular, $\text{III}_K(\hat{A}_r^{\text{ord}})$ is finite. \(\square\)

**Remark 13.3.** If $\text{rank} E(\mathbb{Q}) \geq 2$ for a rational elliptic curve $E/\mathbb{Q}$, $\text{Ker}_{MW}$ has divisible part of corank $\text{rank} E(\mathbb{Q}) - 1$ as $\text{rank}_{\mathbb{Z}_p} E(\mathbb{Q}_p) = \dim E = 1$. In particular, $\text{III}_1^1(T_p E) = \lim_{\to \mathbb{Q}} \text{III}^1_1(E[p^n]) = \lim_{\to \mathbb{Q}} \text{III}^1_1(E[p^\infty])$ contains $T_p \text{Ker}_{MW}$. Therefore $\text{rank}_{\mathbb{Z}_p} \text{III}^1_1(T_p E) \geq \text{rank} E(\mathbb{Q}) - 1$ independent of the choice of $S$.

Now we prove

**Theorem 13.4.** Let $K$ be a number field. Assume $p > 2$, (F) and (A) for $\mathfrak{w} \in \mathfrak{m}_r$ and that $\text{Ker}_{MW}$ is finite (for example, if $\dim_{\mathbb{Q}}(A_r(K) \otimes_{\mathbb{Z}} \mathbb{Q}) \leq \dim A_r$ and $(\mathfrak{w})$ is a prime ideal). Assume one of the following two conditions

(e1) $E^{\infty}(K_v)_T$ is finite for all $v \mid p$,

(e2) $A_r$ does not have split multiplicative reduction modulo $p$ at all primes $p \mid p$ of $K$.

If $\text{III}_K(\hat{A}_r^{\text{ord}})$ is finite, we have an exact sequence

(13.5)

\[
0 \to \text{III}_K(\hat{A}_r^{\text{ord}}) \to \text{III}_K(J^\infty_{\infty, T})^{\text{ord}} \xrightarrow{\varpi} \text{III}_K(J^\infty_{\infty, T})^{\text{ord}}
\]

with finite error, and $\text{III}_K(J^\text{ord}_{\infty, T})^{\vee}$ is a $\mathbb{T}$-module of finite type. If in addition $\mathbb{T}$ is an integral domain, $\text{III}_K(J^\text{ord}_{\infty, T})^{\vee}$ is a torsion $\Lambda$-module of finite type.

**Proof.** Recall the diagram (8.13) (localized at $T$):

\[
\begin{array}{cccccc}
\text{Ker}(\iota_{\text{III,}S,T}) & \to & \text{III}_K(\hat{A}_r^{\text{ord}}) & \xrightarrow{\iota_{\text{III,}S,T}} & \text{III}_K(J^\text{ord}_{\infty,T}) & \xrightarrow{\varpi_{\text{III,}S,T}} & \text{III}_K(\varpi(J^\text{ord}_{\infty,T})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E^{\infty}(K)_T & \to & H^1(\hat{A}_r^{\text{ord}}) & \xrightarrow{\iota} & H^1(J^\text{ord}_{\infty,T}) & \xrightarrow{\varpi} & H^1(\varpi(J^\text{ord}_{\infty,T})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E^{\infty}_S(K)_T & \to & H^1_S(\hat{A}_r^{\text{ord}}) & \xrightarrow{\iota_{S,T}} & H^1_S(J^\text{ord}_{\infty,T}) & \xrightarrow{\varpi_{S,T}} & H^1_S(\varpi(J^\text{ord}_{\infty,T})).
\end{array}
\]

Since $\mathbb{T}$ is a ring direct summand of $\mathfrak{h}$, tensoring $\mathbb{T}$ does not affect exactness. Thus all the columns and the bottom two rows of the above sequence are exact. By Proposition 9.7, we have $E^{\infty}(K_v) = 0$ if $v \mid p$; so, $E^{\infty}_S(K)_T = \prod_{v \mid p} E^{\infty}(K_v)_T$. Since $\prod_{v \mid p} E^{\infty}(K_v)_T$ is finite by (e1) or by Theorem 17.3 under (e2), the top row is also exact up to finite error (e.g., [BCM, Proposition I.1.4.2 (2)]).

By the exact sequence of Proposition 13.2, finiteness of $\text{III}_K(\hat{A}_r^{\text{ord}})$ and $\text{Ker}_{MW}$ implies finiteness of $\text{III}_K(\hat{A}_r^{\text{ord}}[p^\infty])$. Thus

\[
0 \to \text{III}_K(\hat{A}_r^{\text{ord}}) \to \text{III}_K(J^\text{ord}_{\infty,T}) \xrightarrow{\varpi} \text{III}_K(\varpi(J^\text{ord}_{\infty,T})) \xrightarrow{\text{Proposition 13.2 (1)}} \text{III}_K(J^\text{ord}_{\infty,T}).
\]
is exact up to finite error.

Taking Pontryagin dual, $\Pi_K(J^{\text{ord}}_{\infty})^\vee \otimes \mathbb{T}/\varpi\mathbb{T}$ is isomorphic to $\Pi_K(\hat{A}_r^{\text{ord}})^\vee$ up to finite error, and hence it is finite. Thus by Nakayama’s lemma, $\Pi_K(J^{\text{ord}}_{\infty})^\vee$ is a $\mathbb{T}$-module of finite type. Localizing at a prime factor $P((\varpi))$, we find $\Pi_K(J^{\text{ord}}_{\infty})^\vee \otimes \mathbb{T}_P/P\mathbb{T}_P = 0$; so, again by Nakayama’s lemma, we have $\Pi_K(J^{\text{ord}}_{\infty})^\vee = 0$; so, $\Pi_K(J^{\text{ord}}_{\infty})^\vee$ is a torsion module if $\mathbb{T}$ is an integral domain.

$\square$

**Remark 13.5.** For any arithmetic point $P$, we write $H_P$ for the field in $\text{End}(A_P/K) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by Hecke operators; so, $[H_P : \mathbb{Q}] = \dim A_P$. Since $A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an $H_P$ vector space, we have $\dim_{\mathbb{Q}} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = [H_P : \mathbb{Q}] \dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}$. Similarly, $\dim_{\mathbb{Q}} \ker MW \otimes_{\mathbb{Z}} \mathbb{Q}$ is a multiple of $[H_P : \mathbb{Q}]$. Thus $\dim_{\mathbb{Q}} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \dim A_P \iff \dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1 \Rightarrow \ker MW$ is finite (as the image of $A_P(K)$ in $\prod_{v \mid p} A_P(K_v)$ span at least rank 1 submodule if $\dim_{\mathbb{Q}} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} > 0$).

Under this, we get

**Corollary 13.6.** Assume $(F)$ and $(A)$ for a prime $(\varpi) = P_0$. Suppose that $\mathbb{T}$ is a unique factorization domain (e.g., if $\mathbb{T}$ is regular). Let $A$ be a number field and that $A_r$ does not have split multiplicative reduction at every place over $p$. Suppose that $\dim_{\mathbb{Q}} A_r(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \dim A_r$. If $\Pi_K(\hat{A}_r^{\text{ord}})$ is finite, $\Pi_K(J^{\text{ord}}_{\infty})^\vee$ is a torsion $\Lambda$-module of finite type, and for almost all primes $P \in \Omega_T$, $\Pi_K(\hat{A}_P)$ is finite and $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is constant $\leq 1 \iff \dim_{\mathbb{Q}} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \dim A_P$.

*Proof. *By Remark 13.5, the assumption $\dim_{\mathbb{Q}} A_r(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \dim A_r$ implies $\ker MW$ is finite. Combined with finiteness of $\Pi_K(\hat{A}_r^{\text{ord}})$, we find the finiteness of $\Pi_K(A^{\text{ord}}_r[p\infty])$ by Proposition 13.2. Then by Corollary 12.2, $\Pi(K^S/K, G_\mathbb{T})^\vee$ is a torsion $\mathbb{T}$-module of finite type.

From the exact sequence $\Pi_K(J^{\text{ord}}_{\infty})^\vee \xrightarrow{\pi_2} \Pi_K(J^{\text{ord}}_{\infty})^\vee \rightarrow \Pi_K(A^{\text{ord}}_r)^\vee = 0$ up to finite error, $\Pi_K(J^{\text{ord}}_{\infty})^\vee$ is a torsion $\mathbb{T}$-module. Thus the support of $\Pi_K(J^{\text{ord}}_{\infty})^\vee$ contains only finitely many prime divisors. Let $\lambda \in \Lambda$ be an element prime to $P_0$ which kill both $\Pi_K(J^{\text{ord}}_{\infty})^\vee$ and $\Pi(K^S/K, G_\mathbb{T})^\vee$. Write $\Omega_{\mathbb{T}[\lambda]}$ for the complement in $\Omega_{\mathbb{T}}$ of the finite set of prime factors of $(\lambda)$. By Corollary 12.2, for all $P \in \Omega_{\mathbb{T}[\lambda]}$, $\Pi_K(\hat{A}_P^{\text{ord}}[p\infty])$ is finite. Then by Proposition 13.2, $\Pi_K(\hat{A}_P^{\text{ord}})$ is finite.

Write simply $M := J^{\text{ord}}_{\infty}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p$. From the exact sequence $0 \rightarrow M \rightarrow \text{Sel}_K(J^{\text{ord}}_{\infty}) \rightarrow \Pi_K(J^{\text{ord}}_{\infty}) \rightarrow 0$ (the limit sequence of the standard one for $J^{\text{ord}}_r$), we have

$$\text{Sel}_K(J^{\text{ord}}_{\infty})^\vee \otimes_{\mathbb{Z}} \Lambda \left[ \frac{1}{\lambda} \right] = M^\vee \otimes_{\mathbb{Z}} \Lambda \left[ \frac{1}{\lambda} \right],$$

where $X[1/\lambda] = X \otimes_\Lambda \Lambda[1/\lambda]$ for a $\Lambda$-module $X$. Note that $D := \Lambda[1/\lambda]$ is a Dedekind domain; so, $M^\vee \cong D^{\text{def}}$ $\cdots M^{\text{def}}_{\text{tor}}$ for a torsion $D$-module $M^{\text{def}}_{\text{tor}}$. Requiring $\lambda$ to kill also $M^{\text{def}}_{\text{tor}}$, we may assume that $M^\vee$ is $D$-free of rank $m$.

For any arithmetic points $P \in \Omega_{\mathbb{T}}$ outside the support, we have $\Pi_K(J^{\text{ord}}_{\infty})^\vee = 0$ (the subscript “$P$” indicating localization at $P$); so, $\text{Sel}_K(J^{\text{ord}}_{\infty})^\vee = J^{\text{ord}}_{\infty}(K)_P$. Since the $\Lambda$-Selmer group is well controlled by Theorem 10.4, we have

$$\text{Sel}_K(\hat{A}_P^{\text{ord}})^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \text{Sel}_K(J^{\text{ord}}_{\infty})^\vee \otimes_{\mathbb{T}} \kappa(\mathbb{P}) = M^\vee \otimes_{\mathbb{T}} \kappa(\mathbb{P})$$

for the residue field $\kappa(\mathbb{P})$ of $P$. Since $\Pi_K(\hat{A}_P^{\text{ord}})$ is finite, from the exact sequence

$$\hat{A}_P^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \rightarrow \text{Sel}_K(\hat{A}_P^{\text{ord}}) \rightarrow \Pi_K(\hat{A}_P^{\text{ord}}),$$

we conclude

$$m = \dim_{\kappa(\mathbb{P})} \text{Sel}_K(J^{\text{ord}}_{\infty})^\vee \otimes_{\mathbb{T}} \kappa(\mathbb{P}) = \dim_{\kappa(\mathbb{P})} \hat{A}_P^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$
Let $B/Q$ be a $\mathbb{Q}$-simple abelian variety of GL(2)-type (as in the introduction). We assume that $O_B = \text{End}(B/Q) \cap H_B$ is the integer ring of its quotient field $H_B$. Then the compatible system of two dimensional Galois representations $\rho_B = \{\rho_B|_l\}$ realized on the Tate module of $B$ has its L-function $L(s, B)$ equal to $L(s, f)$ for a primitive form $f \in S_2(\Gamma_1(C))$ for the conductor $C = C_B$ of $\rho_B$ (see [KW09, Theorem 10.1]). Thus $B$ is isogenous to $A_f$ over $\mathbb{Q}$ (by a theorem of Faltings). The abelian variety $A_f$ is known to be $\mathbb{Q}$-simple as $H_{A_f}$ is generated by $\text{Tr}(\rho_B(Frob_l))$ for primes $l$ outside $Np$. Let $\pi_f$ be the automorphic representation of $\text{GL}_2(\mathbb{A})$ associated to $f$.

Fix a connected component $\text{Spec}(\mathbb{T})$ of $\text{Spec}(\mathfrak{h}_{n, \delta, \xi})$. If $(\alpha, \delta, \xi) \neq (0, 1, \omega_d)$, for $P \in \Omega_T$, the minimal (nearly ordinary) form $\Gamma := f_P$ in $\pi_f$ may not be primitive. Assume that $P$ is principal (i.e. $(A)$) and $f_P$ is on $\Gamma_r$. Then we define $A_f = J_r[a_r]^{\circ}$ as in $(A)$. If $H_{A_f} = H_{A_f} = H_B, A_f$ is $\mathbb{Q}$-simple and is isogenous to $A_f$.

**Lemma 14.1.** Let the notation be as above. If the conductor of $f$ is divisible by $Np$, the abelian variety $A_f$ is isogenous to $B$ and $H_{A_f} = H_{A_f} = H_B$. If the conductor of $f$ is equal to $N$ prime to $p$ and $f|U(p) = \varphi(p)f$, $A_f$ is isogenous to $B \otimes_{O_B} \mathbb{Q}_l[\varphi(p)]$ as abelian varieties of GL(2)-type, which is in turn isogenous to $B \times B$ just as abelian varieties.

**Proof.** Since $a_l := \text{Tr}(\rho_B(Frob_l)) \in H_{A_f}$ for all $l \nmid Np$, we have $H_B \subset H_{A_f}$. Write $\pi_f = \hat{\otimes}_\varphi \pi_\varphi$ and $\pi_P = \pi(\varphi, \beta)$ or $\sigma(\varphi, \beta)$ with $p$-adic unit $i_p(\varphi(p))$. Note that the $f$ is characterized by (see [H89, §2])

$$(14.1) \quad f \in H^0(\hat{\Gamma}_1(Np^\nu), \pi) \subset S_2(\hat{\Gamma}_1(Np^\nu)), \quad f|T(l) = a_l f \text{ for all } l \nmid Np, \quad f|U(p) = \varphi(p)f$$

and $\pi((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))f = \varphi(d)\beta(a)f$ for $a, d \in \mathbb{Z}_p^\times$, writing $T(l)$ for $U(l)$ if $l|N$. Moreover for the member $\rho_f$ of $\rho_B$ associated to the place $p_A$ induced by $i_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$, we have (cf. [H89, §2])

$$(14.2) \quad \rho_f|_{I_p} \cong \left( \begin{smallmatrix} 0 & \psi \\ \varphi & 0 \end{smallmatrix} \right) \quad \text{with } \beta = |_p^{-1}(i_p^{-1} \circ \psi) \text{ (} \psi \text{ has finite order over } I_p)$$

for the inertia subgroup $I_p \subset \text{Gal}(\mathbb{Q}_p/\mathbb{Q})$, regarding $\varphi, \psi$ as characters of $I_p$ by local class field theory. Then $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}})$ implies $\rho_B^\sigma = \rho_B \Leftrightarrow (\pi(\sigma)) \cong \pi(\overline{\sigma})$. This shows the minimal field of definition of $\pi(\sigma)$ is $H_B$ (a result of Waldspurger), and by (14.2), $H_B$ contains the values of $\varphi|_{I_p}$. Thus $H_{A_f} = H_B(\varphi)$ generated over $H_B$ by the values of $\varphi$, as the central character $\psi_P$ of $\pi$ has values in $H_B$ over $\mathbb{A}_\mathbb{Q}$ (which follows from the fact that $\text{det} \rho_B = \psi_P$). The nearly ordinary vector $f$ is characterized by the above properties (14.1) without $f|U(p) = \varphi(p)f$. Thus in this case, $\varphi^P \cong \varphi$ for $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}})$ implies $f^P = f$. In particular, $H_{A_f} = H_B$ as desired. If $f$ has conductor $N$, $f$ is $p$-stabilized (i.e., $f(z) = f(z) - \beta(p)f(pz)$), then $H_{A_f} = H_B(\varphi(p))$. Since $\varphi(p)$ satisfies $X^2 - a_pX + \varphi_p(p)p = 0$ for the $T(p)$ eigenvalue $a_p$ of $f$, we have $|H_{A_f} : H_B| \leq 2$, and $A_f$ is isogenous to $B \otimes_{O_B} \mathbb{Q}_l[\varphi(p)]$ (as an abelian variety of GL(2)-type).

If the central character $\psi_P$ of $\varphi$ is trivial, $H_B$ is totally real, and $H_B(\varphi(p))$ is totally imaginary; so, $A_f$ is isogenous to $B \times B$ if the conductor of $B$ is prime to $p$. Even if the central character is not trivial, choosing a square root $\zeta := \sqrt{\psi(p)}$, $T(p)\zeta^{-1}$ is self adjoint on $S_2(\Gamma_0(N), \psi_P)$ (e.g., [MFM, Theorem 4.5.4]), and hence $a_p\zeta^{-1}$ is totally real, but for the root $\varphi(f)\zeta^{-1}$ of $X^2 - a_p\zeta^{-1}X + p$, $Q(\varphi(f)\zeta^{-1})$ is totally imaginary as with $|a_p| \leq \sqrt{|N|}$ combined with $|\beta(p)| < |\varphi(p)|$). This shows that $H_{A_f}$ is a quadratic extension of $H_B$, and hence $A_f$ is isogenous to $B \times B$. \hfill \square

Let $A$ be another $\mathbb{Q}$-simple abelian variety of GL(2)-type. Thus $A$ is isogenous to $A_g$ for a primitive form $g \in S_2(\Gamma_1(C_A))$ of conductor $C_A$. Let $\pi_g$ be the automorphic representation of $g$, and write $g$ for the minimal nearly $p$-ordinary form in $\pi_g$. Without losing generality, we may (and do) assume that $O_A = \text{End}(A_f/Q) \cap H_A$. Note that $H_B \cong Q(f) \subset \overline{\mathbb{Q}}$ and $H_A \cong Q(g)$. Suppose $A$ is congruent to $B$ modulo $p$ with $(B[p_B] \otimes \kappa(p_B) \mathbb{F}_p)^{\ast \ast} \cong (A[p_A] \otimes \kappa(p_A) \mathbb{F}_p)^{\ast \ast}$ as $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. Here, for any ring $R$ and a prime ideal $P$ of $R$, $\kappa(p)$ is the residue field of $p$. 

**Analytic Variation of Tate-Shafarevich Groups**

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14. Preliminary lemmas
Write $O_{A}$ for the $p_{A}$-adic completion of $A$, and let $T_{p_{A}}A = \varprojlim A[p^{n}_{A}]$ (the $p_{A}$-adic Tate module of $A$). We call that $A$ is of $p_{A}$-type $(\alpha, \delta, \xi)$ if we have an exact sequence of $I_{p}$-modules

$$0 \to V(\nu^{-\delta}, \xi^{-1}) \to T_{p_{A}}A \to V(\mu_{p_{A}}^{-1}, \xi^{-1}) \to 0$$

with $V(\nu^{-\delta}, \xi^{-1}) \cong V(\epsilon, \xi^{-1}) \cong O_{p_{A}}$ as $O_{p_{A}}$-modules, where $\epsilon$ is a character of $\text{Gal}(\mathbb{Q}_{p}/\mu_{p_{A}}$ with values in $\mu_{p_{A}}$ such that $u \in \mathbb{Z}_{p}^{\times}$ (resp. $[\zeta, \mathbb{Q}_{p}]$ for $\zeta \in \mu$) acts on $V(\nu^{-\delta}, \xi^{-1})$ by $\mu_{n}, \xi^{-1}(1, \zeta)$ and on $V(\epsilon, \xi^{-1})$ by $\epsilon(u)^{\alpha}$ (resp. by $\xi^{-1}(1, \zeta)$). Here $[\mathbb{Q}_{p}]$ is the local Artin symbol. If $\xi(\zeta, \zeta') = \xi(\zeta)$ for $(\zeta, \zeta') \in \mu_{2}$ and $\alpha = 0$, this is just a $p_{A}$-ordinarity.

Choosing $g$ (resp. $f$) well in the Galois conjugacy class of $g$ (resp. $f$), we may assume that $p_{A}$ and $p_{B}$ are both induced by the fixed embedding $i_{p} : \mathcal{O} \mapsto \mathcal{O}_{p}$.

**Lemma 14.2.** Let the notation be as above. Suppose that $C_{A}/C_{B}$ is in $\mathbb{Z}_{[p]}^{\times}$ and that $B$ (resp. $A$) is of $p_{A}$-type (resp. $p_{A}$-type) $(\alpha, \delta, \xi)$. Write $C_{B} = Np_{r}$. Then there exists a connected component $\text{Spec}(\mathcal{T})$ of $\text{Spec}(\mathcal{O}_{\alpha, \delta, \xi}(N))$ such that for some primes $P, Q \in \text{Spec}(\mathcal{T})$, $f = f_{P}$ and $g = f_{Q}$.

**Proof.** Let $\mathcal{P}$ be the two dimensional Galois representation into $GL_{2}(\mathbb{F})$ realized on $B[p_{B}]$ for $\mathbb{F} = O_{B}/p_{B}$. Write $N$ for the prime-to-$p$ part of $C_{B}$ (and hence of $C_{A}$). Replacing $\mathcal{P}$ by its semi-simplification, we may assume that $\mathcal{P}$ is semi-simple. Since $(B[p_{B}]_{\otimes_{\mathbb{Q}_{p}}(\mathcal{O}_{p})})^{\otimes_{\mathbb{Q}_{p}}(\mathcal{O}_{p})} = (A[p_{B}]_{\otimes_{\mathbb{Q}_{p}}(\mathcal{O}_{p})})^{\otimes_{\mathbb{Q}_{p}}(\mathcal{O}_{p})}$, $L(s, A) = L(s, g)$ and $L(s, B) = L(s, f)$ imply $f \bmod p_{B} = g \bmod p_{B}$. Since $f_{B} := f$ is nearly $p$-ordinary with nearly ordinary character given by $[u_{\zeta}, \mathbb{Q}_{p}] : = \epsilon_{B}(u)^{\alpha} \xi^{-1}(1, \zeta)$ ($u \in \Gamma$ and $\zeta \in \mu$) for a character $B : \mathbb{Z}_{p}^{\times} \rightarrow \mu_{p_{A}}(\mathcal{O}_{p})$ and has central character $z \mapsto \epsilon_{B}(z)^{\alpha} \xi^{-1}(1, \zeta)(z)$ for $z \in \mathbb{Z}_{p}^{\times}$, $f_{B}$ generates an automorphic representation whose $p$-component $\pi_{p}$ is given by the principal series $\pi(\phi, \varphi)$ (or the Steinberg representation $\pi(\phi', \varphi')$) with $\varphi(u_{\zeta}) = \epsilon_{B}(u)^{\alpha} \xi^{-1}(1, \zeta)$ and $\varphi(u_{\zeta}) = [u_{\zeta}B(u)^{\alpha} \xi^{-1}(1, \zeta)]$. Moreover, $f_{B}(U(p)) = \varphi(p)f_{B}$ with $\text{ord}_{\mathbb{P}}(\varphi(p)) = 0$. See [H189, §2] for these facts (in particular, the $p$-component of $f_{B}$ is proportional to the nearly ordinary vector $v$ in $\pi_{p}$, fixed by the $p$-component of $\pi_{H, r}$ characterized by $\pi_{p}((\sigma_{a}^{0}f_{B})) = \phi(a)\varphi(d)v$ for $a, d \in \mathbb{Q}_{p}^{\times}$ and $U(p)v = \varphi(p)v$).

The form $f_{A} := g$ associated to $A$ has similar property whose $p$-component is given by $\pi(\phi', \varphi')$ (or the Steinberg representation $\pi(\phi', \varphi')$) with $\varphi'(u) \bmod p_{A} = \varphi \bmod p_{B}$ and $\varphi'(u) \bmod p_{A} = \varphi \bmod p_{B}$. More precisely, we have $\varphi'(u_{\zeta}) = \epsilon_{A}(u)^{\alpha} \xi^{-1}(1, \zeta)$ and $\varphi'(u_{\zeta}) = [u_{\zeta}B(u)^{\alpha} \xi^{-1}(1, \zeta)]$. Thus $g = f_{A}$ (resp. $f = f_{B}$) is lifted to a $p$-adic analytic family (of type $(\alpha, \delta, \xi)$) parameterized by an irreducible component $\text{Spec}(\mathcal{T})$ (resp. $\text{Spec}(\mathcal{J})$) of $\text{Spec}(\mathcal{O}_{\alpha, \delta, \xi}(N))$. Since $f \bmod p_{B} = g \bmod p_{B}$, the algebra homomorphisms $\lambda : \mathcal{O}_{\alpha, \delta, \xi}(N) \rightarrow \mathcal{O}_{p}$, realized as $f_{T}(n) = \lambda_{T}(n)f_{T}$ and $g_{T}(n) = \lambda_{T}(n)g_{T}$ satisfy $\lambda_{T} = \lambda_{g} \bmod m$ for a maximal ideal $m$ of $\mathcal{O}_{\alpha, \delta, \xi}(N)$. Then, $P = \text{Ker}(\lambda_{T})$ and $Q = \text{Ker}(\lambda_{g})$ belong to the connected component $\text{Spec}(\mathcal{T})$ given by $\mathcal{T} = \mathcal{O}_{\alpha, \delta, \xi}(N)_{m}$, since the local rings of $\mathcal{O}_{\alpha, \delta, \xi}(N)$ corresponds one-to-one to the maximal congruence classes modulo $\mathcal{P}$ ($\mathcal{P} := \{x \in \mathbb{Z}_{p}^{\times} : |x|_{p} < 1\}$) of Hecke eigenforms of prime-to-$p$ level $N$ and of type $(\alpha, \delta, \xi)$ just because the set of maximal ideals $\Sigma_{\mathcal{O}}$ of $\mathcal{O}_{\alpha, \delta, \xi}(N)$ is made of $\text{Ker}(\lambda_{T})$ for $\lambda \in \Sigma = \text{Hom}_{alg}(\mathcal{O}_{\alpha, \delta, \xi}(N), \mathcal{P})$. The maximal ideal $m$ is given by $\text{Ker}(\lambda_{T} \bmod \mathcal{P}) = \text{Ker}(\lambda_{g} \bmod \mathcal{P})$ for $\mathcal{P} = \{x \in \mathbb{Z}_{p}^{\times} : |x|_{p} < 1\}$. \hfill $\square$

The following result is just a combination of the above Lemma 14.2 and Theorem 5.3.

**Corollary 14.3.** Let the notation and the assumptions be as in Lemma 14.2 and Theorem 5.3 (in particular, we assume $F$). Assume that the abelian variety $B$ has conductor $N$ prime to $p$. Let $f \in S_{2}(\Gamma_{0}(N), \chi)$ be the primitive form with conductor $N$ prime to $p$ (so, $\xi = 1$) whose $L$-function gives $L(s, B)$. Write $\chi_{\lambda}^{-1}$ for the central character of the automorphic representation generated by $f$. Write $f(T(p)) = a_{f}.f$. If $p \not\mid 6D_{\chi}N\varphi(N)$ and $a_{p} \bmod p_{B} \notin \Omega_{B, p} := \{0, 1, \pm \sqrt{\chi(p)}\}$, then $\mathbb{T}$ is a regular integral domain and $f$ and $\mathcal{P}$ belongs to $\text{Spec}(\mathcal{T})$.

Again we can replace the condition: $p \not\mid 6D_{\chi}N\varphi(N)$ by $p \not\mid 2D_{\chi}N\varphi(N)$ in the case where $h_{\alpha, \delta, \xi}(N)$ is $\Lambda$-free (see Proposition 18.2 for such cases).
15. Proof of a generalized version of Theorem B including exotic towers

Let $B \mathbb{Q}$ be a $\mathbb{Q}$-simple abelian variety of GL(2)-type of conductor $N$ such that $O_B = \text{End}(B \mathbb{Q}) \cap H_B$ is the integer ring of its quotient field $H_B$. Let $B = \{\rho_B, 1\}$ be the two dimensional compatible system of Galois representations associated to $B$. Then $\rho_B$ comes from a Hecke eigenform $f = \sum_{n=1}^\infty \eta_n q^n \in S_2(\Gamma_0(N), \chi)$ by [KW09, Theorem I.10.1]; so, $L(s, B) = L(s, \rho_B) = L(s, f)$. Fix an embedding $O_B \hookrightarrow \mathbb{Q}$ and write $p_B$ for the prime ideal of $O_B$ induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p$. Then we realize the Hecke algebra $h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi])$ inside $\text{End}_\mathbb{C}(S_2(\Gamma_0(N), \chi))$ which is generated over $\mathbb{Z}[\chi]$ by all Hecke operators $T(n)$ and $U(l)$. Then this Hecke algebra is free of finite rank over $\mathbb{Z}$, and hence its reduced part (modulo the nilradical) has a well defined discriminant $D_\chi$ over $\mathbb{Z}$.

**Definition 15.1.** Let $S = S_B$ be the set of prime factors of $6D_\chi N \varphi(N)$ for the conductor $N$ of $\rho_B$, where $D_\chi$ is the discriminant of the reduced part of $h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi])$.

We could include $p = 3$ defining $S = S_B$ be the set of prime factors of $2D_\chi N \varphi(N)$ if $h_{\alpha, \delta, 1}$ is $\Lambda$-free (see remarks after Proposition 4.1 and see also Proposition 18.2).

The prime $p$ is admissible for $B$ if $B$ has good reduction modulo $p$ (so, $p \nmid N$) and $(a_p, \text{mod } p) \not\in \Omega_{B,p} := \{0, 1, \pm \sqrt{\chi(p)}\}$ (so, $B$ has potential partially $p$-ordinary reduction modulo $p$) and write $p_B$-type of $B$ as $(\alpha, \delta, 1)$. We have started with $B$ of conductor prime to $p$. This choice imposes some restriction to the $p_B$-type. In other words, we need to have the identity $\mathbf{1}_1 \cong h_2^{p_0}(\Gamma_0(Np), \chi; W)$ to have $P \in \text{Spec}(h_{\alpha, \delta, 1}(N))$ with $f = f_P$. More precisely, since $\rho_B$ is unramified at $p$, $\xi$ has to be the identity character $1$ of $\mu \times \mu$ (on the other hand, $(\alpha, \delta)$ can be freely chosen). We prove the following result more general than Theorem B including abelian varieties $A$ of $p_A$-type $(\alpha, \delta, 1)$ (not just those $p_A$-ordinary ones):

**Theorem 15.2.** Assume (F) for $(\alpha, \delta, 1)$, and let $K$ be a number field. Let $p \not\in S_B$ be a prime admissible for $B$ and $N$ be the conductor of $B$. Suppose that $B$ is isogenous to $A_{\rho_B}$ and $|\prod K(B)^\text{ord}| < \infty$ and $\dim_{\mathcal{O}_B} B(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1$. Consider the set $\mathcal{A}_{B,p}$ made up of all $\mathbb{Q}$-isogeny classes of $\mathbb{Q}$-simple abelian varieties $A_{/\mathbb{Q}}$ of $p_A$-type $(\alpha, \delta, 1)$ congruent to $B$ modulo $p$ over $\mathbb{Q}$ with prime-to-$p$ conductor $N$. Then, almost all members $A \in \mathcal{A}_{B,p}$ have finite $\prod K(A)^{p_A}$ and $\dim_{\mathcal{H}_A} A(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a constant independent of $A$ given by $0$ or $1$. If further $B^{\text{ord}} \cong A_{\rho_B}^{\text{ord}}$ with $\text{Sel}(K(B))_{p_A} = 0$ and all prime factors of $p$ in $K$ has residual degree $1$, then $\text{Sel}(K(A))_{p_A}$ is finite for all $A \in \mathcal{A}_{B,p}$ without exception.

As is well known, there are density one (partially) ordinary admissible primes in $O_B$ if $B$ does not have complex multiplication (e.g., [H13b, Section 7])

**Proof.** Suppose that $p$ is outside $S_B$, by Theorem 5.3, $T$ is a regular integral domain $\mathbb{I}$. For any $P \in \mathbb{I}$, we have $P = (w) \in \mathbb{I}$ and $(w, A, P)$ satisfies (A).

Since $B[p_\infty^\infty]$ is an ordinary Barsotti–Tate group by our assumption, $A[p_\infty^\infty]$ is potentially ordinary by the congruence modulo $p$ between $A$ and $B$. Here we say $A[p_\infty^\infty]$ “potentially ordinary” if $H_0(k, A[p_\infty^\infty] \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ is non-trivial $p$-divisible rank and $A[p_\infty^\infty] \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ extends to a Barsotti–Tate group with non-trivial étale quotient over the integer ring of a finite extension $k$ of $\mathbb{Q}_p$. Choosing the embedding $O_A \hookrightarrow \overline{\mathbb{Q}}$ well, we may assume that $p_A$ is induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}$. Then by Lemma 14.2, $A$ is isogenous to a modular abelian variety $A_P$ for $P \in \Omega_\tau$ of a connected component $\text{Spec}(\mathbb{T})$ of $\text{Spec}(h_{\alpha, \delta, 1}(N))$ for the big $p$-adic Hecke algebra $h_{\alpha, \delta, 1}(N)$. Since $B$ is of GL(2)-type, we have $B \sim A_{\rho_B}$ (an isogeny) for $P_0 \in \Omega_\tau$ with $P_0 = (w_0)$. Thus we conclude, up to isogeny,

$$\mathcal{A}_{B,p} = \{A_{\mathbb{Q}}| Q \in \Omega_\tau\}$$

by the theorem of Khare–Wintenberger [KW09, Theorem I.10.1] (combined with the proof of the Tate conjecture for abelian varieties by Faltings). Since $O_A$ is the integer ring of $H_A$, we can factor $O_{A,p} = O_A \otimes_{\mathbb{Z}} \mathbb{Z}_p$ into the product $O_{A,p} = O_{A, p}^{\text{ord}} \oplus O_{A, p}^{\text{red}}$ so that for the idempotent $e$ of the factor $O_{A, p}^{\text{ord}}$, $e[A[p_\infty^\infty]]$ is the maximal $p$-ordinary Barsotti–Tate group which becomes étale and multiplicative after étale extension. Since $O_{A,p}$ acts on $\hat{A}$, we can define $\hat{A}^{\text{ord}} := e(\hat{A})$. Since $A$ is isogenous to $A_P$, $\hat{A}^{\text{ord}}$ is isogenous to $\hat{A}_p^{\text{ord}}$; so, $\prod K(A)^{\text{ord}}$ is isogenous to $\prod K(\hat{A}_p^{\text{ord}})$. Then if $\prod K(\hat{A}_p^{\text{ord}})$ is finite, $\prod K(\hat{A}_p^{\text{ord}})$ is finite as it is isogenous to the finite $\prod K(\hat{A}_p^{\text{ord}})$. Since $\prod K(\hat{A})^{\text{ord}}$, $\prod K(\hat{A}_p^{\text{ord}})$,
by Corollary 13.6 (taking $A_r$ there to be $A_{B,p}$), we conclude finiteness of $\text{III}_K(\hat{A})_{p,\epsilon}$ for almost all $A \in A_{B,p} \cong \Omega_T$. The assertion for the Mordell–Weil rank also follows from Corollary 13.6.

Suppose $\text{Sel}_K(\hat{B})^{\text{ord}} = 0$ and $K_v$ for all $v|p$ has residue field $\mathbb{F}_p$. Then $|\varphi(\text{Frob}_v)| - 1 = |a_p - 1| = 1$ as $p \not\in \Omega_{B,p}$. Thus by Schneider [Sc83, Proposition 2, Lemma 3] (see also [Sc82, Proposition 2]), we have, for all $v|p$,

$$|H^1(K_v[\mu_{p^\infty}]/K_v, \hat{A}_r^{\text{ord}}(K_v[\mu_{p^\infty}]))| = |\hat{A}_r^{\text{ord}}(\mathbb{F}_p)|^2 = |\hat{A}_r(\mathbb{F}_p)|^2 = 1.$$  

(15.1)

Strictly speaking, Schneider assumes in [Sc83, §7] that $A_r$ has ordinary good reduction, but his argument works well without change replacing $(A_r(p) := A_r[p^\infty], A_r)$ there by $(A_r[p^\infty]^{\text{ord}}, \hat{A}_r^{\text{ord}})$. Indeed, he later takes care of the general case of formal Lie groups in [Sc87, Theorem 1] (including the case of the ordinary part of the formal group of $A_r$). So, $E^\infty(K_v)^\hat{\epsilon} = E_{\text{Sel}}(K_v) = 0$ for all $v|p$ (see Theorem 17.3 for more details of this fact). Then by the same argument as above, using Corollary 10.5 (2) in place of Corollary 13.6, we conclude $\text{Sel}_K(A_{\text{P},A})$ is finite for all $P \in \Omega_\epsilon$.  

**Remark 15.3.** If we start with an elliptic curve $E$ as in Theorem B, by the modularity, we find a modular factor $B \subset J_1(Np^r)$ isogenous to $E$. Choose $(\alpha, \delta, 1) = (0, 1, 1)$. The finiteness of $\text{III}_K(E)_p$ implies the finiteness of $\text{III}_K(B)$; so, the above theorem implies the statements of Theorem B. The rational elliptic curves listed in Corollary C give examples for such curves with Mordell–Weil Q-rank 0 and finite Tate–Shafarevich group, and the elliptic curve factor of $J_0(37)$ with root number $-1$ give a Mordell–Weil Q-rank 1 example with finite $\text{III}_Q(E)$.

Here is a conjecture:

**Conjecture 15.4.** Suppose $(\alpha, \delta) = (1, 1)$ and $\xi(a, a) = 1$ for all $a \in \mathbb{Z}_p^\times$. Let $\text{Spec}(\mathbb{I})$ be a primitive irreducible component of $\text{Spec}(h)$ whose root number is $\epsilon := \pm 1$ over a totally real number field $K$. Then for almost all $P \in \Omega_\epsilon$, we have $\dim_{H_P} A_{\text{P}(K)} \otimes \mathbb{Q} = \frac{1}{2\epsilon}.$

16. $p$-LOCAL COHOMOLOGY OF FORMAL LIE GROUPS

We gather some technical Galois cohomology computation for proving vanishing of the error terms when $l = p$ in Theorem 17.3. Just for finiteness of the error term, as will be explained in the proof of the theorem, it follows from the computation of the universal norm by P. Schneider in [Sc83, Proposition 2 and Lemma 3, §7] and [Sc87, Theorem 1], and therefore, perhaps, for the first reading, the reader may skip this section. Here we reproduce some of the results of Schneider in our setting in an elementary manner (without using flat cohomology) and prove some possibly new vanishing results.

Suppose $l = p$, and let $K$ be a discretely valued algebraic extension of $\mathbb{Q}_p$. Write $K_s = K[\mu_{p^r}]$ and $X^{ur}$ for the maximal unramified extension of $X = K, K_s$.

If $K$ is unramified over $\mathbb{Q}_p$, by [I60, Theorem 1] and [I73, 12.2]), we have

$$\hat{G}_m(K_s) \cong \mu_{p^r} \oplus \mathbb{Z}_p \oplus a_s \oplus \mathbb{Z}_p[\text{Gal}(K_s/K)]^{[K_s:K_p]^{-1}}$$  

as $\text{Gal}(K_s/K)$-module, where $a_s$ is the augmentation ideal of the group algebra $\mathbb{Z}_p[\Gamma/\Gamma^{p^{s-r}}]$.

**Lemma 16.1.** Assume that $K$ is a finite unramified extension over $\mathbb{Q}_p$. Then we have

$$H^1(K_s/K_r, \hat{G}_m(W_s)) \cong \mathbb{Z}/p^{s-r}\mathbb{Z}$$  

for all $s \geq r$.

**Proof.** The factor $\mathbb{Z}_p[\text{Gal}(K_s/K)]^{[K_s:K_p]^{-1}}$ of (16.1) is given by the kernel of norm map $\hat{G}_m(W_s) \rightarrow \hat{G}_m(\mathbb{Z}_p[\mu_{p^r}])$ modulo torsion (see [I60]). Note that

$$H^1(K_s/K_r, \hat{G}_m(W_p)) = \text{Hom}(\text{Gal}(K_s/K_r), \mathbb{Z}_p) = 0.$$

We have an exact sequence $0 \rightarrow a_s \rightarrow \mathbb{Z}_p[\text{Gal}(K_s/K)] \xrightarrow{\text{deg}} \mathbb{Z}_p \rightarrow 0$ given by $\text{deg}(\sum_{\sigma} a_s \sigma) = \sum_{\sigma} a_s \sigma$. Note that $\text{Coker}(H^0(K_s/K_r, \mathbb{Z}_p[\text{Gal}(K_s/K)])) \xrightarrow{\text{deg}} \mathbb{Z}_p = \mathbb{Z}_p/[K_s : K_r] \mathbb{Z}_p = \mathbb{Z}_p/p^{s-r}\mathbb{Z}_p$. We have
the corresponding cohomology exact sequence:

\[ 0 \to \mathbb{Z}_p/p^{s-r}\mathbb{Z}_p \to H^1(\text{Gal}(K_s/K_r), a_s) \to H^1(\text{Gal}(K_s/K_r), \mathbb{Z}_p[\Gamma/\Gamma_s]) = 0 \]

as \( \mathbb{Z}_p[\text{Gal}(K_s/K)] \) is a \( \mathbb{Z}_p[\text{Gal}(K_s/K_r)] \)-free module. Thus we have

\[ (16.2) \quad H^1(\text{Gal}(K_s/K_r), a_s) \cong \mathbb{Z}/p^{s-r}\mathbb{Z}. \]

We also have, for the generator \( \sigma \) of \( \text{Gal}(K_s/K_r) \),

\[ (16.3) \quad H^1(\text{Gal}(K_s/K_r), \mu_{p^r}) \cong \text{Ker}(N_{K_s/K_r} : \mu_{p^r} \to \mu_{p^r})/\text{Im}(\sigma - 1 : \mu_{p^r} \to \mu_{p^r}) \cong 0, \]

as the norm map \( N_{K_s/K_r} : \mu_{p^r} \to \mu_{p^r} \) is onto. From (16.1), this shows the result. \( \square \)

Let \( A \) be an abelian variety defined over \( K \). We suppose that \( \text{End}(A/K) \) contains a reduced commutative algebra \( O_A \). Assume that

(A1) \( A/K \) has semi-stable reduction over the integer ring \( W \) of \( K_r \);

(A2) The formal Lie group of the Néron model of \( A \) over \( W \) has a maximal multiplicative factor \( \mathbb{A} \) (see [Sc87, §1] for the maximal multiplicative factor);

(A3) Writing \( O_A \) for the \( p \)-adic closure of the image of \( O_A \) in \( \text{End}(A/W_r) \), we have \( A \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbb{A} \) over \( W_{ur}^{W} \) as formal \( O_A \)-modules, where \( \mathbb{A} \) is the \( O_A \)-lattice in \( O_A \otimes_{\mathbb{Z}_p} Q_p \) (i.e., \( \mathbb{A} \otimes_{\mathbb{Z}_p} Q_p \cong O_A \otimes_{\mathbb{Z}_p} Q_p \)).

We have the formal logarithm \( \text{Log} : A \to \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbb{A} \) defined over \( Q_p[\mu_{p^r}] \) and hence defined over \( K_r \). At the same time, we have the \( p \)-adic logarithm \( \log_p \circ \text{id} : \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbb{A} \to \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbb{A} \) defined over \( Q_p \). Since \( A_{/W_{ur}} \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_A_{/W_{ur}} \) for \( s \geq r \), the two logarithms are the same over \( W_{ur}^{W} \); in particular, the radius of convergence in \( \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbb{A}(W_{ur}^{W}) \subset W_{ur}^{W} \otimes_{\mathbb{Z}_p} \mathbb{A} \) of the exponential maps \( \exp_A := \text{Log}^{-1} \) and \( \exp_p := (\log_p \circ \text{id})^{-1} \) are equal; so, by Galois equivariance of the exponential map, we have

\[ A(W_s)/A[p^\infty](W_s) \cong \text{Im}(\text{Log} : A(W_s) \to W_s \otimes_{\mathbb{Z}_p} \mathbb{A}) \]

\[ = \text{Im}(\log \circ \text{id} : \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbb{A}/\mu_{p^\infty}(W_s) \otimes_{\mathbb{Z}_p} \mathbb{A}) \to W_s \otimes_{\mathbb{Z}_p} \mathbb{A}) \]

\[ \cong \widehat{\mathbb{G}}_m(W_s) \otimes_{\mathbb{Z}_p} \mathbb{A}/\mu_{p^\infty}(W_s) \otimes_{\mathbb{Z}_p} \mathbb{A} \cong (\mathbb{Z}_p \otimes a_s \otimes \mathbb{Z}_p[\text{Gal}(K_s/K)]^{[K : Q_p]^{-1}}) \otimes_{\mathbb{Z}_p} \mathbb{A} \]

as \( \text{Gal}(K_s/K_r) \)-modules. In particular, we have

**Corollary 16.2.** Assume that \( K \) is a finite unramified extension over \( Q_p \). Then we have

\[ H^1(K_s/K_r, A(W_s)/A[p^\infty](W_s)) \cong (\mathbb{Z}/p^{s-r}\mathbb{Z}) \otimes_{\mathbb{Z}_p} \mathbb{A} = \mathbb{A}/p^{s-r}\mathbb{A} \]

for all \( s \geq r \).

We now study the \( \text{Gal}(K_s/K_r) \)-module structure and the cohomology of \( A(W_s) \). The Barsotti-Tate group \( \widehat{A}^{\text{ord}}[p^\infty]/q_s \) has a filtration \( A[p^\infty] \hookrightarrow \widehat{A}^{\text{ord}}[p^\infty] \hookrightarrow \widehat{A}^{\text{ord}}[p^\infty]^{\text{per}} \), where \( \widehat{A}^{\text{ord}}[p^\infty]^{\text{per}} \) becomes unramified over \( Q_p[\mu_{p^r}] \). On \( T_{p^s}A[p^\infty] \), \( \text{Gal}(K_r/K_r) \cong \text{Gal}(Q_p[\mu_{p^r}]/Q_p) \) acts by a character \( \psi \) with values in \( O_A^{\text{et}} \). Identifying \( \psi \) with the corresponding character of \( \text{Gal}(Q_p[\mu_{p^r}]/Q_p) \), we twist the Galois action on the group functor \( R \to A(R) \) so that

\[ (16.4) \quad \sigma \cdot x := \psi^{-1}(\sigma[\psi[\mu_{p^r}]]))(\sigma(x)) \]

for \( Q_p[\mu_{p^r}] \)-algebras \( R \), where \( \sigma \in \text{Aut}(R_{/\mathbb{Q}_p}) \). Since \( \psi(\sigma)^{-1} \in \text{Aut}(A/R_{/\mathbb{Q}_p}) \) gives a descent datum (see [GME, §1.11.3, (DS2)]), we can twist \( A \) by this cocycle, and get another abelian variety \( A_{/\mathbb{Q}_p} \) (see [Mi72, (a)]).

Similarly, on \( T_{p^s}A[p^\infty]^{\text{per}} \), \( \text{Gal}(K_r/K_r) \cong \text{Gal}(Q_p[\mu_{p^r}]/Q_p) \) acts by a character \( \varphi \) with values in \( O_A^{\text{et}} \). Identifying \( \varphi \) with the corresponding character of \( \text{Gal}(Q_p[\mu_{p^r}]/Q_p) \), via the new action \( \sigma \cdot x := \varphi^{-1}(\sigma[\varphi[\mu_{p^r}]]))\)(\sigma(x)) \), we get another abelian variety \( A_{\text{et}/\mathbb{Q}_p} \). Thus the Galois action on \( A_{\text{et}/\mathbb{Q}_p}[p^\infty]^{\text{per}} \) is unramified over \( Q_p \).

For a scheme \( X/S \), and finite flat morphism \( S' \to S \), we write \( \text{Res}_{S'/S}X \) for the Weil restriction of scalars; so, \( \text{Res}_{S'/S}X \) is a scheme over \( S \) such that \( \text{Res}_{S'/S}X(T) = X(S' \times_S T) \) for all \( S \)-schemes.
Thus $K$ acts by $A_{\mu} = A_{\mu} \circ \ord$ isomorphic to $(\text{Res}_1)$. If $A \circ \ord$ is an endomorphism of schemes (by Yoneda’s lemma), $A \rightarrow \text{Res}_{K_r/K} A$. Since $A$ and $\text{Res}_{K_r/K} A$ are projective, we find that $A \rightarrow \text{Res}_{K_r/K} A$ is a closed immersion. In the same way, we have another closed immersion $A_{\mu} \rightarrow \text{Res}_{K_r/K} A_{\mu} \equiv \text{Res}_{K_r/K} A$.

Since $K_r \otimes K_r \equiv \prod_A \text{Gal}(K_r/K) \otimes K_r$ by sending $x \otimes y$ to $(x \sigma)(y \sigma)$, for any variety $X$ defined over $K_r$, we have $\text{Res}_{K_r/K} X \equiv \prod_\sigma X^\sigma$, where $X^\sigma = X \otimes \text{Gal}(K_r/K)$. Thus $\tau \in \text{Gal}(K_r/K)$ acts on $\text{Res}_{K_r/K} X_{\text{ur}}$ by a permutation: $x = (x_\sigma) \mapsto \tau \cdot x = (x_{\tau \sigma})$, and $\text{Gal}(K_r/K) \hookrightarrow \text{Aut}(\text{Res}_{K_r/K} X)$. Thus $O_A[\text{Gal}(K_r/K)] \subset \text{End}(\text{Res}_{K_r/K} A_{\mu})$ by embedding $\text{Gal}(K_r/K)$ in this way. For $x = (x_\sigma) \in \text{Res}_{K_r/K} X_{\text{ur}}(\overline{\mathbb{Q}}_p)$, we have $x^\tau = \tau \cdot (x_\sigma^\tau)$. Then the image of $A$ in $\text{Res}_{K_r/K} A_{\mu}$ is given by $1_{\text{ur}}(\text{Res}_{K_r/K} A_{\mu})$, where $1_{\text{ur}} = [K_r : K]^{-1} \sum_\sigma \psi^{-1}(\sigma) \sigma \in O_A[\text{Gal}(K_r/K)]$. Since $\tau \in \text{Gal}(\overline{\mathbb{Q}}_p/K)$ acts on $\text{Res}_{K_r/K} A_{\mu}$ by $(x_\sigma) \mapsto (x_{\tau \sigma})$, writing the Galois action on $A_{\mu}$ as $x \mapsto x^\sigma$, the action of $\sigma$ in $\text{Gal}(\overline{\mathbb{Q}}_p/K)$ on $x \in \text{Res}_{K_r/K} A_{\mu}$ is $x \mapsto \varphi(\sigma) \cdot (x^\sigma)$. In particular, $A_{\text{et}}[\mathfrak{p}_{\text{ur}}] \otimes \text{ord}(K_{\text{ur}})$ is unramified, and the action of $\text{Gal}(K_{\text{ur}}/K)$ on $A_{\mu} \otimes \mathfrak{p}_{\text{ur}}(K_{\text{ur}})$ is via the $p$-adic cyclotomic character. Here $A_{\mu} = \prod_\mu \text{Gal}(K_r/K)$ is the formal Lie group whose Barsotti–Tate group is the potentially connected part of the Barsotti–Tate group of $A_{\mu}$. This formal Lie group descends to $W$ and is isomorphic to $\widehat{\mathbb{G}}_m \otimes \mathfrak{p}$ over $W^{ur}$ for the integer ring $W$ of $K$.

For an abelian variety $X_{Q_p}(R)$ for the quotient field $Q(R)$ of a Dedekind domain $R$, write $X_{Q_p}(R)$ for the Néron model of $X$ over $R$. By (Res4), $\text{Res}_{W_r/W_r} A$ is the Néron model of $\text{Res}_{K_r/K} A$. Since we have a closed immersion $\mu_{p_r} \otimes \mathfrak{p}_{Q_p} \otimes \mathfrak{p} \equiv \mathfrak{A} \hookrightarrow A_{Q_p}$, by (Res5), we have a closed immersions $\text{Res}_{W_p^{ur}/W^{ur}}(\mu_{p_r} \otimes \mathfrak{p}_{Q_p} \otimes \mathfrak{p} \equiv \mathfrak{A}) \hookrightarrow \text{Res}_{W_r/W_r} A \equiv \text{Res}_{W_r/W_r} A_{\mu}$ and $\mu_{p_r} \otimes \mathfrak{p}_{Q_p} \otimes \mathfrak{p} \equiv \mathfrak{A} \hookrightarrow \text{Res}_{W_{p_r}/W_{p_r}}(\mu_{p_r} \otimes \mathfrak{p}_{Q_p} \otimes \mathfrak{p} \equiv \mathfrak{A})$. The composite factors through $A_{\mu} \hookrightarrow \text{Res}_{K_r/K} A_{\mu}$, which is a closed immersion. This shows that $\mu_{p_r} \otimes \mathfrak{p}_{Q_p} \otimes \mathfrak{p} \equiv \mathfrak{A} \hookrightarrow A_{Q_p}$ is a closed immersion. Thus, we have $A_{\mu} \otimes \mathfrak{A} \equiv \widehat{\mathbb{G}}_m \otimes \mathfrak{p} \equiv \mathfrak{A}$ over $W^{ur}$. Therefore, as $\text{Gal}(K_s/K)$ modules for all $s \geq 0$, for $d = [K : Q_p]$, $A_{\mu}^{\mathfrak{o},[p_{\text{ur}}]}(W_s) \equiv (Z_p + a_s + \mathfrak{p}[\text{Gal}(K_s/K)]^{d-1}) \otimes \mathfrak{A}$.

Thus the Cartier dual of $A_{\mu}^{\mathfrak{o},[p_{\text{ur}}]}(K_{\text{ur}})$ is étale; so, the Cartier dual extends a unique étale Barsotti–Tate group over $W^{ur}$. Taking Cartier dual back, $A_{\mu}^{\mathfrak{o},[p_{\text{ur}}]}$ extends to a Barsotti–Tate group over $W^{ur}$. By étale descent, $A_{\mu}^{\mathfrak{o},[p_{\text{ur}}]}$ extends to a Barsotti–Tate group over $W$. Thus the formal Lie group $A_{\mu}^{\mathfrak{o},[p_{\text{ur}}]}$ over $W$ descends to a formal Lie group $A'$ defined $W$ so that $A'[p_{\text{ur}}]/K$ is isomorphic to $A_{\mu}^{\mathfrak{o},[p_{\text{ur}}]}(K_{\text{ur}})$. By (16.4), for all $s \geq r$, we get, as $\text{Gal}(K_r/K)$-modules,

$$
\frac{A_r(W_s)}{A_{r}^{\mathfrak{o},[p_{\text{ur}}]}(W_s)} \equiv (Z_p + a_s + \mathfrak{p}[\text{Gal}(K_r/K)]^{r-1}) \otimes \mathfrak{A}(\psi),
$$

$$
A_{r}^{\mathfrak{o},[p_{\text{ur}}]}(W_s) \equiv A_{r}^{\mathfrak{o},[p_{\text{ur}}]}(W_s) \otimes \mathfrak{A} \mathfrak{O}_A(\psi)
$$

with $A_{r}^{\mathfrak{o},[p_{\text{ur}}]} \equiv \mu_{p_r} \otimes \mathfrak{p} \equiv \mathfrak{A}$ over $W^{ur}$, where $\mathfrak{A}(\psi)$ is an $O_A$-module of rank 1 on which $\text{Gal}(K_r/K)$ acts by $\psi$. From this, we get
Proposition 16.3. Assume that \( K \) is a finite unmramified extension over \( \mathbb{Q}_p \), and let \( \sigma \) be a generator of \( \text{Gal}(K_s/K) \) for \( s \geq s \). Then we have
\[
H^1(K_s/K, \mathcal{A}(\psi)/\mathcal{A}[p^\infty]) = \mathfrak{A} \end{equation}
for all \( s \geq r \geq 1 \), where \( (\psi(\sigma) - 1, p^s - 1) \) is the ideal of \( \mathcal{O}_A \) generated by \( p^s - 1 \) and \( \psi(\sigma) - 1 \).

Proof. We study \( H^1(K_s/K, X) \) for each of direct summand of \( \mathcal{A}(W_s)/\mathcal{A}[p^\infty](W_s) \) in (16.5). We first deal with \( \mathfrak{A}(\psi) \). We simply write \( O \) for \( \mathcal{O}_A \). Note that
\[
H^1(K_s/K, \mathfrak{A}(\psi)) = \begin{cases} \text{Hom}(\text{Gal}(K_s/K), \mathfrak{A}) = 0 & \text{if } \psi = 1, \\ \mathfrak{A}/(\psi(\sigma) - 1)\mathfrak{A} & \text{if } \psi \neq 1. \end{cases}
\]
We have an exact sequence
\[
0 \to a_s \otimes_{\mathbb{Z}_p} \mathfrak{A}(\psi) \to \mathfrak{A}(\mathfrak{A})(\text{Gal}(K_s/K)) \to \mathfrak{A}(\psi) \to 0,
\]
where \( \mathfrak{A}(\psi)[\text{Gal}(K_s/K)] := \mathfrak{A}(\psi) \otimes_{\mathcal{O}_A} \text{Gal}(K_s/K) \) with \( \deg(\sum a_{\sigma} \sigma) = \sum a_{\sigma} (a_{\sigma} \in \mathfrak{A}(\psi)) \). If \( \psi \neq 1 \), we have the corresponding cohomology exact sequence:
\[
0 \to H^1(K_s/K, a_s \otimes_{\mathbb{Z}_p} \mathfrak{A}(\psi)) \to H^1(K_s/K, \mathfrak{A}(\psi)[\text{Gal}(K_s/K)]) = 0
\]
as \( \mathfrak{A}(\psi)[\text{Gal}(K_s/K)] \cong \mathfrak{A}[\text{Gal}(K_s/K)] \) by sending \( \sigma \psi(\sigma) \to \sigma \). If \( \psi = 1 \), we have the following exact sequence:
\[
0 \to \mathfrak{A}/[\mathfrak{A}[\text{Gal}(K_s/K)] \mathfrak{A}] \to H^1(K_s/K, \mathfrak{A}(\psi)[\text{Gal}(K_s/K)]) = 0
\]
Thus we have
\[
H^1(K_s/K, a_s \otimes_{\mathbb{Z}_p} \mathfrak{A}(\psi)) \cong \mathfrak{A}/(p^{s-1})\mathfrak{A} \end{equation}
if \( \psi \neq 1 \),
\[
\mathfrak{A}/p^{s-1}\mathfrak{A} \end{equation}
if \( \psi = 1 \).
Since \( \mathfrak{A}(\psi)[\text{Gal}(K_s/K)] \) is cohomologically trivial and \( p^{s-1}\mathfrak{A} \subset (\psi(\sigma) - 1)\mathfrak{A} \) if \( \psi \neq 1 \) and \( s > 1 \), we get the desired result. \( \square \)

Lemma 16.4. Assume \( p > 2 \). Let \( a \in \mathcal{O}_A \) be given by the action of Frob. Then we have, for \( s \geq r \),
\[
H^1(\text{Gal}(K_s/K), \mathcal{A}[p^r](W_s)) \cong \mathfrak{A}/(p^{s-1}, \nu_p \psi(\sigma) - 1)\mathfrak{A} \end{equation}[a - 1]
which is finite and bounded independent of \( s \geq r \).

Proof. The Frobenius element Frob acts on \( \mathcal{A}[p^r] \) via multiplication by \( a \). Note that, for \( s \geq r \)
\[
\mathcal{A}[p^r](W_s) = (\mu_{p^r}(W_s^{ur}) \otimes_{\mathbb{Z}_p} \mathfrak{A}(\psi))[a - 1]
\]
\[
= \{ x \in \mu_{p^r}(W_s^{ur}) \otimes_{\mathbb{Z}_p} \mathfrak{A}(\psi)(a - 1)x = 0 \} \cong \mathfrak{A}(\psi)/p^{s-1}\mathfrak{A}(\psi)[a - 1]
\]
as \( \text{Gal}(K_s/K) \)-modules. Thus we get
\[
H^1(\text{Gal}(K_s/K), \mathcal{A}[p^r](W_s)) \cong \mathfrak{A}/(p^{s-1}, \nu_p \psi(\sigma) - 1)\mathfrak{A} \end{equation}[a - 1]
as desired. \( \square \)

17. Finiteness of the \( p \)-local error term

We continue to use notation introduced in the previous Section 16. We assume \( (F) \) and \( p > 2 \). Here \( K/\mathbb{Q}_p \) is a finite extension with \( p \)-adic integer ring \( W \), but when we use the results in Section 16, we assume that \( K \) is an unramified over \( \mathbb{Q}_p \). Put \( K_s = K[\mu_{p^r}] \) with integer ring \( W_s \).

We studied the \( \Lambda \)-BT group \( G_{1,0,\omega} \) associated to the tower \( \{ X_1(Np^r) \} \) in [H14a, §5], which is defined over \( \mathbb{Z}_p[\mu_{p^\infty}] \). Here \( \omega(a, d) = \omega(d) \). For the general tower \( \{ X_r \} \) determined by the fixed data \( (a, \delta, \xi) \), \( J_r \) is a factor of \( \text{Res}_{F_1/\mathbb{Q}_p}(Np^r) \) again over \( \mathbb{Q}[\mu_{p^r}] \) if \( r \geq \epsilon \), since \( F_1 \subset \mathbb{Q}[\mu_{p^r}] \).

Thus taking the tower of regular model \( X_r/\mathbb{Z}_p[\mu_{p^r}] \) made out of the regular model \( X_1(Np^r)/\mathbb{Z}_p[\mu_{p^r}] \) (via the corresponding Weil restriction of scalars) and considering \( J_r/\mathbb{Z}_p[\mu_{p^r}] := \text{Pic}(X_r/\mathbb{Z}_p[\mu_{p^r}]) \) over \( \mathbb{Z}_{(p)}[\mu_{p^\infty}] \), \( G = G_{a,\delta,\xi/\mathbb{Z}_p}[\mu_{p^\infty}] := J_r^\infty[p^\infty]/\mathbb{Z}_p[\mu_{p^r}] \) is a \( \Lambda \)-direct factor of \( G_{1,0,\omega}^F \) over \( \mathbb{Z}_p[\mu_{p^r}] \).

Thus \( G_{/\mathbb{Z}_p}[\mu_{p^r}] \) is a \( \Lambda \)-BT group in the sense of [H14a, §3] (replacing (CT) by (ct) in [H14a,
Remark 5.5 [if $\xi = 1$]. Though it is assumed that $p > 3$ in [H14a, §3], the result there is valid for $p = 2, 3$. This is because the ordinary or nearly ordinary part is trivial if $N p \leq 3$ (and the assumption $p > 3$ is imposed to have $N p \geq 4$ for the representability of the elliptic moduli problem).

We take its connected component $G_s/\mathbb{Z}_p[p_{\infty}]$ and put $G_s/\mathbb{Z}_p[p_{\infty}] = G_s^p[\gamma^p - 1]$ which is a connected Barsotti–Tate group defined over $\mathbb{Z}_p[p_{\infty}]$. Write $G_s/\mathbb{Z}_p[p_{\infty}]$ for the formal Lie group associated to the connected Barsotti–Tate group $G^\circ_s/\mathbb{Z}_p[p_{\infty}]$ [GME, 1.13.5].

We put $G_s/\mathbb{Z}_p[p_{\infty}] = \lim_{s \to s} G_s$, where the projection $G_{s+1} \to G_s$ is induced by the natural trace map $\pi^s_{s'} : G_{s'} \to G_s$ for $s' > s$. We study Case A: $\text{Coker}(J_s^{\text{ord}} \xrightarrow{\pi_s} \varpi(J_s^{\text{ord}}))$ and Case B: $\text{Coker}(J_s^{\text{ord}} \xrightarrow{\pi_s} \varpi(J_s^{\text{ord}}))$ for $\pi_A = \varpi$ and $\pi_B$ is the natural projection. Identify $\varpi(J_s^{\text{ord}})$ with $\varpi(J_s^{\text{ord}})$ by $\pi^s_{s'}$ and $\varpi(J_s^{\text{ord}})$ by $\pi^s_{s'}$. Let $B = B$s (resp. $A = A_s$) be the connected formal Lie group over $\mathbb{Z}/[p_{\infty}]$ associated to (the connected component of) the Barsotti–Tate group of $\hat{G}_s^{\text{ord}}[p_{\infty}]$ in Case A and that of $\varpi(J_s^{\text{ord}})[p_{\infty}]$ (resp. $A_s[p_{\infty}]$ in Case A and $\varpi(J_s^{\text{ord}})[p_{\infty}]$ in Case B).

We first study $\text{Coker}(G_s(W) \to \varpi(G_s(W)))$ in Case A and $\text{Coker}(G_s(W) \to B_s(W))$ in Case B. We have a commutative diagram of Barsotti–Tate groups over $\mathbb{Z}_p[p_{\infty}]$ with exact rows $\text{[H14a, §5]}$:

\[ 0 \to A_s[p_{\infty}] \to G_s^\circ \to G_s^\circ/A_s[p_{\infty}] \to 0. \]

This produces to the following commutative diagram of formal Lie groups over $\mathbb{Z}_p[p_{\infty}]$ with exact rows for $\pi^s = (A, B)$:

\[
\begin{array}{ccccccccc}
A_s & \longrightarrow & G_s & \longrightarrow & G_s/A_s & \longrightarrow & \pi^s(G_s) \\
\downarrow \pi^s_{s'} & & \downarrow & & \downarrow & & \\
A_s' & \longrightarrow & G_s' & \longrightarrow & G_s'/A_s' & \longrightarrow & \pi^s(G_s').
\end{array}
\]

Since $G_s^\circ/A_s[p_{\infty}]$ is a Barsotti–Tate group over $W_s$ by [H14a, Theorem 5.4], $G_s/A_s$ is a smooth formal group over $W_s$ (e.g., [Se87, Lemma 1]). Thus $G_s \cong (G_s/A_s) \times_{W_s} A_s$ as formal schemes (but not necessarily as formal groups). Anyway, this shows that $G_s(W_s) \to \pi^s(G_s(W_s))$ is surjective for all $s' \geq s$. Therefore, we get an exact sequence

\[ (17.1) \quad 0 \to A_s(W_s) \to G_s(W_s) \to \pi^s(G_s(W_s)) \to 0 \quad \text{for all} \ s' \geq s \geq r \quad \text{including} \ s' = \infty. \]

Assume now that $K$ is a finite extension of $\mathbb{Q}_p$. Since $\text{Gal}(\overline{K}/K_{p^r})$ acts on $A$ by the p-adic cyclotomic character, we find $A \cong \hat{G}_m$ over $W_{p^r}$ for $d = \dim A$. Thus $A_s$ (resp. $B_s$) is the maximal multiplicative part of the formal group of $A_s$ (resp. $B_s$ isogenous to $A_s$) over $W_s$. In Corollary in the introduction of [Oh00] (see also [H13a, Lemma 4.2]), Ohita shows that $TG_s := \lim_{s \to s} T_s G_s^\circ \cong h$ (and hence $TG_s^{\circ} \cong \mathbb{T}$) canonically as $h$-modules. Assuming (F), we have $T_pG_s^\circ \cong h_s$; so, $G_s \cong \hat{G}_m \otimes_{\mathbb{Z}_p} h_s$ over $W_{p^r}$. Define $\mathfrak{A} \subset h_s$ by the annihilator of $G_s/A$ and $\mathfrak{B} : = \operatorname{Ker}(h_s \to \text{End}(A/K_s))$. Hence we have an exact sequence of formal groups:

\[ (17.2) \quad 0 \to \hat{G}_m \otimes_{\mathbb{Z}_p} \mathfrak{A} \to G_s \xrightarrow{\varpi} G_s \to \hat{G}_m \otimes_{\mathbb{Z}_p} h_s/\mathfrak{B} \to 0 \]

since $0 \to \mathfrak{A} \to h_s \xrightarrow{\varpi} h_s \to h_s/\mathfrak{B} \to 0$ is an exact sequence of $\mathbb{Z}_p$-free modules. Thus we have $A \cong \hat{G}_m \otimes_{\mathbb{Z}_p} \mathfrak{A}$ over $W_{p^r}$, and $\mathfrak{A}$ is an $h_s$-ideal and is an $O_A$-module. This shows that $A_s$, $B_s$, $\varpi(J_s) = J_s/A_s$ and $J_s$ all satisfy (A1–3) in Section 16.

The action of the Frobenius $[p : \mathbb{Q}_p]$ on $[p_{\infty}]$ is the multiplication by $a_p^{-1} \in O'_A$ (where $a_p$ is the image of $U(p)$ in $O_A$). Thus $A(W_{p^r}) = A \otimes_{\hat{G}_m}(W_{p^r}) \cong A \otimes_{\mathbb{Z}_p}(1 + m_{W_{p^r}})$ on which the natural Galois action on $\hat{G}_m(W_{p^r})$ is twisted by a character $\psi : \text{Gal}([\mathbb{Q}_p]/\mathbb{Q}) \to O'_A$ induced by the ordinary character $\psi$ sending $z$ in $\mathbb{Z}_p^\times$ to the image in $O_A'$ of the Hecke operator in $h_s$ of the class of $(\frac{1}{2})$ in $\hat{G}_m$. Let $A$ be the abelian variety $A_s$ in Case A and the Hecke-stable complement of $A$ in $J_s$ in Case B. Then $A^\text{ord} = \varpi(J_s^{\text{ord}})$ in Case B. Let $O_A := \text{End}(A/K)$, which is an order of the Hecke algebra generated over $\mathbb{Q}$ by Hecke operators $T(n)$ in $\text{End}^0(A/K) = \text{End}(A/K) \otimes_{\mathbb{Z}} \mathbb{Q}$. 


Recall the Galois representation \( \rho_A \) of \( \text{Gal}(\overline{\mathbb{Q}}_p/K) \) realized on \( T_{p,\text{ord}} \) into \( GL_2(W) \) for a finite flat extension \( W \) of \( \mathbb{Z}_p \). Take the connected component \( \text{Spec}(\mathbb{T}) \) of \( \text{Spec}(\mathfrak{h}) \) such that \( \mathfrak{h}/\varpi\mathfrak{h} = \mathbb{T}/\varpi\mathbb{T} \). Write \( \rho_{A|\text{Gal}(\overline{\mathbb{Q}}_p/K)} \cong \begin{pmatrix} \nu_p & \star \\ 0 & \varphi \end{pmatrix} \) and \( \rho_T = \begin{pmatrix} \nu & \star \\ 0 & \varphi \end{pmatrix} \) for a deformation \( \psi : \text{Gal}(K_{ur}/K) \to \mathbb{T}^\times \) of \( \psi \).

We write \( \text{Frob} \in \text{Gal}(K_{ur}/K) \) for the Frobenius element inducing the generator of \( \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}) \) and an appropriate power of the identity id = \([p, \mathbb{Q}_p]\) on \( K_\infty/K \).

**Proposition 17.1.** Suppose \( \mathfrak{F} \) and unramified of \( K/\mathbb{Q}_p \). Let \( \mathcal{G}^\circ \) be the connected component of \( \mathcal{G} = \mathcal{G}_{\alpha, \delta, \xi} \). Then we have \( H^1(K_{\infty}/K, \mathcal{G}^\circ_{s, T}(W_{\infty})) = (\mathbb{T}_s/\langle \nu_p\psi(\sigma_\infty) - 1 \rangle_{\mathbb{T}_s})[\psi(\text{Frob}) - 1] \).

If either \( |\nu_p\psi(\sigma_\infty)| - 1\mathbb{Z}_p \) or \( |\psi(\text{Frob}) - 1|_{\mathbb{T}_s} = 1 \), then we have the vanishing \( H^1(K_{\infty}/K, \mathcal{G}^\circ_{s, T}(W_{\infty})) = 0 \).

**Proof.** As we have seen, under \( \mathfrak{F} \), we have \( \mathcal{G}^\circ_{s, T}(W_{\infty}) \cong \mathfrak{m}_p \mathcal{T}_{s, \mathbb{Z}}(\psi) \mathcal{G}_{s, T} \) as \( \mathcal{G}(K_{ur}/K) \)-modules, where \( \text{Gal}(K_{ur}/K) \) acts on \( \mathbb{T}_{s, \mathbb{Z}}(\psi) \) by \( \psi \). We apply Lemma 16.4 to the formal Lie group \( \mathcal{A} \) with \( \mathcal{A}[p�] = \mathcal{G}^\circ_{s, T} \). Note that \( a \) in the Lemma is the image of \( \psi(\text{Frob}) \) in \( \mathcal{O}_{\mathcal{A}} \). From this, the cohomology of \( \mathcal{G}_{s, T} \) vanishes if either \( |\nu_p\psi(\sigma_\infty)| - 1|_{\mathbb{Z}_p} = 1 \) or \( |\psi(\text{Frob}) - 1|_{\mathbb{T}_s} = 1 \). We have then \( H^1(K_{\infty}/K, \mathcal{G}^\circ_{s, T}(W_{\infty})) = \lim_{\rightarrow} H^1(K_{\infty}/K, \mathcal{G}^\circ_{s, T}(W_{\infty})) = 0 \).

**Remark 17.2.** Since \( \psi(\text{Frob})\psi(\text{Frob}) = \det(\rho_T)([p, \mathbb{Q}_p]) \), we may replace \( \psi(\text{Frob}) \) in Proposition 17.1 by \( \psi(\text{Frob})^{-1}\det(\rho_T)([p, \mathbb{Q}_p]) \). Note that \( \det(\rho_T)([p, \mathbb{Q}_p]) \) is of finite order.

**Theorem 17.3.** Let the notation be as in Theorem 10.4. Let \( K \) be a finite extension of \( \mathbb{Q}_p \) for \( p > 2 \), and put \( K_s = K[\mu_p] \) \((s = 1, 2, \ldots, \infty)\). If \( A_r \) does not have split multiplicative reduction over \( W_r \), then the error term \( E^\infty(K)_{\mathbb{T}} \) in Case A is finite. Let \( \sigma \) be a generator of \( \text{Gal}(K_{\infty}/K) \). For the vanishing of the error term, we have the following assertions:

**Case A** Assume one of the following two conditions:

(a) \( A_r \) has good reduction over \( W_1 = W[\mu_p] \) with \( |\varphi(\text{Frob}) - 1|_{\mathbb{Z}_p} = 1 \);

(b) \( K_{/\mathbb{Q}_p} \) is unramified, \( |\psi(\sigma)|_{\mathbb{T}_s} = 1 \) and either \( |\varphi(\text{Frob}) - 1|_{\mathbb{T}_s} = 1 \) or \( |\psi(\text{Frob}) - 1|_{\mathbb{T}_s} = 1 \). Then \( E^\infty(K)_{\mathbb{T}} \) vanishes.

**Case B**

(c) Suppose that \( K_{/\mathbb{Q}_p} \) is unramified. If \( \psi(\sigma) \neq 1 \), \( \nu_p\psi(\sigma) \neq 1 \) and \( \varphi(\text{Frob}) \neq 1 \), then

\[ E^\infty_{\text{Sel}}(K)_{\mathbb{T}} : = \text{Coker}(J^\infty_{\text{ord}}(K)_{\mathbb{T}} \to \hat{B}^\text{ord}_r(K) \otimes \mathbb{Q}_p) = 0. \]

(d) Without assuming \( K_{/\mathbb{Q}_p} \) is unramified, if \( A_r \) has potential good reduction modulo \( m_W \) for all \( p \in \Omega_{\mathbb{T}} \), we have \( E^\infty_{\text{Sel}}(K)_{\mathbb{T}} = 0 \).

**Proof.** We start with Case A. Suppose that \( A_r \) has good reduction over \( W \). Then we write \( \varpi(J_s) \) for the abelian variety quotient \( J_s/A_s \), since \( J_s/A_s \) is \( \text{ord}(J_s) \) by definition. By (15.1), we have \( |H^1(K_{\infty}/K, \hat{A}^\text{ord}_r(K_{\infty}))| = |A_r[p�](\overline{\mathbb{F}})|^2 \). We then have an exact sequence \( \hat{A}^\text{ord}_r(K_{\infty}) \to \hat{J}^\text{ord}_r(K_{\infty}) \to \mathcal{E}(\hat{J}^\text{ord}_r(K_{\infty})), \) as \( J_s \) and \( \varpi(J_s) \) has semi-stable reduction over \( W_r \). By cohomology sequence of this short exact sequence, we have the claimed finiteness. In the non-split multiplicative case (over \( W_r \)), we have a similar exact sequence of the formal Lie groups, and applying the formal version [Sc87, Theorem 1], we get the finiteness for the connected part. The surjectivity (up to finite error) for the special fiber can be easily shown (see below). If \( A_r \) has either good or non-split multiplicative reduction over \( W_r \), we have finiteness of \( H^1(K_{\infty}/K_r, \hat{A}^\text{ord}_r(K_{\infty})) \) as above. Then by the inflation-restriction exact sequence:

\[ H^1(K_r/K, \hat{A}^\text{ord}_r(K_r)) \to H^1(K_{\infty}/K_r, \hat{A}^\text{ord}_r(K_{\infty})) \to H^0(K_r/K, H^1(K_{\infty}/K_r, \hat{A}^\text{ord}_r(K_{\infty}))) \to H^2(K_r/K, \hat{A}^\text{ord}_r(K_r)), \]

finiteness of \( H^j(K_r/K, \hat{A}^\text{ord}_r(K_r)) \) \((j = 1, 2)\) and \( H^1(K_{\infty}/K_r, \hat{A}^\text{ord}_r(K_{\infty})) \) tells us finiteness of the cohomology \( H^1(K_{\infty}/K, \hat{A}^\text{ord}_r(K_{\infty})), \) from which we conclude the finiteness of \( E^\infty(K)_{\mathbb{T}} \). If in addition \( |\varphi(\text{Frob}) - 1|_{\mathbb{Z}_p} = 1 \) and \( A_r \) has good reduction over \( W_r \), from \( |H^1(K_{\infty}/K_r, \hat{A}^\text{ord}_r(K_{\infty}))| = |A_r[p�](\overline{\mathbb{F}})|^2 = 0 \), we conclude

\[ (17.3) \quad H^1(K_r/K, \hat{A}^\text{ord}_r(K_r)) \cong H^1(K_{\infty}/K, \hat{A}^\text{ord}_r(K_{\infty})). \]
If \( r = 1, p \nmid [K_1 : K] \) and this group vanishes so, \( E^\infty(K)^r = 0 \).

We reprove finiteness and vanishing, assuming that \( K/\mathbb{Q}_p \) is unramified, via the result in the previous Section 16. We first look into the connected component. By (17.1),

\[
0 \to \mathcal{A}(W_s) \to G_s(W_s) \to \pi_2(G_s)(W_s) \to 0
\]
is exact for all \( s' \geq s \geq r \). Taking its Galois cohomology sequence, we get another exact sequence

\[
0 \to \mathcal{A}(W) \to G_s(W) \to \pi_2(G_s)(W) \to H^1(Gal(K_s/K), \mathcal{A}(W_s)).
\]

Since \( H^1(K_s/K, \mathcal{A}[p^\infty](W_s)) \to H^1(K_s/K, \mathcal{A}(W_s)) \to H^1(K_s/K, \mathcal{A}(W_s)[p^\infty](W_s)) \) is exact and the two end terms are finite whose order are bounded independent of \( s \) by Proposition 16.3 and Lemma 16.4, we find \( \text{Coker}(G_s^\infty(W_s)^{Gal(K_s/K)} \overset{\pi_2}{\to} \pi_2(G_s)(W_s)^{Gal(K_s/K)}) \) is finite. Under the extra conditions in the theorem, the cokernel vanishes.

Similarly, we have the exact sequence:

\[
0 \to \tilde{\mathcal{A}}_{s'}^{\text{ord}}(F) \to \tilde{\mathcal{J}}_{s'}^{\text{ord}}(F) \to \varpi(\tilde{\mathcal{J}}_{s'}^{\text{ord}}(F)) \to H^1(F, \tilde{\mathcal{A}}_r^{\text{ord}}).
\]

If \( \varphi(\text{Frob}) \neq \pm 1 \), \( A_r \) has good reduction (not just semi-stable one) over \( W_r \); so, by Lang’s theorem [56], \( H^1(F, \tilde{\mathcal{A}}_r^{\text{ord}}) \subset H^1(F, A_r) = 0 \). Even if \( \varphi(\text{Frob}) = \pm 1 \), from the exact sequence

\[
0 \to A_0(\tilde{F}) \to A_r(\tilde{F}) \to \pi_0(A_r(\tilde{F})) \to 0
\]
for the connected component \( A_0 \) of \( A_r \), we find

\[
H^1(F, A_r) \cong H^1(F, \pi_0(A_r)) \quad \text{as Gal}(\tilde{F}/F) \quad \text{cocomological dimension 1. Thus } H^1(F, \tilde{A}_r^{\text{ord}})
\]
is finite. After passing to the limit, we find

\[
|\text{Coker}(J_{\infty}^{\text{ord}}(F) \overset{\pi_2}{\to} \varpi(J_{\infty}^{\text{ord}}(F))| \leq |\pi_0(A_r(\tilde{F}))| < \infty.
\]

We have the following exact sequence:

\[
0 \to G^\infty(W_\infty)_T \to G(K_\infty)_T \overset{\text{red}}{\to} J_\infty(F)[p^\infty]_{T}^{\text{ord}} \to 0.
\]

Indeed, the maximal étale quotient \( G^\text{et} \) of \( G/W_\infty \) is a \( \Lambda \)-BT group by [H14a, Proposition 6.3]; so, its closed points lifts to a \( W_\infty \)-point as \( W_\infty \) is henselian. (Note that \( G_W^\text{et} \) may not be an étale Barsotti–Tate group for finite \( s \).) Taking the fixed point of \( Gal(K_\infty^\text{nr}/K) \), we have

\[
0 \to G^\infty(W_\infty)^{Gal(K_\infty^\text{nr}/K)} \to G(K_\infty)_T \overset{\text{red}}{\to} J_\infty(F)[p^\infty]_{T}^{\text{ord}} \to H^1(K_\infty^\text{nr}/K, G^\infty(W_\infty)_T).
\]

Then by Corollary 17.1, \( \text{Coker}(G(K)_T \overset{\text{red}}{\to} J_\infty(F)[p^\infty]_{T}^{\text{ord}}) = 0 \) (assuming either \( |\varphi(\text{Frob}) - 1|_p = 1 \) or \( |\psi_\nu(p) - 1|_p = 1 \)), and in particular, \( \text{Coker}(J^{\text{ord}}_{\infty}(K)_T \overset{\text{red}}{\to} J_\infty(F)[p^\infty]_{T}^{\text{ord}}) = 0 \).

We have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
G^\infty(W_\infty)^{Gal(K_\infty/K)} & \overset{\text{red}_{\varphi}}{\longrightarrow} & J_\infty(K)_T & \overset{\text{red}_{\varphi}}{\longrightarrow} & J_\infty(F)[p^\infty]_{T}^{\text{ord}} & \overset{\text{red}_{\varphi}}{\longrightarrow} & H^1(K_\infty^\text{nr}/K, G^\infty(W_\infty)_T) \\
\varpi(G^\infty)(W_\infty)^{Gal(K_\infty/K)} & \overset{\text{red}_{\varphi}}{\longrightarrow} & \varpi(J_\infty(K))_T & \overset{\text{red}_{\varphi}}{\longrightarrow} & \varpi(J_\infty(F)[p^\infty]_{T}^{\text{ord}}) & \overset{\text{red}_{\varphi}}{\longrightarrow} & \\
\text{Coker}(\varpi) & \overset{\text{red}_{\varphi}}{\longrightarrow} & \text{Coker}(\varpi) & \overset{\text{red}_{\varphi}}{\longrightarrow} & \text{Coker}(\varpi).
\end{array}
\]

For the Frobenius endomorphism \( \hat{\varphi} (= \varphi(\text{Frob})) \), we have

\[
J_\infty(F)[p^\infty]_{T}^{\text{ord}} = J_\infty(F)[p^\infty]_{T}^{\text{ord}}[\hat{\varphi} - 1].
\]

Since \( \hat{\varphi} \equiv \varphi(\text{Frob}) \mod m_T \), if \( \varphi(\text{Frob}) \neq 1 \mod m_W \) (\( \varphi(\text{Frob}) - 1|_p = 1 \)), \( J_\infty(F)[p^\infty]_{T}^{\text{ord}} = 0 \), and \( \text{Coker}(\varpi) \) is finite under \( \psi(\sigma) \neq 1 \) and vanishes if \( \text{Coker}(\varpi) = 0 \) under the extra conditions (i.e., \( |\psi(\sigma) - 1|_p = 1 \) and either \( |\varphi(\text{Frob}) - 1|_p = 1 \) or \( |\psi_\nu(p) - 1|_p = 1 \)).

We apply \( X \mapsto X^\vee := \text{Hom}(X, \mathbb{Q}_p/\mathbb{Z}_p) \) to the above diagram. Since \( \mathbb{Q}_p/\mathbb{Z}_p \) is \( \mathbb{Z}_p \)-injective, \( X \mapsto X^\vee \) is an exact contravariant functor, all arrows of (17.4) are reversed, but exactness is kept. Since \( \text{Coker}(\varpi) \hookrightarrow H^1(K, A_r)_p \), its Pontryagin dual module \( \text{Coker}(\varpi)^\vee \) is a \( \mathbb{Z}_p \)-module.
of finite type. Since this module killed by arithmetic prime \((\varpi)\), we need to show the vanishing of the \((\varpi)\)-localization \(\Coker(\varpi_\infty)_{(\varpi)}^\vee = 0\). Note that we have a surjective morphism of \(A\)-module: \(\Coker(\red) \to \Coker(\red_J)\) and that \(\Coker(\red)\) is killed by \((\nu_p g(\sigma) - 1)\)\((\gamma - 1)\) by Corollary 17.1. Since \((\varpi)\) is prime to \(\gamma - 1\), we have the vanishing of the localization \(\Coker(\red_J)_{(\varpi)}^\vee = 0\). From the diagram obtained by applying \(X \mapsto X^\vee\), the localized sequence \(0 = \Coker(\varpi_\infty)_{(\varpi)}^\vee \to \Coker(\varpi_\infty)_{(\varpi)}^\vee \to \Coker(\varpi_\infty)_{(\varpi)}^\vee\) is exact. Since \(\Coker(\varpi_\infty)_{(\varpi)}^\vee\) is finite under \(\psi(\sigma) \neq 1\) and \(\varphi(\text{Frob}) \neq 1\), we conclude \(\Coker(\varpi_\infty)_{(\varpi)}^\vee = 0\); so, \(\Coker(\varpi_\infty)_{(\varpi)}^\vee = 0\) by the finiteness of \(\Coker(\varpi_\infty)\). Since \(\Coker(\varpi_\infty)_{(\varpi)}^\vee = 0\) is finite \(\mathbb{Z}_p\)-module, dualizing back, this shows finiteness of \(\Coker(\varpi_\infty)\) as desired.

Now we deal with Case B. First suppose \(|\psi(\sigma) - 1|_p = |\nu_p \psi(\sigma) - 1|_p = |\varphi(\text{Frob}) - 1|_p = 1\). Recall \(\mathcal{B}_s\) the formal Lie group of \(\mathcal{B}_s[p^\infty]^{\text{ord}}\). Similarly to the argument in Case A, we have the following commutative diagram with exact rows and columns for \(s \geq r\):

\[
\begin{array}{cccc}
G_s(W_{\infty})_{T(G_{\infty})}^{\text{Gal}(K_{\infty}/K)} & \rightarrow & \tilde{J}_s(\mathbb{K})_{T} & \rightarrow & J_s([p^\infty]_{T})_{T} \\
\beta_s & \downarrow & \beta_s & \rightarrow & \beta_s \\
\mathcal{B}_s(W_{\infty})_{T(G_{\infty})}^{\text{Gal}(K_{\infty}/K)} & \rightarrow & \tilde{B}_s^{\text{ord}}(\mathbb{K})_{T} & \rightarrow & B_s[p^\infty]^{\text{ord}}(\mathbb{K})_{T} \\
\downarrow & & \downarrow & & \downarrow \\
\Coker(\beta_s) & \rightarrow & \Coker(\beta_s) & \rightarrow & \Coker(\beta_s)
\end{array}
\] (17.5)

The \(\Coker(\red, s, r)\) vanishes if either \(|\varphi(\text{Frob}) - 1|_p = 1\) or \(|\psi \nu_p(\sigma) - 1|_p = 1\) by the argument in Case A. Assume \(|\varphi(\text{Frob}) - 1|_p = |\psi \nu_p(\sigma) - 1|_p = |\psi(\sigma) - 1|_p = 1\). Thus the bottom horizontal sequence is exact, and \(\Coker(\tilde{\beta}_s)\) vanishes as \(\mathcal{B}_s[p^\infty]^{\text{ord}}(\mathbb{K})_{T} \cong B_s[p^\infty]^{\text{ord}}(\mathbb{K})_{T} = 0\) if \(|\varphi(\text{Frob}) - 1|_p = 1\). Since \(\Coker(J_s[p^\infty]^{\text{ord}}(\mathbb{K})_{T} = \tilde{B}_s^{\text{ord}}(\mathbb{K})_{T} \otimes _{\mathbb{Z}_p} \mathbb{Q}_p\), we have \(\varpi(\mathcal{G}) = \mathcal{G}\) (as \(\mathcal{G}\) is the torsion part of \(J_{s,p}\)).

Thus by Proposition 17.1, \(|\nu_p \psi(\sigma) - 1|_p = 1\) implies \(H^1(K_{\infty}/K, \varpi(\mathcal{G})(W_{\infty})) = 0\). Similarly, by \(|\psi(\sigma) - 1|_p = 1\), Proposition 16.3 tells us \(H^1(K_{\infty}/K, \varpi(G_s)(W_{\infty})) = 0\), and hence \(\Coker(\tilde{\beta}_s) = 0\).

This shows \(\Coker(\beta_\infty) = \lim_s \Coker(\beta_s) = 0\), which implies \(\Coker(\beta_\infty) = 0\).

Suppose now that \(\varphi(\text{Frob}) \neq 1\), \(\psi \nu_p(\sigma) \neq 1\) and \(\psi(\sigma) \neq 1\). Thus we may assume that \(\psi \nu_p(\gamma, \mathbb{Q}_p) \neq 1\) and write \(\psi(\sigma) = \zeta' c\) so that \(\zeta' = p\)-power of root of unity and \(\zeta'^{m} = 1\) for \(m\) prime to \(p\). Then, by Propositions 17.1 and 16.3, \(\Coker(\beta_s)\) for all \(s\) is killed by \((t^* - 1)(\gamma t^* - 1)\) as \(\varpi([\gamma, \mathbb{Q}_p]) = t^* (t = 1 + T \in \mathbb{Z}_p[[T]] = A)\) for \(z \in \mathbb{Z}_p^*\) determined by the \(A\)-module structure of \(J_s^{\text{ord}}\), and hence its image in \(\Coker(\beta_s)\) is killed by the ideal \((\varpi, (t^* - 1)(\gamma t^* - 1))\).

Since \(\varpi(t^* - \zeta)\), \(\Coker(\beta_s)\) is the surjective image of \(B_s^{\text{ord}}(K)/(\zeta - 1)(\gamma - 1)\), which is finite with order independent of \(s\). Thus \(\Coker(\beta_\infty) = \lim_s \Coker(\beta_s)\) has bounded order. Since \(\Coker(\beta_\infty) = \Coker(J_s[p^\infty]^{\text{ord}}(K) \otimes _{\mathbb{Z}_p} \mathbb{Q}_p)\) is \(p\)-divisible, we conclude \(E^s_{\text{Sel}}(K) = \Coker(\beta_\infty) = 0\). This shows the desired finiteness of \(E^s_{\text{Sel}}(K) = \Coker(\beta_\infty)\).

Now we deal with the last case (d). Then by the assumption, \(J_s[p^\infty]^{\text{ord}}(\mathbb{K})_{T}\) is contained in the maximal abelian scheme quotient of the Néron model of \(J_s\) over \(W_s\). Thus we find a abelian scheme quotient \(C_s\) of \(J_s/W_s\) such that \(\tilde{C}_s = \tilde{J}_s[p^\infty]^{\text{ord}}(\mathbb{K})_{T}\) for all \(s\). Write \(\varpi(\mathcal{C}) := C_s/A_s\). Thus we need to deal with the following exact sequence:

\[
\varpi(\tilde{C}_s)^{\text{ord}}(\mathbb{K})_{T} \to \tilde{C}_s^{\text{ord}}(\mathbb{K})_{T} \to \Coker(\beta_s) \to 0
\]

where \(\mathcal{C}_s\) is the formal group of \(C_s/W_s\). Then by Schneider’s theorem already quoted, \(\Coker(\beta_s) \hookrightarrow H^0(K_s/K, H^1(K_{\infty}/K_s, \varpi(C_s))) \hookrightarrow \varpi(C_s)[p^\infty](\mathbb{F})\). Since \(\Coker(\beta_s)\) is covered by \(B_s^{\text{ord}}(K)_{T} \cong B_r(K)_{T}\) with number of generators over \(\mathbb{Z}_p\) bounded independent of \(s\) and killed by \(\varphi(\text{Frob}) - 1\) for the Weil number \(\varphi(\text{Frob})\) of positive weight, its order is also bounded independently of \(s\). Since \(\Coker(\beta_\infty)\) is \(p\)-divisible, this shows \(E^s_{\text{Sel}}(K) = \Coker(\beta_\infty) = 0\).

□
We describe, in down-to-earth terms, how to create $p$-adic analytic family of modular form associated to an irreducible component of $\text{Spec}(\mathfrak{h}_0,\delta,\xi)$ from a $p$-ordinary family coming from an irreducible component of $\text{Spec}(\mathfrak{h}_0,\phi\circ\sigma_d)$ for $\phi\circ\sigma_d(a,d)$ only dependent on $d$; i.e., $\phi\circ\sigma_d(a,d) = \phi(d)$ for some character $\phi$ of $(\mathbb{Z}/p^r\mathbb{Z})^\times$. We show that as a $\Lambda$-algebra $\mathfrak{h}^{\text{ord}} := \mathfrak{h}_0,\phi\circ\sigma_d$ is isomorphic to $\mathfrak{h}_0,\xi,\delta,\theta$ for a specific choice $\xi$ depending on $\phi$ by $T(l) \mapsto \kappa(l)^{-1}T(l)$ regarding $l$ as an idele $l_l$ in $(\mathbb{A}^{(p\infty)})^\times$ supported on $\mathcal{O}_l$. Here $\kappa$ is a suitably chosen character of $(\mathbb{A}^{(\infty)})^\times/\mathbb{Q}^\times$ with values in $\Lambda$.

Let $S_k(\Gamma_0(Np^r), \varepsilon \phi \chi)$ for a character $\varepsilon : \mathbb{Z}_p^\times / \mu \rightarrow \mu_p$, $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{Q}_p^\times$ and $\phi : (\mathbb{Z}/p^r\mathbb{Z})^\times \rightarrow \mathcal{O}_l^\times$ be the space of cusp forms in $S_k(\Gamma_1(Np^r))$ satisfying the following identity

$$f \left( \frac{az + b}{cz + d} \right) = \varepsilon(a_p^{-\delta}d_p^{-\delta})\phi(a \text{ mod } p^r)\chi(a \text{ mod } N)f(x)(cz + d)^k$$

for all $(a,b,c,d) \in \Gamma_0(Np^r)$. Write $\psi$ for the character of $\Gamma_0(Np^r)$ given by

$$\gamma = \left( \frac{a}{c}, \frac{b}{d} \right) \mapsto \varepsilon(a_p^{-\delta}d_p^{-\delta})\phi(a \text{ mod } p^r)\chi(a \text{ mod } N).$$

We lift $f$ to $f : GL_2(\mathbb{Q}) \lbrack \Gamma_{GL_2(\mathbb{A})} \rbrack \rightarrow \mathbb{C}$ as in [H10, page 779]. Here we lift a Dirichlet character $\eta : (\mathbb{Z}/M\mathbb{Z})^\times \rightarrow \mathbb{C}$ to a character of $\eta_k : \mathbb{A}^\times / \mathbb{Q}_p^\times \rightarrow \mathbb{C}$ so that $\eta_k(l_l) = \eta(l)$ for all primes $l \mid M$, where $l_l \in \mathbb{A}^\times$ has $l$-component $l$ and outside $l$, it is trivial. Then $f$ satisfies

$$f \left( xzu \right) = \phi_k\chi_k\mathcal{F}_k(\varepsilon_k \delta_k)(z)\varepsilon_k(a_p^{-\delta}d_p^{-\delta})\phi_k\psi_k(d_p)\chi_k\chi_N(d_N)f(x)$$

for $u = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(Np^r)$ and $z$ in the center $Z(\mathbb{A})$ of $GL_2(\mathbb{A})$. Here we regard $\mathcal{F}_k(\varepsilon_k \delta_k) = \phi_k\chi_k\mathcal{F}_k(\varepsilon_k \delta_k)$ as a character of $Z(\mathbb{A})\mathcal{O}_l(\mathbb{N}p^r)$, and write it as $\psi_k : Z(\mathbb{A})\mathcal{O}_l(\mathbb{N}p^r) \rightarrow \mathbb{C}$.

Thus the space $S_k(\mathcal{O}_l(\mathbb{N}p^r), \psi_k)$ defined in [H10, page 779]. By the correspondence described in [H10, §1.1],

$$S_k(\Gamma_0(Np^r), \psi) \cong S_k(\mathcal{O}_l(\mathbb{N}p^r), \psi_k).$$

Revisit the classical space $S_k(\Gamma_0(Np^r), \Phi \chi)$ with usual Neben character $\Phi \chi$ for $\Phi : (\mathbb{Z}/p^r\mathbb{Z})^\times \rightarrow \mathcal{O}_l^\times$ and $\chi$ as above. Thus $f \in S_k(\Gamma_0(Np^r), \Phi \chi)$ satisfies

$$f \left( \frac{az + b}{cz + d} \right) = \Phi(a \text{ mod } p^r)\chi(a \text{ mod } N)f(x)(cz + d)^k$$

for all $(a,b,c,d) \in \Gamma_0(Np^r)$. Lift $f$ to $GL_2(\mathbb{A})$ in the same manner as above, we get $f \in S_k(\mathcal{O}_l(\mathbb{N}p^r), \Psi_k)$ (for $\Psi_k = \Phi_k\chi_k\mathcal{F}_k(\varepsilon_k \delta_k)$) satisfying $f \left( xzu \right) = \Phi_k\chi_k\mathcal{F}_k(\varepsilon_k \delta_k)(z)\varepsilon_k(a_p^{-\delta}d_p^{-\delta})\Phi_k\psi_k(d_p)\chi_k\chi_N(d_N)f(x)$. Then, for a character $\varphi : \mathbb{Z}_p^\times \rightarrow \mathbb{Q}_p^\times$, take a unique character $\varphi_k$ of $\mathbb{A}^\times / \mathbb{Q}_p^\times$ with $\varphi_k|_{\mathbb{Z}_p^\times} = \varphi^{-1}$, and define $f \otimes \varphi \in S(\mathcal{O}_l(\mathbb{N}p^r), \Psi_k)\mathcal{O}_l(\mathbb{N}p^r)$ by $f \otimes \varphi(g) = \varphi_k\det(g)\psi(g)$. Then we can go back to $f \otimes \varphi \in S_k(\Gamma_0(Np^r), \Psi_k)$ by the isomorphism $S_k(\Gamma_0(Np^r), \Psi_k) \cong S_k(\Gamma_0(Np^r), \Phi_k\chi_k)$ in [H10, §1.1]. Here we have $\Psi_{k,\varphi}(zu) = \varphi_k(z)^2\varphi_k(a_p\varphi)(d_p\varphi_k\chi_k\chi_N(d_N))$. By definition, we get

**Lemma 18.1.** If $f$ as above satisfies $f(T(n)) = \lambda(T(n))f$ (a Hecke eigenform) with $T(l) = U(l)$ if $l \nmid \mathcal{N}p$, we have $(f \otimes \varphi)|T(l) = \varphi_k(l)\lambda(T(l))f \otimes \varphi$ for all primes $l$ prime to $p$, and for $U(p)$, we have $(f \otimes \varphi)|U(p) = \lambda(U(p))(f \otimes \varphi)$. Thus this operation $f \mapsto f \otimes \varphi$ preserves “ordinarity”.

Here we have used the well known fact that $\varphi$ factors through the $p$-adic cyclotomic character whose value at $p$ is equal to $1$; thus, $(\varphi_k(p)) = 1$ and the formula $(f \otimes \varphi)|U(p) = \lambda(U(p))(f \otimes \varphi)$ is consistent with $(f \otimes \varphi)|T(l) = \varphi_k(l)\lambda(T(l))(f \otimes \varphi)$.

By the lemma, if $f$ has weight 2, from the abelian subvariety $A_f$ attached to $f$, we get an abelian variety $A_{f\otimes \varphi}$ of $J_k$ (for a suitable choice of $H$) which is the $\varphi$-twist of $A_f$ (see (16.4)); i.e., we have an identity of $l$-adic Tate modules $T_l A_{f\otimes \varphi} \cong (T_l A_f) \otimes \varphi$ as Galois modules identifying $\varphi$ as a Galois character via $\frac{1}{p^\infty} = \text{Gal}(\mathbb{Q}(\mu_p^\infty))/\mathbb{Q}$.

If $f$ (or starting $f$) is a Hecke eigenform, the modular form $f : GL_2(\mathbb{Q}) \rightarrow \mathbb{C}$ and its right translations $R(g)(f)(x) = f(xg) = f(xg)$ for $g \in GL_2(\mathcal{O}_l)$ generate an irreducible automorphic representation $\pi_f \otimes \varphi$ of $\mathcal{O}_l$.
representation $\pi = \pi_f$ of $GL_2(\mathbb{A})$. Similarly, the modular form $f \otimes \varphi : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \to \mathbb{C}$ and its right translation $R(g)(f \otimes \varphi)(x) = (f \otimes \varphi)(xg) = f(xg)\varphi_\lambda(\det(xg))$ for $g \in GL_2(\mathbb{A})$ generate an irreducible automorphic representation $\pi_{f \otimes \varphi}$. Plainly, we have $\pi_{f \otimes \varphi} \cong \pi \otimes \varphi_h$. Inside $\pi_{f \otimes \varphi}$, we find a unique new vector which corresponds to a primitive Hecke eigenform $f_\varphi \in S_k(\Gamma_0(C(\pi \otimes \varphi)), \phi\chi\varphi^2)$ for the conductor $C(\pi \otimes \varphi)$. The form $f_\varphi$ is usually not equal to $f \otimes \varphi$ even if $f$ is primitive (as their Neben types are plainly different). As explained in [H09, §3.1], $f \otimes \varphi$ often has level smaller than the primitive form $f_\varphi$. Unless the $p$-component $\pi_p$ is super-cuspidal, $\pi_p \otimes \varphi$ has non-zero $U(p)$-eigenvector with non-zero eigenvalue. Indeed, if $\pi_p = \pi(\alpha, \beta)$, there are non-zero eigenspaces in $\pi \otimes \varphi_h$ on which $U(p)$ acts by $\alpha \varphi_h(p_p)$ and $\beta \varphi_h(p_p)$ (if $\alpha(p) \neq \beta(p)$, the eigenspaces of each of the above value is one-dimensional). If $\pi_p$ is special, we have one dimensional eigenspace with non-zero eigenvalue. In any case, even if $\varphi$ is highly ramified at $p$, the eigenvalues of $U(p)$ for $f \otimes \varphi$ and $f$ are equal.

Suppose that

$$\mathcal{F}_1 := \{f_p \in S_2(\Gamma_0(Np^{r(p)}), \phi\chi\varphi_p)\}_{p \in \mathcal{O}_k}$$

is the ordinary $p$-adic analytic family for $\chi$ modulo $N$, $\phi$ modulo $p^r$ and $\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_p\oplus\mathbb{Q}_p$. Pick a positive integer $b$ prime to $p$ and a character $\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_p\oplus\mathbb{Q}_p$. Then we consider the twisted family $\mathcal{F}_1(b) = \{f_p \otimes \varphi^{1/b}_p\varphi\}$. Since $f_p \in S_2(\Gamma_0(Np^{r(p)}), \phi\chi\varphi_p)$, we have $f_p \otimes \varphi^{1/b}_p \in S_2(\Gamma_0(Np^{r(p)}), \psi_p)$, where $\psi_p((a \ b \ c \ d)) = \phi(a \mod p)\chi(c \mod N)\varphi_p(a/b)(d/p)^{1/b}(d/p)^{1/b}(a/b)^{-1}(d/p)$. Thus in this case, $\varphi^{1/b}_p \varphi^{-1}$ factors through $G/H$ for $H$ defined for $(\alpha, \delta) = (1, 1)$ and $\xi(a, d) = \varphi(a)\varphi(d)$. If one starts with $f_{1b} \in S_2(\Gamma_0(N))$ whose $L$-function has root number $\pm 1$, the $L$-function $f_p \otimes \varphi^{1/b}_p \varphi$ has the same root number. Therefore, the most interesting case is when $b = 2$ (so, $p > 2$), $(\alpha, \delta) = (1, 1)$ and $\phi \chi = 1$. This process can be reversed by tensoring back $\varphi^{1/b}_p \varphi^{-1}$.

Thus we have one-to-one correspondence of families of modular forms of $h_{0,1,\xi_{ord}}$ and $h_{1,1,\xi_{ord}}$, where $\xi_{ord}(a, d) = \varphi(d)^{1/b} \varphi(d)$. This shows, writing $\Lambda + \mathbb{Z}_p[T]$ with $T = 1 + T$.

**Proposition 18.2.** Let the notation as above. Then the $\mathbb{Z}_p$-algebra $h_{1,1,\xi_{ord}}$ is isomorphic to $h_{0,1,\xi_{ord}}(\xi_{ord}(a, d) = \varphi(d))$ by $T(l^n) \mapsto t^{-1/b}\log_p(l^n/\log_p(l^n)}\varphi(b(l^n))T(l^n)$ for primes $l$ as $\mathbb{Z}_p$-algebras, where $\gamma = 1 + p^r$ and $\log_p$ is the $p$-adic logarithm and we have written $T(l^n) = U(l^n)$ for $l\mid Np$. The $\Lambda$-algebra structure of $h_{1,1,\xi_{ord}}$ is obtained twisting the $\Lambda$-algebra structure of $h_{0,1,\xi_{ord}}$ by the character $\kappa : \mathbb{Z}_p^* \to \Lambda^*$, which is given by $\kappa(\gamma)^s = t^s$ for $s \in \mathbb{Z}_p$. In particular, the algebra $h_{1,1,\xi_{ord}}$ is free of finite rank over $\Lambda$ for all primes $p$.

**References**

Books


