ANalytic Variation of Tate–Shafarevich Groups

HARUZO HIDA

Abstract. Let $K$ be a number field. For a prime $p$, we study the inductive limit of the $p$-ordinary part of the Tate-Shafarevich groups and the Selmer groups (over $K$) of modular Jacobians of level $Np^r$ as $r \to \infty$ for a fixed integer $N$ prime to $p$. We prove control theorems of the $p$-primary part of $\Sha^1_K(A_p)$ over $p$-adic analytic family of abelian varieties $A_p$. In particular, under mild conditions, we show that if $\Sha^1_K(A_{p_0})^{\text{ord}}$ is finite for one member $A_{p_0}$ of the analytic family and the Mordell–Weil rank of $A_{p_0}$ is $\leq 1$ over its Hecke field, then $\Sha^1_K(A_p)^{\text{ord}}$ is finite for almost all members $A_p$.

1. Introduction

Fix a prime $p$ and a positive integer $N$ prime to $p$ throughout the paper. Let $\text{Spec}(\mathcal{M})$ be an irreducible component of (the spectrum of) the $p$-ordinary big Hecke algebra $\mathcal{H}$. Attached to $\mathcal{M}$ is the Mazur–Kitagawa $p$-adic $L$-function $L(k, s)$ for the weight variable $k$ and the cyclotomic variable $s$. One may regard, essentially, $L$ as an element of the affine ring of the irreducible component of $\text{Spec}(\mathcal{H}^{\text{ord}})$ covering $\text{Spec}(\mathcal{M})$ for the two variable nearly $p$-ordinary big Hecke algebra $\mathcal{H}^{\text{ord}}$. We study in this paper the tower of modular curves $\{X_r\}_r$ whose Jacobians (or more precisely their $p$-ordinary part) correspond to the one variable $p$-adic $L$-function $k \mapsto L(2k+2, \alpha_k+1)$ for a fixed pair of $p$-adic integers $\alpha, \delta \in \mathbb{Z}_p$. In other words, regarding $L$ as a function of the formal torus $\widehat{\mathbb{G}}^{\text{ad}}_m$ (as $\widehat{\mathbb{G}}^{\text{ad}}_m = \text{Spf}(\Lambda)$ for the Iwasawa algebra $\Lambda$), an abelian factor $A$ of $J_r := \text{Pic}^0(X_r)$ belonging to $\mathcal{M}$ corresponds to the $L$-value $L(\zeta^\delta, \zeta^\alpha)$ for a $\rho$-th root of unity $\zeta \in \widehat{\mathbb{G}}^{\text{ad}}_m$. In this introduction, for simplicity, we assume that $\alpha = 0$ and $\delta = 1$; so the corresponding $p$-adic $L$-function is $k \mapsto L(2k+2, 1)$ concentrating to the weight variable, though one of the referees of this paper pointed out that the case $(\alpha, \delta) = (1, 1)$ is more interesting as $k \mapsto L(2k+2, k+1)$ interpolates the central critical values (so the function $k \mapsto L(2k+2, k+1)$ could be identically zero; see Section 18). This case (and also a more general case of an arbitrary $\alpha, \delta$) will be taken care of in the main text, and all the results stated here for $(\alpha, \delta) = (0, 1)$ stand for general $(\alpha, \delta)$, starting with an exotic tower $\{X_r\}_r$ (depending on $(\alpha, \delta)$) different from $\{X_1(Np^r)\}_r$.

Because of our simplifying assumption $(\alpha, \delta) = (0, 1)$, the tower $X_r = X_1(Np^r)/\mathbb{Q}$ is given by the compactified moduli of the classification problem of pairs $(E, \phi)$ of elliptic curves $E$ and an embedding $\phi : \mu_{Np^r} \hookrightarrow E[Np^r]$ as finite flat group schemes. Write $J_r/\mathbb{Q}$ for the jacobian variety of $X_r$ whose origin is given by the infinity cusp $\infty \in X_r(\mathbb{Q})$ of $X_r$.

For a set of places $S$ of a number field $K$, write $K^S/K$ for the maximal extension unramified outside $S$. For a topological Gal($K^S/K$)-module $M$ and $v \in S$, we write $H^\bullet(K^S/K, M)$ (resp. $H^\bullet(K_v, M)$ for the $v$-completion $K_v$ of $K$) for the continuous cohomology of the profinite group Gal($K^S/K$) (resp. Gal($K_v/K$) for an algebraic closure $\overline{K}$ of $K_v$) giving the discrete topology to the coefficients $M$ (so, $H^\bullet(K_v, M)$ and $H^\bullet(K, M)$ is a torsion module if $q > 0$). Define

$$\Sha^1(K^S/K, M) = \text{Ker}(H^3(K^S/K, M) \to \prod_{v \in S} H^2(K_v, M)) \text{ for } j = 1, 2$$

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and put \( \Pi^1(K^S/K, M)_p := \Pi^1(K^S/K, M) \otimes_{\mathbb{Z}} \mathbb{Z}_p \). Often we simply write \( \Pi^1 \) for \( \Pi^1 \).

More generally, for a module \( M \), we define \( M_p \) by \( M \otimes_{\mathbb{Z}} \mathbb{Z}_p \) (so, \( M_p \) is the maximal \( p \)-power torsion submodule \( M[p^{\infty}] \) of \( M \) if \( M \) is \( p \)-torsion, and the maximal \( p \)-profinite quotient if \( M \) is profinite). Throughout the paper, when \( M \) is related to an abelian variety, we always assume that \( S \) contains all finite places at which the abelian variety has bad reduction in addition to all \( p \)-adic and archimedean places of \( K \). Unless otherwise mentioned, we assume \( S \) to be chosen finite.

In addition to the Mordell–Weil group \( J_r(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \), we study the Tate–Shafarevich group \( \Pi_K(J_r, \Pi_K(K^S/K, J_r[p^{\infty}])) \) and the Selmer group

\[
\text{Sel}_K(J_r) = \ker(H^1(K^S/K, J_r[p^{\infty}]) \to \prod_{v \in S} H^1(K_v, J_r)).
\]

The Tate–Shafarevich group and the Selmer group of an abelian variety are independent of \( S \); so, we omitted 

\( K^S/K \)

from the notation. The Hecke operator \( U(p) \) and its dual \( U^*(p) \) acts on \( \Pi_K(J_r) \) and their \( p \)-adic limit \( e = \lim_{n \to \infty} U(p)^n \) and \( e^* = \lim_{n \to \infty} U^*(p)^n \) are well defined on the above groups \( H \). We write \( H^{ord} := e(H) \).

By Picard functoriality, we have injective limits \( G \rightarrow G_r := J_r[p^{\infty}]^{ord} \) (a \( \Lambda \)-BT group in the sense of [H14]), \( R \rightarrow J_r^{ord}(R) = \lim_{f \rightarrow \infty} J_r(f) \) for \( J_r(R) = \lim_{f \rightarrow \infty} J_r(R)/p^n J_r(R) \) as an fpf sheaf over \( K \), \( \Pi_K(J_r^{ord}) = \lim_{f \rightarrow \infty} \Pi_K(J_r)^{ord} \), \( \Pi_K(K^S/K, G) = \lim_{f \rightarrow \infty} \Pi_K(K^S/K, J_r[p^{\infty}]^{ord}) \), and \( \text{Sel}_K(J_r^{ord}) = \lim_{f \rightarrow \infty} \text{Sel}_K(J_r)^{ord} \). We study control under Hecke operators acting on these arithmetic cohomology groups. These groups, we call \( \Lambda \)-BT groups, \( \Lambda \)-MW groups, \( \text{ind} \)-\( \Lambda \)-TS groups and \( \Lambda \)-Selmer groups in order. For each Shimura’s abelian subvariety \( A_f \subset J_1(Np^r) \) associated to a Hecke eigenform \( f \in S_2(\Gamma_1(Np^r)) \) [IAT, Theorem 7.14], we can think of the ordinary part of the Tate–Shafarevich group \( \Pi_K(A_f)^{ord} \) and the Selmer group \( \text{Sel}_K(A_f)^{ord} \) (see Section 8 of the text or [ADT, page 74] for the definition of these groups). Let \( h = h(N) \) be a big ordinary Hecke algebra of prime-to-\( p \) level \( N \), and pick a primitive connected component \( \text{Spec}(T) \subset \text{Spec}(\mathbb{Q}(h(N))) \) in the sense of [H66a, §3]. Then points \( P \in \text{Spec}(T)(\overline{\mathbb{Q}}_p) \) correspond one-to-one to \( p \)-adic Hecke eigenforms \( f_p \) in a slope 0 analytic family. Assuming for example that \( T \) is a unique factorization domain, in a densely populated subset \( \Omega_T \subset \text{Spec}(T)(\overline{\mathbb{Q}}_p) \) of principal primes (indexed by \( (\zeta, \zeta^*) = (\zeta, 1) \) for \( \zeta \in \mu_{p^{\infty}} \), \( f_p \) is classical, at all prime factors of \( N \) and of weight \( 2 \) (at the definition of \( \Omega_T \) will be given below Corollary 10.2). Write \( Np^r(T) \) for the minimal level of \( f_p \). Let \( A_{P/Q} \) (resp. \( B_{P/Q} \)) be Shimura’s abelian subvariety (resp. abelian variety quotient) of \( J_1(Np^r(T)) \) associated to \( f_p \). Write \( H_P \) for the subfield of \( \text{End}(A_{P/Q}) \otimes_{\mathbb{Z}} \mathbb{Q} \) generated by the Hecke operators. In this introduction, for simplicity, we assume that \( A_P \) for every \( P \in \Omega_T \) has potentially good reduction at \( p \) and that \( A_P \) for some \( P \in \Omega_T \) has good reduction over \( \mathbb{Z}_p \). We prove control theorems for these arithmetic cohomology groups which imply

**Theorem A.** Suppose \( p > 2 \), \( |S| < \infty \) and that \( T \) is a unique factorization domain.

1. If \( |\Pi(K^S/K, A_{P_0}[p^{\infty}]^{ord})| < \infty \) for a single point \( P_0 \in \Omega_T \), then \( \Pi(K^S/K, A_{P}[p^{\infty}]^{ord}) \) is finite for almost all \( P \in \Omega_T \) (Corollary 12.2).

2. If \( |\Pi_K(A_{P_0})^{ord}| < \infty \) and \( dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1 \) for a single point \( P_0 \in \Omega_T \), then \( \Pi_K(A_{P})^{ord} \) is finite for almost all \( P \in \Omega_T \) (Theorems 13.4 and 13.6).

3. If \( |\Pi_K(A_{P_0})^{ord}| < \infty \) and \( dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1 \) for a single point \( P_0 \in \Omega_T \), then for almost all \( P \in \Omega_T \), \( 0 \leq dim_{H_P} A_{P}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1 \) is independent of \( P \) (Theorems 13.4 and 13.6).

4. If \( \text{Sel}_K(A_{P_0})^{ord} \) is finite for a single point \( P_0 \in \Omega_T \), then \( \text{Sel}_K(A_{P})^{ord} \) is finite for almost all \( P \in \Omega_T \). Moreover if \( \text{Sel}_K(A_{P_0})^{ord} = 0 \) for single point \( P_0 \in \Omega_T \) such that \( A_{P_0}/Q \) has good reduction modulo each prime factor \( p \) of \( A_{P_0} \), \( 0 \) for the residue field \( \mathbb{F}_p \), \( \text{Sel}_K(A_{P})^{ord} \) is finite for all \( P \in \Omega_T \) without exception (Corollary 10.5).

Here the words “almost all” means “except for finitely many”. In the assertion (2) and (3), the condition that \( \Pi_K(A_{P_0})^{ord} \) is finite and \( dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1 \) for a single point \( P_0 \in \Omega_T \) can
be replaced a shorter condition: \(|\mbox{III}(K^S/K, A_{P_{\infty}}[p^{\infty}\mbox{ord}])| < \infty\) for a single point \(P_0\). The assertion (3) does not mean the identity \(\dim_{H_{P_{\infty}}}(A_{P_{\infty}})_{\otimes_{\mathbb{Z}}} \mathbb{Q} = \dim_{H_{P_{\infty}}}(A_{P_{\infty}})_{\otimes_{\mathbb{Z}}} \mathbb{Q}\) for almost all \(P\), as \(P_0\) might be in a closed subscheme of \(\text{Spec}(\mathbb{T})\) over which the Mordell–Weil rank is \(\geq 1\). Taking the “self-dual” tower associated to \((\alpha, \delta) = (1, 1)\), as the root number is constant in the self-dual family, if we know the \(p\)-Selmer parity conjecture for the family \(\{A_P\}_{P \in \Omega_T}\) over \(K\), our main results (which produce Theorem A) imply the identity \(\dim_{H_{P_{\infty}}}(A_{P_{\infty}})_{\otimes_{\mathbb{Z}}} \mathbb{Q} = \dim_{H_{P_{\infty}}}(A_{P_{\infty}})_{\otimes_{\mathbb{Z}}} \mathbb{Q}\) for most \(P\). The parity conjecture (for the self-dual tower) holds true under good circumstances by the results of Nekovář [N06, Chapter 12], [N07] and [N09] (particularly, the result in [N07] is valid over any number field \(K\)). More general statements covering “exotic modular towers” (including “self-dual towers”) will be given as Theorems and Corollaries indicated in the theorem. The ring \(\mathbb{T}\) is usually a power series ring of one variable over a discrete valuation ring (and hence a unique factorization domain; see Theorem 5.3).

A principle in the formal deformation theory of arithmetic objects is that a naturally discrete/disconnected objects under Archimedean topology (e.g., the cuspidal automorphic spectrum; see, the description in [PAF, §1.1–3]) is surprisingly continuous under a totally disconnected topology. The Mordell–Weil group of modular Jacobians is another example we deal with in this paper (and completing by a profinite topology, they depends continuously on the points of the formal spectrum of the Hecke algebra). Another point is that the Galois module structure of \(G\) determines the complex \(L\)-function of \(A_{P_{\infty}}\) (for \(P\) running over arithmetic points) and by the BSD-type conjectures, it (conjecturally) determines the fppf sheaf \(A_{P_{\infty}}\) to good extent. Since the complex \(L\)-values are interpolated over the family, the arithmetic information of \(G\) should essentially determine the sheaf \(J_{\infty}^{\text{ord}}\) (via the \(p\)-adic BSD type conjectures). As we describe in Theorem B soon, by this continuity, if two abelian varieties (or more precisely their Barsotti–Tate groups) are close

\[
\begin{align*}
\hat{A}(\kappa) &= \lim_{\kappa} A(\kappa)/p^n A(\kappa) \quad \text{for a finite Galois extension } \kappa/k, \\
\hat{A}(\kappa) &= \lim_{\kappa} \hat{A}(F) \quad \text{for an infinite Galois extension } \kappa/k
\end{align*}
\]

with \(F\) running over all finite Galois extensions \(k\) inside \(\kappa\). An explicit description of \(\hat{A}(F)\) for a finite extension \(F/k\) is given at the end of this introduction as (S), and only when \(\kappa/k/\mathbb{Q}\) are finite extensions, we have the identity \(\hat{A}(\kappa) = A(\kappa) \otimes_{\mathbb{Z}} \mathbb{Z}_p\). A key principle is

\[\text{(P) Though } \text{End}(A_{/\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \text{End}(A[p^{\infty}]_{/\mathbb{Q}}) \text{ does not act on the abelian variety } A_{/\mathbb{Q}}, \text{ it acts on the fppf/étale abelian sheaf } \hat{A}_{/\mathbb{Q}}.\]

We consider the Galois cohomology groups \(H^q(\hat{K}^S/K, \hat{A}(\hat{K}^S))\) for a number field \(K\) and \(H^q(K, \hat{A}(\hat{K}))\) for \(k = \mathbb{Q}_l\) putting discrete topology on \(\hat{A}(\kappa)\) for \(\kappa = K^S, K\) and profinite topology on the Galois group. Here a number field means a finite extension of \(\mathbb{Q}\). We write these cohomology groups as \(H^q(\hat{A})\) for a statement valid globally and locally. Recall \(M_p := M \otimes_{\mathbb{Z}} \mathbb{Z}_p\) for a \(p\)-torsion module \(M\). Then we prove, as Lemma 7.2, \(H^1(\hat{A}) \cong H^1(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p =: H^1(A)_p\), where \(H^1(A)\) stands for
The implication (a) is complicated, as we need to control well the $\mathbb{T}$-module $J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. If $|A_P(K)| < \infty$ for infinitely many $P$, $|A_P(K)| < \infty$ for almost all $P$ by [H15, Theorem 6.6], which

(1.2)
implies easily that \( J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \) is well controlled. If \(|A_P(K)| = \infty\) for infinitely many \(P\), again by [H15, Theorem 6.6], \(|A_P(K)| = \infty\) for almost all \(P\), and by a more involved argument, \( J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \) is well controlled; i.e., we get under \((*_{R_b})\) an exact sequence up to finite error (Corollary 13.7):

\[
0 \to \tilde{A}_P^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \to J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{\nu} J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \to 0
\]

for almost all \(P \in \Omega_T\) (see Corollary 13.7). This essentially implies Theorem A (3) and by the exact sequence

\[
0 \to J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \to \text{Sel}_K(J_{\infty}^{\text{ord}}) \to \Pi_K(J_{\infty}^{\text{ord}}) \to 0
\]

combined with the control of the Selmer group, we get under \((*_{R_b})\) an exact sequence up to finite error

\[
0 \to \Pi_K(\tilde{A}_P^{\text{ord}}) \to \Pi_K(J_{\infty}^{\text{ord},T}) \xrightarrow{\nu} \Pi_K(J_{\infty}^{\text{ord}}) \quad \text{(Proposition 13.2 and Theorem 13.8)}
\]

for almost all \(P \in \Omega_T\) which is essential in the proof of Theorem A (2). The control sequence of \(\Pi_K(J_{\infty}^{\text{ord}})\) in (1.4) combined with the assumption \(|\Pi_K(\tilde{A}_P^{\text{ord}})| < \infty\) implies Theorem A (2) (see Theorems 13.4 and 13.6).

The assertion of Theorem A (2) is almost equivalent to the assertion Theorem A (1) combined with the control result of the completed limit Mordell–Weil group \(J_{\infty}^{\text{ord}}(K) := \lim_{\to n} J_{\infty}^{\text{ord}}(K)/p^n J_{\infty}^{\text{ord}}(K)\) given in [H15, Theorem 6.6]. The proof of Theorem A (2) is a non-standard combination of different control results, that of \(J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p\), that of the Selmer group (Theorem A (4)) and the control (1.4) of the limit Tate–Shafarevich group and in this sense, a key is the control/continuity of \(\Pi(K^S/K, \mathcal{G})\) and the \(\Lambda\)-adic BT group \(\mathcal{G}\) (the latter is the base of automorphic deformation theory).

The author knew the assertion Theorem A (1) long ago but, until recently, did not realize its (rather non-standard) implication to (2) and (3). Technically, this point is something new, and the smallness of \(\Pi(K^S/K, \mathcal{G})\) (i.e., its \(\Lambda\)-co-torsion property) is behind the minimalist principle. In other words, the existence of an arithmetic point \(P_0\) satisfying the minimalist conditions: \(|\Pi_Q(\tilde{A}_P^{\text{ord}})| < \infty\) and \(\text{rank}_\mathbb{Z} A_{P_0}(\mathbb{Q}) \leq \dim A_{P_0}\) implies the minimal assertions \(\text{rank}_\mathbb{Z} A_P(\mathbb{Q}) \leq \dim A_P\) for most of arithmetic points \(P\). This is exactly what the co-torsion property of \(\Pi(K^S/K, \mathcal{G})\) means.

Writing \(\rho_T\) for the modular two-dimensional Galois representation associated to \(T\) (see [GME, §4.3.1]), we can think of the Selmer group \(\text{Sel}_Q(\rho_T \otimes \Phi)\) for a Galois character \(\Phi\) with values in \(T^\times\) (cf. [Gr94] or [HMI, §1.2.4]). The group \(\text{Sel}_Q(\rho_T \otimes \Phi)\) has natural relation to the limit Mordell–Weil group studied in [H15] and [H16a] and our ind \(\Lambda\)-Selmer group \(\text{Sel}_K(J_{\infty}^{\text{ord}})\). NekovÁE studied control theorems for this type of Selmer group and Howard [Ho07] studied the Selmer groups via Heegner classes (for \(\Phi = \sqrt{p} \det(\rho_T)^{-1}\) with the \(p\)-adic cyclotomic character \(\nu\)). However variation of the Tate–Shafarevich groups over an analytic family has not been studied in depth for an arbitrary number field \(K\), though when \(K\) is an abelian extension of \(\mathbb{Q}\), Kato's Euler system argument is an obvious exception. In this sense, this is perhaps the first attempt to understand the dependence of the group over a general number field \(K\) on the members \(A_P\) of an ordinary analytic family. Our points are:

(a) The Tate–Shafarevich groups (and the Selmer groups) have precise control over a given \(p\)-adic analytic family relative to the “inductive” limit over the tower without completion (as treated in this paper). This is true even when the corresponding \(p\)-adic L-function is identically zero over the family because of the parity of the root number.

(b) We studied in [H15] the \(p\)-adically completed injective limit of the Mordell–Weil groups in the family (and the completion is necessary in the process). We study here control of the \(p\)-divisible Mordell–Weil groups \(J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p\), \(\text{Sel}_K(J_{\infty}^{\text{ord}}), \Pi_K(J_{\infty}^{\text{ord}})\) and \(\Pi_K(J_{\infty}^{\text{ord}})\). Combining the control theorems of all of these, we get finer results.

The author originally wanted to make this paper almost independent of the results in [H15] (and [H16a]), but the referee has pointed out an erroneous statement which was a key to make this
independent of [H15, Theorem 6.6] and our proof of this version relies on [H15] to good extent. A key ingredient of the proof of [H15, Theorem 6.6] is a finiteness statement [H15, (6.22)] of an error term. In any case, the author is grateful for the comments of the referee.

Here is a description of some key technical tools (and its origin) behind the proof of Theorem A. For each \( h \)-module or \( h \)-sheaf \( M \) defined over the small fpf site over \( K \), we write \( M := M \otimes_h \mathbb{T} \). Long ago, in the 1981–82, the author studied the \( A \)-BT group \( \mathcal{G}_T := \mathcal{G} \otimes_h \mathbb{T} \) and proved exactness of the sheaf sequence

\[
(FB) \quad 0 \to A_p[p^\infty]_{\text{ord}} \to \mathcal{G}_T \xrightarrow{\varpi} \mathcal{G}_T \to 0
\]
as long as \( P \in \Omega_T \) is principal generated by \( \varpi \). This was the start of the study of the Hecke algebra \( h \) and its control, since \( h \) is defined to be the \( \Lambda \)-subalgebra of \( \text{End}_A(\mathcal{G}) \) generated by Hecke operators \( T(n) \). Since the results (described in [H86a] and [H86b]) appeared to be too strong to most number theorists, it took almost 5 years for the papers [H86a] and [H86b] to be published (although now these papers become somewhat classical or even obsolete). Anyway, the assertion (1) of Theorem A is a direct consequence of the exactness of \( 0 \to \mathcal{III}(K^S/K, \hat{A}_P[p^\infty]_{\text{ord}}) \to \mathcal{III}(K^S/K, \mathcal{G}_T) \to \mathcal{III}(K^S/K, \mathcal{G}_T) \) after applying the \( \mathcal{III}_K \)-functor to (FB) (Theorem 12.1). The author knew this control result (under some more restrictive assumptions) even in 1984. Our technique is simple-minded and is just a geometric study of fibers of the arithmetic sheaf made out of Picard groups of modular curves, which does not require any explicit study of the points in \( \mathcal{G}(K) \). This could be said the beauty of Iwasawa theory. Therefore, our treatment is insensitive to the choice of the base number field \( K \). A natural choice of \( K \) could be a CM field (appearing usually as an endomorphism field of an abelian variety) or the field of definition of the abelian variety with some level structure (which is often anti-cyclotomic, an opposite of CM fields), but we can take any number field \( K \) indeed. A weakness of our method is that it requires an input from explicit creation of rational points (i.e., we need some finiteness at one point \( P_0 \in \Omega_T \) all the time). For some Tate–Shafarevich groups, such a finiteness has been proven via an explicit calculation of Heegner points. What the author did not know at the time (of [H86b]) was the second fundamental exact sequence of étale sheaves over \( \mathbb{Q} \) (as a consequence of the theory of Section 2)

\[
(FJ) \quad 0 \to \hat{A}_P^{\text{ord}} \to J_\infty^{\text{ord}} \xrightarrow{\varpi} J_\infty^{\text{ord}} \to \hat{B}_P^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0 \quad \text{(Corollary 6.3)},
\]
and its sibling

\[
(FS) \quad 0 \to A_p[p^\infty]_{\text{ord}} \to \hat{A}_P^{\text{ord}} \oplus \varpi(J_\infty^{\text{ord}}) \to J_\infty^{\text{ord}} \to 0 \quad \text{(Corollary 6.4)},
\]
where \( \hat{X}^{\text{ord}}(R) = e(X(R)) \) for \( X = A_P, B_P \) (and \( \hat{X} \) is the sheaf whose value over a field is as in (1.1) above and (S) below). The sequence (FS) is the limit of factoring \( J_\bullet \) into the sum of \( A_P \) and its complement (written as \( \varpi(J_\bullet) \) symbolically), and a key point is that the complement survives to give the image sheaf \( \varpi(J_\infty^{\text{ord}}) \) of \( \varpi \) after passing to the limit. Though some of these sequences appeared in [H15], they are fully exploited in this paper. Applying the functor \( \text{Sel}_k \) to (FJ), it is relatively easy to prove the control result (4) (for Selmer groups) of Theorem A (see Section 10), since \( \hat{B}_P^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) is a sheaf of \( \mathbb{Q}_p \)-vector spaces without nontrivial cohomology (and we can essentially regard (FJ) as a short exact sequence). A hardest part (done in Section 13) is to prove control of the Tate–Shafarevich group \( III_K(A_P^{\text{ord}}) \cong III_K(A_P^{\text{ord}}) \). We show injectivity of the natural map:

\[
III_K(\varpi(J_\infty^{\text{ord}})) \to III_K(J_\infty^{\text{ord}}) \text{ associated to the inclusion } \varpi(J_\infty^{\text{ord}}) \hookrightarrow J_\infty^{\text{ord}} \text{ in two ways under a slightly different set of assumptions}. \]

Under \((\dagger)\), the \( III_K \)-functor applied to (FS) provides us this injectivity for \( P_0 \) and \( \varpi = \varpi_0 \) generating \( P_0 \) (Proposition 13.2). In a different way, we show the exactness of (1.4) (including the injectivity) for general \( P \) from (1.3) in the proof of Theorems 13.4 and 13.6. Since the co-Selmer group \( \text{Sel}_K(J_\infty^{\text{ord}}) \) is well controlled by Theorem 10.4 (which proves the assertion (4)), control of \( J_\infty^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \) implies that the limit Tate–Shafarevich group \( III_K(J_\infty^{\text{ord}}) \) is also well controlled; so, we obtain the exactness (1.4). The finiteness for almost all \( P \) of \( III_K(A_P^{\text{ord}}) \) claimed in these theorems relies on \((\dagger)\) and the injectivity proven by Proposition 13.2 for \( P_0 \) and for general \( P \) in the second way and a highly nontrivial calculation by P. Schneider of...
the universal norm of an abelian variety done in the early 1980s. This is done by applying the \( \Theta_K \)
functor to \((FJ)\). In other words, if \( \Theta_K(\mathcal{A}_P^\text{ord}) \) is finite, by \((1.4)\), the Pontryagin dual \( \Theta_K(J_\infty^\text{ord}) \) is \( \mathbb{T} \)-torsion of finite type (by Nakayama’s lemma).

As we wrote, our method is insensitive to the base field \( K \). The following example (for which
Theorem A might be useful to the study of \( \Theta_K(E) \) and \( E(K) \) for a generic non CM non-totally real
field \( K \)) is suggested by one of the referees of this paper. We choose \( K \) to be a quintic field over \( \mathbb{Q} \)
with two complex places (whose Galois closure \( K^{\text{gal}} \) has Galois group \( A_5 \) or \( S_5 \)). If we choose a finite
set \( v \) of rational places well (including the infinite place), as \( \text{Gal}(K^{\text{gal}}/\mathbb{Q}) \) is almost \( S_5 \), we can make
\((-1)^{[v]}\), \((-1)^{|V|}) \) to have a given pair of parity for the set \( V \) of all places of \( K \) over \( v \). Start with \( v 
\)
Note that the pair of root numbers of \( E/\mathbb{Q} \) and \( E/K \) is given by \((-1)^{[v]} = 1, (-1)^{|V|} = -1 \) (e.g., [DD11, §1.2]). Choose a prime \( p \) ordinary for \( E \) such that \( E = A_{P_0} \) for \( P_0 \in \Omega_T \) for a
unique factorization domain \( T \) (this holds for primes \( p \) of density 1). If we can further assume \( |E(\mathbb{Q})| < \infty, \dim_{\mathbb{Q}} E(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1 \) and \( |\Theta_K(E)| < \infty \), then Theorem A (combined with the result of Nekovár
[NO7]) tells us that \( \dim_{\mathbb{Q}} A_{P}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = [H_P : \mathbb{Q}] \) and \( |\Theta_K(A_{P})^{\text{ord}}| < \infty \) for most of \( P \in \Omega_T \) without
really studying \( A_{P}(K) \) explicitly, though how we can achieve the above situation (i.e., \( |E(\mathbb{Q})| < \infty, \dim_{\mathbb{Q}} E(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1 \) and \( |\Theta_K(E)| < \infty \)) is another (statistical) question. One would expect that the
K-rank is 0 or 1 for most of \( K \)-rational elliptic curves \( E \) which do not descend to a proper
subfield of \( K \) (in short, when \( E \) is properly defined over \( K \)). On the other hand, the component of the “big” Hecke algebra for \( GL(2)/\mathbb{Q} \) carrying such an elliptic curve perhaps does not have much
arithmetic points as \( K \) has complex places (a conjecture of Calegari–Mazur [CM09]). Thus our way
of studying K-rank of abelian varieties of \( GL(2) \)-type over analytic families would break down for
abelian varieties properly defined over a non-totally real field \( K \).

We may reformulate our result via congruence among abelian varieties introduced in [H16b].
For such reformulation, we recall first the definition of the congruence. An \( F \)-simple abelian variety (with
a polarization) defined over a number field \( F \) is called, in this paper, “of \( GL(2) \)-type” if we have a
subfield \( H_A \subset \text{End}^0(A_F) = \text{End}(A_F) \otimes_{\mathbb{Z}} \mathbb{Q} \) of degree \( \dim A \) (stable under Rosati-involution).
If \( F = \mathbb{Q} \) (or more generally \( F \) has a real place), for the two-dimensional compatible system \( \rho_A \) of Galois
representation of \( A \) with coefficients in \( H_A \), \( H_A \) is generated by traces \( \text{Tr}(\rho_A((\text{Frob}_p))) \) of Frobenius elements \( \text{Frob}_p \) for \( F \)-primes \( f \) of good reduction (i.e., the field \( H_A \) is uniquely determined by \( A \); see
[GME, §5.3.1] and [Sh75, Theorem 0]). We always regard \( F \) as a subfield of the algebraic closure \( \overline{\mathbb{Q}} \).
Thus \( O_A := \text{End}(A_F) \cap H_A \) is an order in \( H_A \). Write \( O_A \) for the integer ring of \( H_A \). Replacing \( A \) by the
abelian variety representing the group functor \( R \mapsto A(R) \otimes_{\mathbb{Q}} O_A \), we may choose \( A \) so that \( O_A \) = \( O \)
the \( F \)-isogeny class of \( A \). Since finiteness of the Tate–Shafarevich group of \( A \) (not necessarily
its exact size) is determined by the \( F \)-isogeny class of \( A \) for a field extension \( K/F \), we hereafter
assume that \( \text{End}(A_F) \cap H_A = O_A \) for any abelian variety of \( GL(2) \)-type over \( F \). For two abelian
varieties \( A \) and \( B \) of \( GL(2) \)-type over \( F \), we say that \( A \) is \( congruent \) to \( B \) modulo a prime \( p \) over \( F \) if
we have a prime factor \( p_A \) (resp. \( p_B \)) of \( p \) in \( O_A \) (resp. \( O_B \)) and field embeddings \( \sigma_A : O_A/p_A \hookrightarrow \overline{\mathbb{F}}_p \)
and \( \sigma_B : O_B/p_B \hookrightarrow \overline{\mathbb{F}}_p \) such that \( (A[p_A] \otimes_{O_A/p_A} \sigma_A \overline{\mathbb{F}}_p)^{\text{ss}} \cong (B[p_B] \otimes_{O_B/p_B} \sigma_B \overline{\mathbb{F}}_p)^{\text{ss}} \) as \( \text{Gal}(\overline{\mathbb{Q}}/F) \)-
modules, where the superscript “ss” indicates the semi-simplification. In this introduction, we
assume that \( A \) is \( p_A \)-ordinary meaning that the Barsotti–Tate group \( A[p_A] \) has nontrivial \( p \)-divisible
\( p \)-unramified quotient. Hereafter in this article, we always assume that the field of definition is \( \mathbb{Q} \), but the coefficient field \( K \) is any number field.

Let \( E/Q \) be an elliptic curve. Writing the Hasse–Weil L-function \( L(s, E) \) as a Dirichlet series
\( \sum_{n=1}^\infty a_n n^{-s} \) with \( a_n \in \mathbb{Z} \), i.e., \( 1 + p - a_p = |E(\mathbb{F}_p)| \) for each prime \( p \) of good reduction for \( E \),
we call \( p \) admissible for \( E \) if \( E \) has good reduction at \( p \) and \( (a_p, \mod p) \) is not in \( \Omega_E := \{\pm 1, 0\} \). Therefore, the maximal étale quotient of \( E[p] \) over \( \mathbb{Z}_p \) is not isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) up to unramified quadratic twists. By the Hasse bound \( |a_p| \leq 2\sqrt{p}, \) \( p \geq 7 \) is not admissible if and only if \( a_p \in \Omega_E \) (so,
2 and 3 are not admissible). Thus if \( E \) does not have complex multiplication, the Dirichlet density
of non-admissible primes is zero by a theorem of Serre as \( L(s, E) = L(s, f) \) for a rational Hecke eigenform \( f \) (see [H16b, §8]). A proto-typical theorem we prove is as follows.

**Theorem B.** Let \( E_{\mathbb{Q}} \) be an elliptic curve with \( |\Pi_{\mathbb{K}}(E)| < \infty \) and \( \dim_{\mathbb{Q}} E(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1 \). Let \( N \) be the conductor of \( E \) and pick an admissible prime \( p \) for \( E \). Consider the set \( \mathcal{A}_{E, p} \) made up of all \( \mathbb{Q} \)-isogeny classes of \( \mathbb{Q} \)-simple \( p \)-ordinary abelian varieties \( A_{\mathbb{Q}} \) of \( \text{GL}(2) \)-type with prime-to-\( p \) conductor \( N \) congruent to \( E \) modulo \( p \) over \( \mathbb{Q} \). Then there exists an explicit (computable) finite set \( S_E \) of primes depending on \( N \) but independent of \( \mathbb{K} \) such that if \( p \notin S_E \), almost all members \( A \in \mathcal{A}_{E, p} \) have finite \( \Pi_{\mathbb{K}}(A)_{p\mathbb{A}} \) and constant dimension \( \dim_{\mathbb{A}} A(\mathbb{K}) = 0 \) (i.e., \( \text{Sel}_{\mathbb{K}}(E)_p = 0 \) in short) and \( E \) can be embedded into \( J_r \), for some \( r > 0 \), then as long as every prime factor of \( p \) in \( \mathbb{K}/\mathbb{Q} \) has residue field \( \mathbb{F}_p \), every \( A \in \mathcal{A}_{E, p} \) has finite \( \Pi_{\mathbb{K}}(A)_{p\mathbb{A}} \) and \( \text{Sel}_{\mathbb{K}}(A)_{p\mathbb{A}} \) as long as \( p \notin S_E \).

Here for the prime \( p | A \), we have \( (A[pP_\mathbb{A}] \otimes_{O_{\mathbb{A}}/P_\mathbb{A}, \sigma_\mathbb{A}} \mathbb{F}_p)_{\text{ss}} \cong (E[p] \otimes_{\mathbb{F}_p, \mathbb{F}_p} \mathbb{F}_p)_{\text{ss}} \), and \( \Pi_{\mathbb{K}}(A)_{p\mathbb{A}} \) (resp. \( \text{Sel}_{\mathbb{K}}(A)_{p\mathbb{A}} \)) is the \( p \)-primary part of \( \Pi_{\mathbb{K}}(A)_{\mathbb{A}} \) (resp. \( \text{Sel}_{\mathbb{K}}(A)_{\mathbb{A}} \)). The definition of the set \( S_E \) will be given in Definition 15.1, and a more general version of this theorem will be given as Theorem 15.2. The Frey–Mazur conjecture for abelian varieties states that if \( A[p] \cong B[p] \) as \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) \)-modules for abelian varieties \( A \) and \( B \) over a number field \( \mathbb{K} \), then \( A \) and \( B \) are isogenous over \( \mathbb{K} \) as long as \( p \) is large enough compared to \( [\mathbb{K} : \mathbb{Q}] \). If \( A \) and \( B \) are isogenous, we have \( |\Pi_{\mathbb{K}}(A)| < \infty \Leftrightarrow |\Pi_{\mathbb{K}}(B)| < \infty \) and \( \text{rank} A(\mathbb{K}) = \text{rank} B(\mathbb{K}) \). Our theorem has a flavour similar to the conjecture (under Principle (P) and taking ordinary parts), replacing \( A \) and \( B \) by \( A^{\text{ord}} \) and \( B^{\text{ord}} \), but requiring \( A \) and \( B \) are defined over \( \mathbb{Q} \) and \( A^{\text{ord}}[pA] \cong B^{\text{ord}}[pB] \) as \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-modules which implies, for example, \( |\Pi_{\mathbb{K}}(A^{\text{ord}})| < \infty \Leftrightarrow |\Pi_{\mathbb{K}}(B^{\text{ord}})| < \infty \) for “most” \( A \) and \( B \) under some extra assumptions.

Taking \( \mathbb{K} = \mathbb{Q} \) and applying the theorem to the modular elliptic curves \( E = X_0(N) \) for small \( N \), we get the following corollary:

**Corollary C.** Let \( N \) be one of 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49 (all the cases when \( X_0(N) \) is an elliptic curve with finite \( X_0(N)(\mathbb{Q}) \)). Pick an admissible prime \( p \) for \( X_0(N) \). Then we have \( |\Pi_{\mathbb{Q}}(A)_{p\mathbb{A}}| < \infty \) and \( |\text{Sel}_{\mathbb{Q}}(A)_{p\mathbb{A}}| < \infty \) for almost all \( A \) in \( \mathcal{A}_{X_0(N), p} \). If further \( X_0(N)(\mathbb{Q})_p = \Pi_{\mathbb{Q}}(X_0(N))_p = 0 \), \( \text{Sel}_{\mathbb{Q}}(A)_{p\mathbb{A}} \) and \( \Pi_{\mathbb{Q}}(A)_{p\mathbb{A}} \) are both finite for all \( A \) in \( \mathcal{A}_{X_0(N), p} \) without exception.

By a celebrated theorem of Kolyvagin [K88] (with modularity of rational elliptic curves [BCDT01]), as long as the algebraic \( \mathbb{Q} \)-rank and the analytic \( \mathbb{Q} \)-rank of \( E \leq 1 \), we have \( |\Pi_{\mathbb{Q}}(E)| < \infty \). For the modular elliptic curves \( X_0(N) \) listed above, the point of the above corollary is that we have \( |X_0(N)(\mathbb{Q})| < \infty \) and that the exceptional set \( S_E \) can be taken to be empty (or more precisely, \( S_E \) is contained in non-admissible primes); so, the statement becomes more transparent. This corollary produces infinitely many examples of simple abelian varieties of (unbounded) dimension (\( > 1 \)) with finite Tate–Shafarevich group \( \Pi_{\mathbb{Q}}(A)_{p\mathbb{A}} \) as we know \( \text{dim}_P A \) grows indefinitely in an analytic family [H11].

We have a version of this corollary for some elliptic curve factors \( E/Q \) of \( J_0(N) \) (e.g., \( N = 37 \)) of root number \(-1\) assuming \( \text{rank}_Q E(\mathbb{Q}) = 1 \) and \( |\Pi_{\mathbb{Q}}(E)| < \infty \) for the analytic family with constant root number \(-1\) containing \( E \) (such a family is associated to an exotic tower; see Theorem 15.2).

Instead of starting with the modular elliptic curve as listed above, we can start with a CM elliptic curve \( E \) with finite Tate–Shafarevich group over \( \mathbb{Q} \) (studied by Rubin [R87]) and then we get a similar result for the CM family of abelian varieties containing the starting elliptic curve (although some CM cases are also covered by Corollary C).

For an extension \( X \) of an abelian variety by a finite group scheme defined either over a number field \( K \) or a local field \( K \) of characteristic 0, we define the fpf abelian sheaf \( \tilde{X} \) explicitly as follows:

\[
\tilde{X}(R) = \begin{cases} 
X(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \text{if } [K : \mathbb{Q}] < \infty, \\
X[p^{\infty}]\langle R \rangle & \text{if } [K : \mathbb{Q}_l] < \infty \ (l \neq p) \text{ or } [K : \mathbb{R}] < \infty, \\
(X/X^{(p)})\langle R \rangle \text{ as a sheaf quotient} & \text{if } [K : \mathbb{Q}_p] < \infty
\end{cases}
\]
for fppf algebras $R/K$, where $X^{(p)}$ is the maximal prime-to-$p$ torsion subgroup of $X$. If $R$ is a finite extension field of $K$ (except for the case of $K = \mathbb{R}, \mathbb{C}$), $\hat{X}(R) = \varprojlim_n X(R)/p^n X(R)$ as already mentioned. Therefore, we could have defined $\hat{X}(R) := \varprojlim_n X(R)/p^n X(R)$ except in the case where $K = \mathbb{R}, \mathbb{C}$ (and using this definition, the value $\hat{X}(R)$ is computed in [H15, (S) in page 228] as specified in (S) above). For $K = \mathbb{R}, \mathbb{C}$, this is just a convention as $H_q(K,?)$ with coefficients in a $\mathbb{Z}_p$-module just vanishes if $p > 2$. Throughout the paper, we write $M^\vee$ for the Pontryagin dual module $\text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ for a $\mathbb{Z}_p$-module $M$.

Contents

1. Introduction 1
2. $U(p)$-isomorphisms 9
3. Exotic modular curves 12
4. Hecke algebras for exotic towers 16
5. Abelian factors of $J_r$ 18
6. Limit abelian factors 22
7. Generality of Galois cohomology 25
8. Diagrams of Selmer groups and Tate–Shafarevich groups 27
9. Vanishing of the error term for $l$-adic fields with $l \neq p$. 29
10. Control of $\Lambda$-Selmer groups 32
11. Control of ind $\Lambda$-MW groups 35
12. Control of $\Lambda$-BT groups and its cohomology 36
13. Control of $\Lambda$-TS groups 36
14. Parameterization of congruent abelian varieties 43
15. A generalized version of Theorem B including exotic towers 45
16. $p$-Local cohomology of formal Lie groups 47
17. Finiteness of the $p$-local error term 48
18. Twisted family 52
References 53

2. $U(p)$-Isomorphisms

Replacing fppf cohomology we described in [H15, §3] by étale cohomology, we reproduce the results and proofs in [H15, §3] as it gives the foundation of our control result, though we need later to adjust technically the method described here to get precise control of the limit Tate–Shafarevich group. Let $S = \text{Spec}(K)$ for a field $K$. Let $X \rightarrow Y \rightarrow S$ be proper morphisms of noetherian schemes. We study

$$H^0_{\text{fppf}}(T, R^1 f_* \mathbb{G}_m) = H^0_{\text{ét}}(T, R^1 f_* \mathbb{G}_m) = R^1 f_* O_X^*(T) = \text{Pic}_X/S(T)$$

for $S$-scheme $T$ and the structure morphism $f : X \rightarrow S$. Write the morphisms as $X \xrightarrow{\pi} Y \xrightarrow{s} S$ with $f = g \circ \pi$. We note the following general fact:

**Lemma 2.1.** Assume that $\pi$ is finite flat. Then the pull-back of line bundles: $\text{Pic}_Y/S(T) \ni \mathcal{L} \mapsto \pi^* \mathcal{L} \in \text{Pic}_X/S(T)$ induces the Picard functoriality which is a natural transformation $\pi^* : \text{Pic}_Y/S \rightarrow \text{Pic}_X/S$ contravariant with respect to $\pi$. Similarly, we have the Albanese functoriality sending $\mathcal{L} \in \text{Pic}_X/S(T)$ to $\bigwedge^{\text{deg}(X/Y)} \pi_* \mathcal{L} \in \text{Pic}_Y/S(T)$ as long as $X$ has constant degree over $Y$. This map $\pi_* : \text{Pic}_X/S \rightarrow \text{Pic}_Y/S$ is a natural transformation covariant with respect to $\pi$.

Hereafter we always assume that $\pi$ is finite flat with constant degree.

In [H15, §3], we assumed that $f$ and $g$ have compatible sections $S \xrightarrow{s_g} Y$ and $S \xrightarrow{s_f} X$ so that $\pi \circ s_f = s_g$. However in this paper, we do not assume the existence of compatible sections, but
we limit ourselves to $T = \text{Spec}(\kappa)$ for an étale extension $\kappa$ of the base field $K$. Then we get (e.g., [NMD, Section 8.1] and [ECH, Chapter 3])

\[
\text{Pic}_X/S(T) = H^0_{\text{fppf}}(T, R^1f_*\mathcal{G}_m) \cong H^1_{\text{fppf}}(X_T, O^\times_{X_T}) = H^1_{\text{ét}}(X_T, O^\times_{X_T})
\]

\[
\text{Pic}_Y/S(T) = H^0_{\text{fppf}}(T, R^1g_*\mathcal{G}_m) \cong H^1_{\text{fppf}}(Y_T, O^\times_{Y_T}) = H^1_{\text{ét}}(Y_T, O^\times_{Y_T})
\]

for any $S$-scheme $T$. The identity at (*) follows from the fact: $\text{Pic}_T = 0$, since $T$ is a union of points (i.e., $\kappa = k_1 \oplus \cdots \oplus k_m$ for finite separable field extensions $k_j/K$). Here $X_T = X \times_S T$ and $Y_T = Y \times_S T$. We suppose that the functors $\text{Pic}_X/S$ and $\text{Pic}_Y/S$ are representable by group schemes whose connected components are smooth (for example, if $X, Y$ are smooth proper and geometrically irreducible (and $S = \text{Spec}(K)$ for a field $K$); see [NMD, 8.4.2–3]). We then write $J_j = \text{Pic}^0_{X/S}(\sim = X, Y)$ for the identity connected component of $\text{Pic}_j/S$. Anyway we suppose hereafter also that $X, Y, S$ are varieties (in the sense of [ALG, II.4]).

For an fppf covering $U \to Y$ and a presheaf $P = P_Y$ on the fppf site over $Y$, we define via Čech cohomology theory an fppf presheaf $U \mapsto \hat{H}^q(U, P)$ denoted by $\hat{H}^q(P_Y)$ (see [ECH, III.2.2 (b)]). The inclusion functor from the category of fppf sheaves over $Y$ into the category of fppf presheaves over $Y$ is left exact. The derived functor of this inclusion of an fppf sheaf $F = F_Y$ is denoted by $\hat{H}^q(F_Y)$ (see [ECH, III.1.5 (c)]). Thus $\hat{H}^q(G_{m/Y})(U) = H^q_{\text{fppf}}(U, O^\times_Y)$ for a $Y$-scheme $U$ as a presheaf (here $U$ varies in the small fppf site over $Y$).

Instead of the Hochschild-Serre spectral sequence used in [H68b, §4] to get a control of modular group cohomology, assuming that $f, g$ and $\pi$ are all faithfully flat of finite presentation, we use the spectral sequence of Čech cohomology of the flat covering $\pi : X \to Y$ in the fppf site over $Y$ [ECH, III.2.7]:

\[
\hat{H}^p(X_T/Y_T, H^q(G_{m/Y})) \Rightarrow H^p_{\text{fppf}}(Y_T, O^\times_{Y_T}) \Rightarrow H^p(Y_T, O^\times_{Y_T})
\]

for each $S$-scheme $T$. Here $F \mapsto H^p_{\text{fppf}}(Y_T, F)$ (resp. $F \mapsto H^p(Y_T, F)$) is the right derived functor of the global section functor: $F \mapsto F(Y_T)$ from the category of fppf sheaves (resp. Zariski sheaves) over $Y_T$ to the category of abelian groups. The canonical isomorphism $\iota$ is the one given in [ECH, III.4.9].

Write $H^\bullet_{\text{fppf}}(G_{m/Y_T})$ and $\hat{H}^\bullet(Y_T/X_T, H^0_{\text{fppf}})$ for $\hat{H}^\bullet(Y_T/X_T, H^0_{\text{fppf}})$. From this spectral sequence, we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccccc}
\hat{H}^1(H^0_{Y_T}) & \cong & \text{Pic}_Y/S(T) & \xrightarrow{a} & \hat{H}^0(\mathcal{X}_{Y_T}, H^1(G_{m,Y})) & \xrightarrow{\iota} & \hat{H}^2(H^0_{Y_T}) \\
\uparrow & & \uparrow \iota & & \uparrow \iota & & \uparrow \iota \\
\hat{H}^1(H^0_{Y_T}) & \cong & \text{Pic}_Y/S(T) & \xrightarrow{b} & \hat{H}^0(\mathcal{X}_{Y_T}, \text{Pic}_Y(T)) & \xrightarrow{\iota} & \hat{H}^2(H^0_{Y_T}) \\
\uparrow ?_1 & & \uparrow \iota & & \uparrow \iota & & \uparrow \iota \\
J_Y(T) & \xrightarrow{c} & \hat{H}^0(\mathcal{X}_{Y_T}, J_X(T)) & \xrightarrow{\iota} & ?_2.
\end{array}
\]

(2.2)

Here the horizontal exactness at the top two rows follows from the spectral sequence (2.1) (see [ECH, Appendix B]).

Take a correspondence $U \subset Y \times_Y Y$ given by two finite flat projections $\pi_1, \pi_2 : U \to Y$ of constant degree (i.e., $\pi_j_*\mathcal{O}_U$ is locally free of finite rank $\deg(\pi_j)$ over $\mathcal{O}_Y$). Consider the pullback $U_X \subset X \times_S X$ given by the Cartesian diagram:

\[
U_X = U \times_{Y \times Y} (X \times S X) \xrightarrow{} X \times_S X \\
\downarrow \\
U \xrightarrow{a} Y \times_S Y
\]

Let $\pi_{j,X} = \pi_j \times_S \pi : U_X \to X$ ($j = 1, 2$) be the projections.
Consider a new correspondence \( U_X^{(q)} = \overline{U_X \times_Y U_X \times_Y \cdots \times_Y U_X} \), whose projections are the iterated product
\[
\pi_{j,X^{(q)}} = \pi_{j,X} \times_Y \cdots \times_Y \pi_{j,X} : U_X^{(q)} \to X^{(q)} \quad (j = 1, 2).
\]

Here is a first step to get a control result of \( \Lambda\)-TS groups:

**Lemma 2.2.** Let the notation and the assumption be as above. In particular, \( \pi : X \to Y \) is a finite flat morphism of geometrically reduced proper schemes over \( S = \text{Spec}(K) \) for a field \( K \). Suppose that \( X \) and \( U_X \) are proper schemes over a field \( K \) satisfying one of the following conditions:

1. \( U_X \) is geometrically reduced, and for each geometrically connected component \( X^\circ \) of \( X \), its pull back to \( U_X \) by \( \pi_{2,X} \) is also connected; i.e., \( \pi^0(X) \xrightarrow{\sim} \pi^0(U_X) \);
2. \( f \circ \pi_{2,X} \).\( \mathcal{O}_{U_X} = f_\ast \mathcal{O}_X \).

If \( \pi_2 : U \to Y \) has constant degree \( \deg(\pi_2) \), then, for each \( q > 0 \), the action of \( U^{(q)} \) on \( H^0(X, \mathcal{O}_X^\times) \) factors through the multiplication by \( \deg(\pi_2) = \deg(\pi_{2,X}) \).

This result is given as [H15, Lemma 3.1, Corollary 3.2]. Though in [H15, §3], an extra assumption of requiring the existence of compatible sections to \( X \to Y \to S \), this assumption is nothing to do with the proof of the above lemma, and hence the proof there is valid without any modification.

To describe the correspondence action of \( U \) on \( H^0(X, \mathcal{O}_X^\times) \) in down-to-earth terms, let us first recall the \( \check{\text{C}} \)ech cohomology: for a general \( S \)-scheme \( T \),

\[
(2.3) \quad \check{H}^q(T, H^0(G_m/Y)) = \frac{\{(c_{i_0,\ldots,i_q})c_{i_0,\ldots,i_q} \in H^0(X_T^{(q+1)}, \mathcal{O}_X^{\times}) \text{ and } \prod_j(c_{i_0,\ldots,i_j} \circ p_{i_0,\ldots,i_{j+1}})^{(-1)^j} = 1\} \}
\]

where we agree to put \( H^0(X_T^{(0)}, \mathcal{O}_X^{(0)}) = 0 \) as a convention,

\[
X_T^{(q)} = X \times_Y X \times_Y \cdots \times_Y X \times S T, \quad O_{X_T^{(q)}} = O_X \times_{O_Y} O_X \times_{O_Y} \cdots \times_{O_Y} O_X \times_{O_Y} O_T
\]

the identity \( \prod_j(c \circ p_{i_0,\ldots,i_{j+1}})^{(-1)^j} = 1 \) takes place in \( O_X^{\times(q+2)} \) and \( p_{i_0,\ldots,i_{j+1}} : X_T^{(q+2)} \to X_T^{(q+1)} \) is the projection to the product of \( X \) the \( j \)-th factor removed. Since \( T \times_T T \equiv T \) canonically, we have \( X_T^{(q)} \cong X_T \times_T \cdots \times_T X_T \) by transitivity of fiber product.

Consider \( \alpha \in H^0(X, \mathcal{O}_X) \). Then we lift \( \pi_{1,X}^\ast \alpha = \alpha \circ \pi_{1,X} \in H^0(U_X, \mathcal{O}_{U_X}) \). Put \( \alpha_U := \pi_{1,X}^\ast \alpha \). Note that \( \pi_{2,X}^\ast \mathcal{O}_{U_X} \) is locally free of rank \( d = \deg(\pi_2) \) over \( \mathcal{O}_X \), the multiplication by \( \alpha_U \) has its characteristic polynomial \( P(T) \) of degree \( d \) with coefficients in \( \mathcal{O}_X \). We define the norm \( N_U(\alpha_U) \) to be the constant term \( P(0) \). Since \( \alpha \) is a global section, \( N_U(\alpha_U) \) is a global section, as it is defined everywhere locally. If \( \alpha \in H^0(X, \mathcal{O}_X^\times) \), \( N_U(\alpha_U) \in H^0(X, \mathcal{O}_X^\times) \). Then define \( U(\alpha) = N_U(\alpha_U) \), and in this way, \( U \) acts on \( H^0(X, \mathcal{O}_X^\times) \).

For a degree \( q \) \( \check{\text{C}} \)ech cohomology class \( [c] \in \check{H}^q(X/Y, H^0(G_m/Y)) \) with a \( \check{\text{C}} \)ech \( q \)-cocycle \( c = (c_{i_0,\ldots,i_q}) \) is given by the cohomology class of the \( \check{\text{C}} \)ech cocycle \( U(c) = (U(c_{i_0,\ldots,i_q})) \), where \( U(c_{i_0,\ldots,i_q}) \) is the image of the global section \( c_{i_0,\ldots,i_q} \) under \( U \). Indeed, \( (\pi_{1,X}^\ast c_{i_0,\ldots,i_q}) \) plainly satisfies the cocycle condition, and \( (N_U(\pi_{1,X}^\ast c_{i_0,\ldots,i_q})) \) is again a \( \check{\text{C}} \)ech cocycle as \( N_U \) is a multiplicative homomorphism. By the same token, this operation sends coboundaries to coboundaries, and define the action of \( U \) on the cohomology group. Thus we get the following vanishing result:

**Proposition 2.3.** Suppose that \( S = \text{Spec}(K) \) for a field \( K \). Let \( \pi : X \to Y \) be a finite flat covering of (constant) degree \( d \) of geometrically reduced proper varieties over \( K \), and let \( Y \xrightarrow{\pi_2} U \xrightarrow{\pi_1} Y \) be two finite flat coverings (of constant degree) identifying the correspondence \( U \) with a closed subscheme
$U \xrightarrow{\pi_1 \times \pi_2} Y \times S$. Write $\pi_{j,X} : U_X = U \times_Y X \to X$ for the base-change to $X$. Suppose one of the conditions (1) and (2) of Lemma 2.2 for $(X,U)$. Then

(1) The correspondence $U \subset Y \times S$ sends $\hat{H}^q(\mathcal{H}_Y^0)$ into $\deg(\pi_2)(\hat{H}^q(\mathcal{H}_Y^0))$ for all $q > 0$.

(2) If $d$ is a $p$-power and $\deg(\pi_2)$ is divisible by $p$, $\hat{H}^q(\mathcal{H}_Y^0)$ for $q > 0$ is killed by $U^M$ if $p^M \geq d$.

(3) The cohomology $\hat{H}^q(\mathcal{H}_Y^0)$ with $q > 0$ is killed by $d$.

This follows from Lemma 2.2, because on each Čech $q$-cocycle (whose value is a global section of iterated product $X_Y^{(q+1)}$), the action of $U$ is given by $U^{(q+1)}$ by (2.3). See [H15, Proposition 3.3] for a detailed proof.

Assume that a finite group $G$ acts on $X/Y$ faithfully. Then we have a natural morphism $\phi : X \times G \to X \times_Y X$ given by $\phi(x,\sigma) = (x,\sigma(x))$. In other words, we have a commutative diagram

$$
\begin{array}{ccc}
X \times G & \xrightarrow{(x,\sigma) \mapsto \sigma(x)} & X \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y,
\end{array}
$$

which induces $\phi : X \times G \to X \times_Y X$ by the universality of the fiber product. Suppose that $\phi$ is surjective; for example, if $Y$ is a geometric quotient of $X$ by $G$; see [GME, §1.8.3]). Under this map, for any fpqc abelian sheaf $F$, we have a natural map $\hat{H}^0(X/Y,F) \to \hat{H}^0(G,F(X))$ sending a Čech 0-cocycle $c \in \hat{H}^0(X,F) = F(X)$ (with $p_1^*c = p_2^*c$) to $c \in \hat{H}^0(G,F(X))$. Obviously, by the surjectivity of $\phi$, the map $\hat{H}^0(X/Y,F) \to \hat{H}^0(G,F(X))$ is an isomorphism (e.g., [ECH, Example III.2.6, page 100]). Thus we get

**Lemma 2.4.** Let the notation be as above, and suppose that $\phi$ is surjective. For any scheme $T$ fpqc over $S$, we have a canonical isomorphism: $\hat{H}^0(X_T/Y_T,F) \cong \hat{H}^0(G,F(X_T))$.

We now assume $S = \text{Spec}(K)$ for a field $K$ and that $X$ and $Y$ are proper reduced connected curves. Then we have from the diagram (2.2) with the exact middle two columns and exact horizontal rows:

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\hat{H}^1(\mathcal{H}_Y^0) & \longrightarrow & \text{Pic}_{Y/S}(T) & \longrightarrow & \hat{H}^0(G,F(X_T)) & \longrightarrow & \hat{H}^2(\mathcal{H}_Y^0) \\
\uparrow & & \uparrow \cup & & \uparrow \cup & & \uparrow \cup \\
?_1 & \longrightarrow & J_Y(T) & \longrightarrow & \hat{H}^0(G,F(X_T)) & \longrightarrow & ?_2,
\end{array}
$$

Thus we have $?_j = \hat{H}^j(\mathcal{H}_Y^0)$ ($j = 1, 2$).

By Proposition 2.3, if $q > 0$ and $X/Y$ is of degree $p$-power and $p | \deg(\pi_2)$, $\hat{H}^q(\mathcal{H}_Y^0)$ is a $p$-group, killed by $U^M$ for $M \gg 0$.

### 3. Exotic modular curves

We study a more general tower $\{X_r\}_r$ different from the standard one $\{X_1(Np^r)\}_r$ considered in the introduction (thus hereafter, $X_r$ could be no longer $X_1(Np^r)$). We introduce open compact subgroups of $\text{GL}_2(\mathbb{A}^{(\infty)})$ giving rise to the general tower $\{X_r\}_r$.

Let $\Gamma := 1 + p^r \mathbb{Z}_p \subset \mathbb{Z}_p^\times$, where $\epsilon = 2$ if $p = 2$ and $\epsilon = 1$ otherwise. Let $\gamma = 1 + p^r\sigma$, which is a topological generator of $\Gamma = \mathbb{Z}_p^\times$. We define the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\Gamma]] = \lim_n \mathbb{Z}_p[\Gamma/\Gamma^p^n]$ and identify it with the power series ring $\mathbb{Z}_p[[T]]$ sending $\gamma$ to $t = 1 + T$. The group $\Gamma$ is a maximal torsion-free subgroup of $\mathbb{Z}_p^\times$. Fix an exact sequence of profinite groups $1 \to H_p \to \Gamma \times \Gamma \xrightarrow{\pi_1} \Gamma \to 1$, and regard $H_p$ as a subgroup of $\Gamma \times \Gamma$. This implies

$$
\pi_1(a,d) = a^n d^{-\delta}
$$

for a pair \((\alpha, \delta) \in \mathbb{Z}_p^2\) with \(\alpha \mathbb{Z}_p + \delta \mathbb{Z}_p = \mathbb{Z}_p\) and hence \(H_p = \{(a, d) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times | a^\alpha d^\delta = 1\}\). Writing \(\mu\) for the maximal torsion subgroup of \(\mathbb{Z}_p^\times\), we pick a character \(\xi : \mu \times \mu \to \mathbb{Z}_p^\times\) and define \(H = H_\xi = H_{\alpha, \delta, \xi} := H_p \times \text{Ker}(\xi) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times = \Gamma \times \Gamma \times \mu \times \mu\). We can take \(\xi(\zeta, \zeta') = \zeta^{\alpha'}\zeta'^{-\delta'}\) for \((\alpha', \delta') \in \mathbb{Z}^2\). Write \(\pi = \pi_n \times \xi : \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times\) and the image of \(H\) in \((\mathbb{Z}_p^\times)^2/(\Gamma_p^{\alpha, \delta})^2\) as \(H_r\).

Then define, for \(\widehat{\mathbb{Z}} = \prod_{\text{primes } \mathbb{Z}_p}\)

\[
\tilde{\Gamma}(M) := \left\{ \begin{pmatrix} z & d \\ \xi \end{pmatrix} \in \text{GL}_2(\widehat{\mathbb{Z}}) | c \in M \widehat{\mathbb{Z}} \right\},
\]

\[
\tilde{\Gamma}_1(M) := \left\{ \begin{pmatrix} z & d \\ \xi \end{pmatrix} \in \Gamma_0(M) | d - 1 \in M \widehat{\mathbb{Z}} \right\},
\]

\[
\tilde{\Gamma}_s = \tilde{\Gamma}_{H, s} := \left\{ \begin{pmatrix} z & d \\ \xi \end{pmatrix} \in \tilde{\Gamma}_0(\mathcal{P}) \cap \tilde{\Gamma}_1(\mathcal{N}) | (a_p, d_p) \in H_s \right\},
\]

By definition, \(\tilde{\Gamma}_s \cap \text{SL}_2(\mathbb{Q}) = \Gamma_1(\mathcal{N}_p^r)\) as in the introduction if \(H_p = \mathbb{Z} \times \{1\}\) (i.e., \((\alpha, \delta) = (0, 1)\)) and \(\xi(a, d) = \omega(d)\) for \(\omega(a) = \lim_{n \to \infty} a^{p^n}\) if \(p\) is odd and otherwise \(\omega(a) = \left(\frac{q_2(\sqrt{-1})}{a}\right)\) (the quadratic residue symbol). We write this \(\xi\) as \(\omega_d\).

Consider the moduli problem over \(\mathbb{Q}_d\) of classifying the following triples

\[(E, \mu_N \xrightarrow{\phi_N} E, \mu_p^\alpha) \to (E, \mu_p^\alpha \xrightarrow{\phi'} E[p^r] \xrightarrow{\psi} \mathbb{Z}/p^r\mathbb{Z})_R,\]

where \(E\) is an elliptic curve defined over a \(\mathbb{Q}\)-algebra \(R\) and the sequence \(\mu_p^\alpha \to E[p^r] \to \mathbb{Z}/p^r\mathbb{Z}\) is meant to be exact in the category of finite flat group schemes. As is well known (e.g., [AME]), the triples are classified by a modular curve \(U_r/\mathbb{Q}\), and we write \(Z_r\) for the compactification of \(U_r\) smooth at cusps. In Shimura’s terminology, writing \(Z_r'\) for the canonical model attached to \(U_r := \tilde{\Gamma}(\mathcal{P}) \cap \tilde{\Gamma}_1(\mathcal{N})\), the curve \(Z_r\) is defined over \(\mathbb{Q}(\mu_p^\alpha)\) and is geometrically irreducible, while we have \(Z_r = \text{Res}_{\mathbb{Q}(\mu_p^\alpha)/\mathbb{Q}}Z_r'\) (when \(N \geq 4\)) which is not geometrically irreducible. We have the identity of the complex points \(Z_r(\mathbb{C}) - \{\text{cusps}\} = \text{GL}_2(\mathbb{Q}) \setminus \text{GL}_2(\mathbb{A})/U_r \times \mathbb{R}^\times \times \mathbb{R}^2(\mathbb{R})\).

Each element \((u, a, d)\) of the group \(G := (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times\) acts on \(Z_r\) by sending \((E, \mu_N \xrightarrow{\phi_N} E, \mu_p^\alpha \xrightarrow{\phi'} E[p^r] \xrightarrow{\psi} \mathbb{Z}/p^r\mathbb{Z})_R\) to

\[(E, \phi_N \circ u : \mu_N \xrightarrow{\phi_N \circ u} E, \mu_p^\alpha \xrightarrow{\phi' \circ \psi} E[p^r] \xrightarrow{\alpha \circ \phi' \circ \psi} \mathbb{Z}/p^r\mathbb{Z}),\]

where \(a \circ \phi'\) is \(a \phi'\) and the action of \(\phi_N\) and \(\phi_N\) is the one we have described in the introduction. For \(z = (z_N, z_p) \in (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times \times \mathbb{Z}_p^\times\), we write the action of \((u, a, d) = (z_N, z_p, z_p)\) as \((z)\). Via the inclusion \(\Gamma \times \Gamma \subset G\), the two variable Iwasawa algebra \(\Lambda := \mathbb{Z}_p[\Gamma \times \Gamma]\) is embedded into \(\mathbb{Z}_p[[\mathcal{G}]] = \Lambda[[\mathbb{Z}/\mathbb{N}\mathbb{Z}]^\times \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times\] for the maximal torsion subgroup \(\mu\) of \(\mathbb{Z}_p^\times\).

We consider the quotient curves \(X_r := Z_r/H\). The complex points of \(X_r\) removed cusps is given by \(Y_r(\mathbb{C}) = \text{GL}_2(\mathbb{Q}) \setminus \text{GL}_2(\mathbb{A})/\tilde{\Gamma}_r \times \mathbb{R}^\times \times \mathbb{R}^2(\mathbb{R})\). Indeed, the action of \((a_p, d_p) \in H\) regarded as an element \(\left(\begin{pmatrix} a_p & 0 \\ 0 & d_p \end{pmatrix}\right) \in \text{GL}_2(\mathbb{Z}_p) \subset \text{GL}_2(\mathbb{Z}^\times)\) is given by \((\phi_\mathcal{P}, \phi_p) \mapsto (\phi_\mathcal{P} \circ d_p, \phi_p \circ a_p)\).

If \(\text{det}(\tilde{\Gamma}_r) = \mathbb{Z}^\times\), by [IAT, Chapter 6], \(X_r\) is a geometrically connected curve canonically defined over \(\mathbb{Q}\). We have an adelic expression of their complex points.

\[
X_r' = \tilde{\Gamma}_r' \cap \text{SL}_2(\mathbb{Q}) \cap \tilde{\Gamma}_r \cap \text{SL}_2(\mathbb{Q}) = \tilde{\Gamma} \setminus \mathbb{H}\] for Shimura’s geometrically irreducible canonical model \(V_S\) defined over \(\tilde{\Gamma}_r\) for \(S = \tilde{\Gamma}_r' \cap \tilde{\Gamma}_r\) (see [IAT, Chapter 6]). In any case, these curves are geometrically reduced curves defined over \(\mathbb{Q}\) with equal number of geometrically connected components (i.e., it is \([F_\xi : \mathbb{Q}]\) for Shimura’s field of definition \(F_\xi \subset \mathbb{Q}^\text{ab}\) fixed by \(\text{det}(\tilde{\Gamma}_r) \subset \mathbb{Z}^\times \cong \text{Gal}(\mathbb{Q}^\text{ab}/\mathbb{Q})\)).
We fix a $\mathbb{Z}_p$ basis $(\zeta_{Np'}) = \exp(\frac{2i\pi}{Np'}) \in \mathbb{Z}_p(1) \times (\mathbb{Z}/N\mathbb{Z})(1)$. Then we identify $\mu_{Np'}$ with $(\mathbb{Z}/Np'\mathbb{Z})$ by $\zeta_{Np'} \mapsto (m \mod Np')$. For a triple $(E, \mu_N \phi_N \to E, \mu_{p'} \phi_{p'} \to E[p'] \xrightarrow{\phi_{p'}} \mathbb{Z}/p'\mathbb{Z})$, by the canonical duality $(\cdot, \cdot)$ on $E[Np']$, we have a unique generator $v \in E[Np']/\text{Im}(\phi_{Np'})$ for $\phi_{Np'} = \phi_N \times \phi_{p'}$ such that $(v, \phi_{Np'}(\zeta_{Np'})) = \zeta_{Np'}$. Then the quotient $E' := E/\text{Im}(\phi_{Np'})$ has an inclusion $\mu_{Np'} \phi_{Np'} \to E'$ given by sending $\zeta_{Np'}$ to $(av \mod \text{Im}(\phi_{Np'})) \in E'$. This gives a new triple $(E', \phi_N', \phi_{p'}', \varphi_{p'})$, where $\varphi_{p'}$ is determined by $(x, \phi_{p'}'(\zeta_{p'})) = \zeta_{p'}(x)$ for $x \in E'/p'$. We define an operator $w_r = w_{\zeta_{Np'}}$ acting on $Z_r$ by sending $(E, \mu_N \phi_N \to E, \mu_{p'} \phi_{p'} \to E[p'] \xrightarrow{\phi_{p'}} \mathbb{Z}/p'\mathbb{Z})$ to the above $(E', \phi_N', \phi_{p'}', \varphi_{p'})$. We have the following fact from the definition:

**Lemma 3.1.** The tower $\{X_r/\mathbb{Q}\}_r$ with respect to $(\alpha, \delta, \xi)$ is isomorphically sent by $w_r$ defined over $\mathbb{Q}$ to the tower over $\mathbb{Q}$ with respect to $(-\delta, -\alpha, \xi')$ for $\xi'(a, d) = \xi(d, a)$. In other words, $H$ defining the tower $\{X_r\}_r$ is send to $H'$ defining the other by the involution $(a, d) \mapsto (d, a)$. Regarding $w_r$ as an involution of $X_r$ defined over $\mathbb{Q}(\mu_{Np'})$, if $\sigma_r \in \text{Gal}(\mathbb{Q}(\mu_{Np'})/\mathbb{Q})$ for $z \in (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times$ is given by $\sigma_r(z) = \zeta_{Np'}$, we have $w_{\sigma_r} = (z) \circ w_r = w_r \circ (z)^{-1}$.

The last assertion of the lemma follows from $w_{\sigma_r} \zeta_{Np'} = w_{\sigma_r} \zeta_{Np'} = w_{\sigma_r} \zeta_{Np'} = (z) \circ w_{\zeta_{Np'}}$ and $w_z^2 = 1$.

The group $\hat{\Gamma}_x$ (s $\neq r$) normalizes $\hat{\Gamma}_x$, and we have $\hat{\Gamma}_x^0 = \Gamma_x^0$ is canonically isomorphic to $(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)/H \mod p^s$ by sending coset $(\sigma, \delta) \hat{\Gamma}_x$ to $(\sigma, \delta)$ $p^s \equiv H \mod p^s$, and the moduli theoretic action of $H$ coincides with the action of $\text{Gal}(X_\mathbb{Q}) = ((\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)/H \mod p^s)$. Through $\Gamma \cong (\Gamma \times \Gamma)/H/\mathbb{Z}_p$ (resp. $(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)/H \cong \mathbb{Z}_p^\times$), the one variable Iwasawa algebra $\Lambda$ (resp. $\mathbb{Z}_p[[Z_p]]$) acts on the tower $\{X_r\}_r$ as correspondences.

If det$(\hat{\Gamma}_{H, r}) = \hat{\mathbb{Z}}^\times$, as explained in ([IAT, Chapter 6]), $X_{r/\mathbb{Q}}$ and $X_{s/\mathbb{Q}}$ is geometrically irreducible. Though we do not need geometric irreducibility, we indicate here an easy criterion when geometric irreducibility holds. We note that det$(\hat{\Gamma}_r) \supset (\hat{\mathbb{Z}}(\mathbb{Q}))^\times$, where $\hat{\mathbb{Z}}(\mathbb{Q}) = \prod_{l \neq p} \mathbb{Z}_l \cong \hat{\mathbb{Z}}/\mathbb{Z}_p$. Thus the problem is reduced to the study of the determinant map at $p$. By $\alpha \mathbb{Z}_p + \delta \mathbb{Z}_p = \mathbb{Z}_p$, it is easy to see by definition, embedding diagonally $H$ into GL$_2(\mathbb{Z}_p)$, that

$$\text{det} : H_p \to \Gamma$$

is an isomorphism if and only if $p \nmid (\alpha + \delta)$ or $\alpha \cdot \delta = 0$.

If $(\alpha', \delta') \in \mathbb{Z}^2$ with $\alpha'Z + \delta'Z = Z$ and $\xi(a, d) = \omega(a)\alpha' \omega(d) - \delta'$,

$$\text{det} : (H \cap \mu \times \mu) \to \mu$$

is an isomorphism if $\alpha' + \delta'$ is prime to $2 \cdot (p - 1)$ or $\alpha' \cdot \delta' = 0$.

The second condition becomes also a necessary condition if we replace $\alpha' \cdot \delta' = 0$ by $\alpha' \cdot \delta' \equiv 0 \mod p - 1$ if $p$ is odd and by $\alpha' \cdot \delta' \equiv 0 \mod 2$ if $p \neq 2$. If $\alpha' = \delta' = i$, then Ker$(\xi) \supset (\zeta, \zeta)$, and hence $\text{det}(H) \supset \mu^2$. Thus to have a non-trivial element in Ker$(\omega^i)$ in $\mu \setminus \mu^2$, $\omega^i$ has to have odd order.

$$\text{det} : (H \cap \mu \times \mu) \cong \mu$$

has odd order.

The image $\text{det}(H)$ can be a proper subgroup in $\mathbb{Z}_p^\times$, and the curve $X_r$ and $X_{s/\mathbb{Q}}$ becomes reducible over the subfield $F = F_{\zeta}/\mathbb{Q}$ fixed by $\text{det}(H)$ identifying $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ with $\mathbb{Z}_p^\times$ by the $p$-adic cyclotomic character.

The most interesting case is when $\xi(a, d) = \omega^i(a) \omega^{-i}(d)$ ($i = 0, 1, \ldots, p - 2$) and $\alpha = \delta = 1$. Suppose $\alpha' = \delta' = i$ for $0 \leq i < p$ (so, $\alpha'Z + \delta'Z = iZ$). In this case, the L-function $L(s, f_p)$ can have root number $\pm 1$ (so, Birch–Swinnerton Dyer conjecture would force the non-triviality of the Mordell–Weil group of $A_p$ if the root number is $-1$). By (3.6), $\text{det}(\hat{\Gamma}_{H, r}) = \mathbb{Z}_p^\times$ if $p > 2$ and $\omega^i$ has odd order. Otherwise, if $p > 2$, $F_{\mathbb{Q}}$ is a unique quadratic extension of $\mathbb{Q}$ inside $\mathbb{Q}(\mu_p)$. If $p = 2$, if $\alpha = \delta = 1$ and $\alpha' = \delta' = 0$, $F_{\mathbb{Q}} = \mathbb{Q} \sqrt{2}$, and if $\alpha = \delta = 1$ and $\alpha' = \delta' = 1$, then $F_{\mathbb{Q}} = \mathbb{Q}(\sqrt{-1}, \sqrt{2})$.

Taking $(X, Y, U)/S$ to be $(X_{s/\mathbb{Q}}, X_{s/\mathbb{Q}}(p)/\mathbb{Q})$ for $s > r \geq 1$, to the projection $\pi : X_r \to X_s$, the result of the previous section is plainly applicable if $X_{s/\mathbb{Q}}$ is geometrically irreducible, since $U(p)$ is also geometrically irreducible as it is the image of $X_{s+1} := \mathfrak{F}/(\Gamma_s \cap \Gamma_0(p^{s+1}))$ by the diagonal product.
of two degeneration maps from $X_{s+1}^r$ in $X_r^* \times X_r^*$. If not, writing $X_{r/F_{l}}^r = \bigcup_i X_{s,i}^r$ for geometrically irreducible components $X_{s,i}^r$, then $U(p)$ restricted in each $X_{s,i}^r \times X_{s,i}^r$ is geometrically irreducible by the same argument above and its degree is a $p$-power independent of the components; so, we can apply the argument in Section 2 in this geometrically reducible cases.

**Corollary 3.2.** Let $F$ be a number field or a finite extension of $\mathbb{Q}_l$ for a prime $l$. Then we have, for integers $r, s$ with $s \geq r \geq \epsilon$,

\[(u) \pi^* : J_{s/R}^r(F) \rightarrow \hat{H}^0(X_s/X_s^r, J_{s/Q}(F)) \cong J_{s/Q}(F)[\gamma^{p^{r-1}} - 1] \text{ is a } U(p)\text{-isomorphism,}
\]

where $J_{s/Q}(F)[\gamma^{p^{r-1}} - 1] = \text{Ker}(\gamma^{p^{r-1}} - 1 : J_s(F) \rightarrow J_s(F))$ and $\epsilon = 1$ if $p > 2$ and $\epsilon = 2$ if $p = 2$.

Here the identity at $(\ast)$ follows from Lemma 2.4. The kernel $A \mapsto \text{Ker}(\gamma^{p^{r-1}} - 1 : J_s(A) \rightarrow J_s(A))$ is an abelian fpf sheaf (as the category of abelian fpf sheaves is abelian and regarding a sheaf as a presheaf is a left exact functor), and it is represented by the scheme theoretic kernel $J_{s/Q}[\gamma^{p^{r-1}} - 1]$ of the endomorphism $\gamma^{p^{r-1}} - 1$ of $J_s/Q$. From the exact sequence $0 \rightarrow J_s[\gamma^{p^{r-1}} - 1] \rightarrow J_s \xrightarrow{\gamma^{p^{r-1}} - 1} J_s$, we get another exact sequence

\[0 \rightarrow J_s[\gamma^{p^{r-1}} - 1](F) \rightarrow J_s(F) \xrightarrow{\gamma^{p^{r-1}} - 1} J_s(F).
\]

Thus

\[(3.7) \quad J_{s/Q}(F)[\gamma^{p^{r-1}} - 1] = J_{s/Q}[\gamma^{p^{r-1}} - 1](F).
\]

By a simple Hecke operator identity (e.g., [H15, (3.1)]), we have the following contraction property of $U(p)$-operator:

\[(3.8) \quad J_{r/R}^r \xrightarrow{\pi^*} J_{s/R}^r, \quad J_{r/R}^r \xrightarrow{\pi} J_{r/R}^s,
\]

where the middle $u'$ is given by $U_{r}^r(p^{s-r}) = \left[ \Gamma_s^r \left( \begin{smallmatrix} 1 & 0 \\ 0 & p^{r-r} \end{smallmatrix} \right) \right] \Gamma_r$ and $u$ and $u''$ are $U(p^{s-r})$. Thus

\[(u1) \quad \pi^* : J_{r/R}^r \rightarrow J_{s/R}^r \text{ is a } U(p)\text{-isomorphism (for the projection } \pi : X_r^* \rightarrow X_s^*).\]

The above $(u)$ combined with $(u1)$ and $(3.7)$ implies the sheaf identity $(u2)$ below for integers $r, s$ with $s \geq r \geq \epsilon$:

\[(u2) \quad \pi^* : J_{s/Q}^r \rightarrow J_{s/Q}[\gamma^{p^{r-1}} - 1] = \text{Ker}(\gamma^{p^{r-1}} - 1 : J_s/Q \rightarrow J_s/Q) \text{ is a } U(p)\text{-isomorphism.}
\]

We reformulate the above statement $(u2)$ into a commutative diagram (and get also the dual diagram):

**Lemma 3.3.** For integers $r, s$ with $s \geq r \geq \epsilon$, we have morphisms

\[\iota_s^t : J_{s/Q}[\gamma^{p^{r-1}} - 1] \rightarrow J_{s/Q}^r \quad \text{and} \quad \iota_s^{t*} : J_{s/Q}^r \rightarrow J_{s/Q}[(\gamma^{p^{r-1}} - 1)(J_{s/Q})]
\]

satisfying the following commutative diagrams:

\[(3.9) \quad J_{s/Q}^r \xrightarrow{\pi^*} J_{s/Q}[\gamma^{p^{r-1}} - 1] \quad \text{and} \quad J_{s/Q}^r \xrightarrow{\pi^*} J_{s/Q}[\gamma^{p^{r-1}} - 1],
\]

and

\[(3.10) \quad J_{s/Q}^r \xrightarrow{\pi^*} J_{s/Q}/(\gamma^{p^{r-1}} - 1)(J_{s/Q}) \quad \text{and} \quad J_{s/Q}^r \xrightarrow{\pi^*} J_{s/Q}/(\gamma^{p^{r-1}} - 1)(J_{s/Q}),
\]
where $u$ and $u''$ are $U(p^{s-r}) = U(p)^{s-r}$ and $u^*$ and $u''^*$ are $U^*(p^{s-r}) = U^*(p)^{s-r}$. In particular, for an fppf extension $T/Q$, the evaluated map at $T$: $(J_s/Q/(\gamma^{p^{s-r}} - 1)(J_s/Q))(T) \cong J'_s(T)$ (resp. $J'_s(T) \cong J_s[\gamma^{p^{s-r}} - 1](T)$) is a $U^*(p)$-isomorphism (resp. a $U(p)$-isomorphism).

Proof. We first prove the assertion for $\pi^*$. We note that the category of groups schemes fppf over a base $S$ is a full subcategory of the category of abelian fppf sheaves. We may regard $J'_s/Q$ and $J_s[\gamma^{p^{s-r}} - 1]/Q$ as abelian fppf sheaves over $Q$ in this proof. Since these sheaves are represented by (reduced) algebraic groups over $Q$, we can check being $U(p)$-isomorphism by evaluating the sheaf at a field $K$ of characteristic 0 (e.g., [EAI, Lemma 4.18]). Since the degree of $X_s$ over $X'_s (r \geq \epsilon)$ is a $p$-power, the kernel $K := \text{Ker}(J'_{s/Q} \to J_s/Q[\gamma^{p^{s-r}} - 1])$ is a $p$-abelian group scheme. By Proposition 2.3 (2) applied to $X = X_{s/Q}$ and $Y = X'_{s/Q}$ (with $S = \text{Spec}(Q)$ and $s \geq r$),

$$K := \text{Ker}(J'_{s/Q} \to J_s/Q[\gamma^{p^{s-r}} - 1])$$

is killed by $U(p)^{s-r}$ as $d = p^{s-r} = \text{deg}(X_s/X'_s)$. Thus we get

$$K \subset \text{Ker}(U(p)^{s-r} : J'_{s/Q} \to J_s/Q).$$

Since the category of fppf abelian sheaves is an abelian category (because of the existence of the sheafification functor from presheaves to sheaves under fppf topology described in [ECH, §II.2]), the above inclusion implies the existence of $\iota^*_s$ with $\pi^* \circ \iota^*_s = U(p)^{s-r}$ as a morphism of abelian fppf sheaves. Since the category of group schemes fppf over a base $S$ is a full subcategory of the category of abelian fppf sheaves, all morphisms appearing in the identity $\pi^* \circ \iota^*_s = U(p)^{s-r}$ are morphism of group schemes. This proves the assertion for $\pi^*$.

Take a number field so that $X_s(K) \neq \emptyset$ (for example, the infinity cusp of $X_s$ is rational over $Q(\mu_p)$). Then $\text{Pic}_0^0(J_s/K) \cong J'_s$ for any $s \geq r \geq 0$ by the self-duality of the jacobian variety. Note that the second assertion is the dual of the first under this self-duality; so, over $K$, it can be proven reversing all the arrows and replacing $J_s[\gamma^{p^{s-r}} - 1](J_s)$ as fppf abelian sheaves (resp. $\pi_s, U(p)$) by the quotient $J_s/(\gamma^{p^{s-r}} - 1)J_s$ as fppf abelian sheaves (resp. $\pi_s, U^*(p)$). By Lemma 2.1, every morphism and every abelian variety of the diagram in question are all well defined over $Q$. In particular $J_s/(\gamma^{p^{s-r}} - 1)(J_s)$ is an abelian variety quotient over $Q$ (cf., [NMD, Theorem 8.2.12] combined with [ARG, §V.7]). Then by Galois descent for projective varieties (e.g., [GME, §1.11]), the diagram descends to $Q$. Since being $U^*(p)$-isomorphism or $U(p)$-isomorphism is insensitive to the descent process, we get the final assertion. □

Remark 3.4. For a finite extension $k$ of $Q$ or $Q_l$ and an extension $A/k$ of abelian variety by a finite étale group scheme, recall that $\widehat{A}(k) = \lim_{n} A(k)/p^nA(k)$ and for an infinite Galois extension $\kappa/k$, $\widehat{A}(\kappa) = \lim_{n} A(k)/p^nA(k)$ for $F$ running over all finite Galois extensions $k$ inside $\kappa$ (here note that $\widehat{A}(k)$ is not equal to $A(k) \otimes Z_p$ if $k$ is a finite extension of $Q_l$; see (S) in the introduction). Thus taking projective limit $\lim_{n} A(k)/p^nA(k)$ and then possibly an inductive limit $\lim_{F} \widehat{A}(F)$ for $A = J_r, J'_r$ and $J_s[\gamma^{p^{s-r}} - 1]$ preserves the commutative diagrams (3.8) and (3.9), and the statements Corollary 3.2, (u), (u1), (u2) and Lemma 3.3 are also valid replacing $A$ in each statement by $\widehat{A}$.

4. HECKE ALGEBRAS FOR EXOTIC TOWERS

Hereafter, we fix the data $(\alpha, \delta, \xi)$ which defines the exotic tower $\{X_r\}_r$. We introduce the Hecke algebra $h_{\alpha,\delta,\xi}$ for the exotic tower $\{X_r\}_r$ defined for $(\alpha, \delta, \xi)$. We assume in the rest of the paper the following condition:

(F) The Hecke algebra $h_{\alpha,\delta,\xi}$ is $\Lambda$-free.

In practice, if the local ring $T$ of $h_{\alpha,\delta,\xi}$ we are dealing with is $\Lambda$-free, our argument works. However there is not a good way to confirm directly $\Lambda$-freeness of $T$; so, we assume (F). If $(\alpha, \delta) = (0,1)$ and $\xi(a,d)$ only depends on $d$, this is always true, and as we see in this section, the $\Lambda$-freeness of $h_{\alpha,\delta,\xi}$ holds for $p \geq 5$ without any other assumptions, and even for $p = 3$, for most of $(\alpha, \delta)$ including the self-dual case of $\alpha = \delta = 1$, the $\Lambda$-freeness of $h_{\alpha,\delta,\xi}$ holds still true (see Proposition 18.2).
Let \( \{X_r/\mathbb{Q}\} \) be the exotic tower as in Section 3. As described in (3.3), \( z \in \mathbb{Z}_p^\times \) acts on \( X_r \). Recall that \( J_r/\mathbb{Q} \) (resp. \( J'_r/\mathbb{Q} \)) is the Jacobian of \( X_r \) (resp. \( X'_r \)). We regard \( J_r \) as the degree 0 component of the Picard scheme of \( X_r \). For an extension \( K/\mathbb{Q} \), we consider the group of \( K \)-rational points \( J_r(K) \).

For each prime \( l \), we consider \( \varpi_l := (1 \quad 0 \quad 0 \quad 0) / \text{GL}_{2}(\mathbb{Q}_l) \), and regard \( \varpi_l \in \text{GL}_{2}(\mathbb{A}) \) so that its component at each place \( v \nmid l \) is trivial. Then \( \Delta := \varpi_l^{-1} \Gamma^r \varpi_l \cap \Gamma^r \) gives rise to a modular curve \( X(\Delta) \) whose \( \mathbb{C} \)-points (outside cusps) is given by \( \text{GL}_{2}(\mathbb{Q}) \backslash \text{GL}_{2}(\mathbb{A}^{\infty}) \times (\mathfrak{c} \cup \mathfrak{c}) / \Delta \). We have a projection \( \pi^r_1 : X(\Delta) \to X^r_+ \) given by \( \mathfrak{c} \ni z \mapsto z/l \in \mathfrak{c} \) in addition to the natural one \( \pi_1 : X(\Delta) \to X^r_+ \) coming from the inclusion \( \Delta \subset \Gamma^r \). Then embedding \( X(\Delta) \) into \( X^r_+ \times X^r_+ \) by these two projections, we get the modular correspondence written by \( T(l) \) if \( l \nmid Np \) and \( U(l) \) if \( l|Np \). We can extend this definition to \( T(n) \) for all \( n > 0 \) prime to \( Np \) via Picard/Albanese functoriality (see Lemma 2.1). We use the same symbol \( T(n) \) and \( U(l) \) to indicates the endomorphism (called the Hecke operator) given by the corresponding correspondence \( T(n) \) and \( U(l) \). The Hecke operator \( U(p) \) acts on \( J_r(K) \) and the \( p \)-adic limit \( e = \lim_{n \to \infty} U(p)^n \) is well defined on the Barsotti–Tate group \( J_r[p^\infty] \) and the completed Mordell–Weil group \( \tilde{J}_r(K) \) as defined in (S) above.

Let \( \Gamma \) be the maximal torsion-free subgroup of \( \mathbb{Z}_p^\times \) given by \( 1 + p^s \mathbb{Z}_p \) for \( s > 2 \) if \( p = 2 \) and \( \epsilon = 2 \) if \( p = 2 \). Writing \( \gamma = 1 + p^s \in \Gamma \), \( \gamma \) is a topological generator of the multiplicative group \( \mathbb{Z}_p^\times \). As described in (3.3), \( (u, a, d) \in G = (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \) acts on \( J_r \) through the quotient \( G/H \cong (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \). This action of \( (u, a, d) \in G \), we write as \( (u, a, d) ; \) so, for a prime \( l \nmid Np \), \( (l) = (u, a, d) \), for \( u = (l \mod N) \) and \( a = d = l \) in \( \mathbb{Z}_p^\times \).

Define \( h_r(Z) \) by the subalgebra of \( \text{End}(J_r) \) generated by \( T(n) \) with \( n\mathbb{Z} + Np\mathbb{Z} = 1 \). Put \( h_r(R) = h_r(\mathbb{Z}) \otimes_{\mathbb{Z}} R \) for any commutative ring \( R \). Then we write \( h_r = h_{r, \alpha, \delta, \xi} := e(h_r(Z)) \). The restriction morphism \( h_{s}(\mathbb{Z}) \ni h \mapsto h_{|L} \in h_{s}(\mathbb{Z}) \) for \( s > r \) induces a projective system \( \{h_{r}\}_r \) whose limit gives rise to the big ordinary Hecke algebra

\[ h = h_{r, \alpha, \delta, \xi}(N) := \varinjlim_{r} h_r. \]

Writing \( (l) \) (the diamond operator) for the action of \( l \in (\mathbb{Z}/Np\mathbb{Z})^\times \) identified with \( \text{Gal}(X_r/X_0(Np^r)) \), we have an identity \( l(l)^2 - T(l^2) \in h_r(\mathbb{Z}) \) for all primes \( l \nmid Np \).

Since \( \Gamma \subset \mathbb{Z}_p^\times \subset G/H \), we have a canonical \( \Lambda \)-algebra structure \( \Lambda = \mathbb{Z}_p[[\Gamma]] \leftarrow h \) sending \( \gamma \) to \( (1, a, d) \) for \( a, d \in \Gamma \) such that \( \pi_1(a, d) = \gamma \) as in (3.1). If \( (\alpha, \delta) = (0, 1) = (\alpha', \delta') \), it is now well known that \( h \) is a free of finite rank over \( \Lambda \) and \( h_r = h \otimes_{\Lambda} \Lambda / (\gamma p^{r-s} - 1) \) (cf. [H68], [GK13] or [GME, §3.2.6]). More generally, by [PAF, Corollary 4.31], assuming \( \Delta \geq 5 \), the same facts hold (and we expect this to be true without any assumption on primes). Anyway, if \( p = 2, 3 \), the specialization map \( h \otimes_{\Lambda} \Lambda / (\gamma p^{r-s} - 1) \to h_r \) is onto with finite kernel, and \( h \) is a torsion-free \( \Lambda \)-module of finite type. We will prove the \( \Lambda \)-freeness of \( h_{s, \alpha, \delta, \xi}(N) \) and isomorphisms \( h \otimes_{\Lambda} \Lambda / (\gamma p^{r-s} - 1) \cong h_r \) for most cases of \( p = 3 \) in Section 18 for the sake of completeness.

A prime \( P \) in \( \omega_h := \bigcup_{r > 0} \text{Spec}(h_r)(\overline{\mathbb{Q}_p}) \subset \text{Spec}(h)(\overline{\mathbb{Q}_p}) \) is called an arithmetic point of weight 2 in \( \text{Spec}(h) \). For a connected (resp. irreducible) component \( \text{Spec}(\mathbb{T}) \) (resp. \( \text{Spec}(\mathbb{I}) \)) of \( \text{Spec}(h) \), we put

\[ \omega_T := \omega_h \cap \text{Spec}(\mathbb{T}) \quad \text{(resp. } \omega_I := \omega_h \cap \text{Spec}(\mathbb{I})\text{)}. \]

In this paper, we only deal with the weight of positive weight 2; so, we often omit the word “weight 2” and just call them arithmetic points/primes. Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism \( \lambda : h \to \overline{\mathbb{Q}_p} \) killing \( \gamma p^{r-s} - 1 \) for \( r \geq 0 \) to a classical Hecke eigenform, we need to fix (once and for all) an embedding \( \overline{\mathbb{Q}} \xrightarrow{1_p} \overline{\mathbb{Q}_p} \) of the algebraic closure \( \overline{\mathbb{Q}} \) in \( \mathbb{C} \) into a fixed algebraic closure \( \overline{\mathbb{Q}_p} \) of \( \mathbb{Q}_p \). We write \( i_\infty \) for the inclusion \( \overline{\mathbb{Q}} \subset \mathbb{C} \).

More generally, we consider the Jacobian variety \( J(Z_r) \) of the curve \( Z_r \) defined above (3.3), and define \( h_{r, \alpha, \delta, \xi}^{\text{ord}} \) to be the maximal \( \Lambda \)-algebra direct summand of \( \text{End}(J(Z_r)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \) in which \( U(p) \) is invertible. Then as before we define \( h_{\alpha, \delta, \xi}^{\text{ord}} = h(N)^{\text{ord}} := \varinjlim_{r} h_{r, \alpha, \delta, \xi}^{\text{ord}}, \) which is a \( \Lambda \)-algebra. We consider

\[ h_{\alpha, \delta, \xi}^{\text{ord}, \varphi} := h_{\alpha, \delta, \xi}^{\text{ord}} \otimes_{\mathbb{Z}_p} W/a_p \otimes_{\mathbb{Z}_p} W, \]
where $a_\varphi$ is the kernel of the algebra homomorphism $W[\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times] \to W[\mathbb{Z}_p^\times] \times$ induced by the character $(a, d) \mapsto \varphi(a, d)\xi(a, d) : \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$. If we take $\varphi(a, d) = a^\alpha d^{-\delta}$ for $(a, d) \in \Gamma \times \Gamma$ and $W = \mathbb{Z}_p$ with $\xi : \mu \times \mu \to \mathbb{Z}_p^\times$, we have $h^{n, \varphi, \varphi} = h_{a, \delta, \xi}(N)$ under present notation. Then by [PAF, Corollary 4.31], $h^{n, \varphi, \varphi}$ is $\Lambda$-free of finite rank for $\Lambda = \mathbb{Z}_p[[\mathrm{Im}(\varphi)]]$. In particular, we have

**Proposition 4.1.** Assume $p \geq 5$ or $(\alpha, \delta, \xi) = (0, 1, \omega_d)$, where $\omega_d(a, d) = \omega(d)$. Then $h_{a, \delta, \xi}(N)$ is $\Lambda$-free of finite rank for $\Lambda = \mathbb{Z}_p[[\Gamma^2/H_p]]$.

**Remark 4.2.** For $p \leq 3$, we will prove in Proposition 18.2 freeness over $\Lambda$ of $h_{a, \delta, \xi}(N)$ if it is obtained by systematic twists of $h_{0, 1, \omega_d}(N)$. This covers the interesting cases of analytic families of abelian varieties, including some corresponding to the $p$-adic L-function $k \mapsto L(2k, k)$ as in the introduction.

Picard functoriality gives injective limits $J_\infty(K) = \lim_{\gamma \to t} J_{\gamma}(K)$ and $J_\infty[p^\infty](K) = \lim_{\gamma \to t} J_{\gamma}[p^\infty](K)$, on which $e$ acts. Write $G = G_{\alpha, \delta, \xi} : = e(J_{\infty}[p^\infty](\Lambda))$, which is called the $\Lambda$-adic Barsotti–Tate group in [H14] and whose arithmetic properties are scrutinized there. Adding superscript or subscript “ord”, we indicate the image of $e$.

The compact cyclic group $\Gamma$ acts on these modules by the diamond operators. Thus $J_\infty(K) \underset{\mathrm{ord}}{\rightarrow}$ is a module over $\Lambda := \mathbb{Z}_p[\Gamma] \cong \mathbb{Z}_p[\Gamma]$ by $\gamma \mapsto t = 1 + T$ for a fixed topological generator $\gamma$ of $\Gamma = \gamma_\infty^\varphi$. The big ordinary Hecke algebra $h$ acts on $J_\infty^{\mathrm{ord}}$ as endomorphisms of functors.

Let $\mathrm{Spec}(\mathbb{T})$ be a connected component of $\mathrm{Spec}(\mathbb{h})$ and $\mathrm{Spec}(\mathbb{I})$ be a primitive irreducible component of $\mathrm{Spec}(\mathbb{T})$. We write $\omega_\delta = \bigcup_{r=0}^{\infty} \mathrm{Spec}(\mathbb{I}/(\gamma^p - 1)/\mathbb{Z}[\mathbb{I}]_p)$ (which is the set of all arithmetic points of weight 2). For $P \in \omega_\delta$ with $P \in \mathrm{Spec}(\mathbb{I}/(\gamma^p - 1)/\mathbb{Z}[\mathbb{I}]_p)$, we write $r(P)$ for the minimal $r$ with this property. Then the corresponding Hecke eigenform $f_P$ belongs to $S_\delta(\Gamma_0(Np^r), \epsilon_P \chi)$ for a character $\epsilon_P : \mathbb{Z}_p^\times \to \mathbb{Z}/N\mathbb{Z}$ and a character $\chi : \mu \times (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Here $f$ in this space satisfies

$$f|((a b) c d) = \epsilon_P(a^{-\alpha} d^\alpha) \xi_P(\xi(a), d) \chi(d) f$$

for $(a b c d) \in \Gamma_0(Np^r)$. Here $\chi_P = \chi|_\mu$ and $\chi_N = \chi|_{\mathbb{Z}/N\mathbb{Z}}$. The corresponding adelic form $f$ satisfies

$$f(\alpha x u) = \epsilon_P(a^{-\alpha} d^\alpha) \chi_P(\xi^{-1}(a_p, d_p)) \chi(d_n(u)) f$$

for all $u = (a b c d) \in \Gamma_0(Np^r)$. Here $\chi(d) = \chi_N(d_n^{-1}) = \chi(d(N\infty))$ regarding $d \in \mathbb{A}^\times$.

For each $h$-module $M$, we put $M_\mathbb{T} := M \otimes_h \mathbb{T}$; in particular, $J_\infty^{\mathrm{ord}} := J_\infty^{\mathrm{ord}} \otimes_h \mathbb{T}$ as an fpfp sheaf.

**5. ABELIAN FACTORS OF $J_r$.**

We give a description of abelian factors $A_s$ and $B_s$ of the modular jacobian varieties $\{J_s\}_s$ of the exotic modular tower which behave coherently in the limit process under the Hecke operator action. Let $\pi_{s, r, s} : J_s \to J_r$ for $s > r$ be the morphism induced by the covering map $\pi_{s, r} : X_s \to X_r$ through Albanese functoriality. Then we define $\pi_{s}^\varphi = w_r \circ \pi_{s, r, s} \circ w_s$. Note that $\pi_{s}^\varphi$ is well defined over $\mathbb{Q}$ (cf. Lemma 3.1), and satisfies $T(n) \circ \pi_{s}^\varphi = \pi_{s}^\varphi \circ T(n)$ for all $n$ prime to $Np$ and $U(q) \circ \pi_{s}^\varphi = \pi_{s}^\varphi \circ U(q)$ for all $q|Np$ (as $w_r \circ h \circ w_r^{-1} = h^\varphi$ for $h \in h_r(\mathbb{Z})$ (? = $s, r$) by [MFM, Theorem 4.5.5]).

Let $\mathrm{Spec}(\mathbb{T})$ be a connected component of $\mathrm{Spec}(\mathbb{h}(\mathbb{N}))$. Write $m_\mathbb{T}$ for the maximal ideal of $\mathbb{T}$ and $\Gamma$ for the idempotent of $\mathbb{T}$ in $h$. We assume the following condition

(A) We have $\varpi \in m_\mathbb{T}$ such that $(\varpi) \cap \Lambda$ is a factor of $(\gamma^p - 1)$ in $\Lambda$ and that $\mathbb{T}/(\varpi)$ is of finite rank over $\mathbb{Z}_p$.

We call a prime ideal $P$ satisfying the above condition (A) a principal arithmetic point of $\mathrm{Spec}(\mathbb{T})$. Write $\varpi_s$ for the image of $\varpi \oplus (1 - 1) \in h_s$ ($s \geq r$) and define an $h_s(\mathbb{Z})$-ideal by

$$a_s = (\varpi_s h_s \oplus (1 - e) h_s(\mathbb{Z}_p)) \cap h_s(\mathbb{Z}).$$

Write $A_s$, for the identity connected component of $J_s[a_s] = \bigcap_{a \in a_s} J_s[a]$, and put $B_s = J_s/a_s J_s$, where $a_s J_s$ is a rational abelian subvariety of $J_s$ given by $a_s J_s(\mathbb{T}) = \sum_{a \in a_s} a(J_s(\mathbb{T})) \subset J_s(\mathbb{T})$.

Taking a finite set $G$ of generators of $a_s$, $a_s J_s$ is the image of $a : \bigoplus_{g \in G} J_s \cong \bigoplus_{g \in G} J_s \otimes \mathbb{Q}$, the kernel $J_s[a_s] = \mathrm{ker}(a)$ is a well defined fpfp sheaves, which is represented by an extension of the
Thus $\mathfrak{a}J_s = (\bigoplus_{g \in G} J_s)/\text{Ker}(\mathfrak{a})$ is well defined as an abelian scheme and is the sheaf fppf quotient. Then again $\widehat{B}_s := J_s/\mathfrak{a}J_s$ is the fppf sheaf quotient and also abelian variety quotient again by [NMD, Theorem 8.2.12]. By definition, $A_s$ is stable under $h_s(\mathbb{Z})$ and $h_s(\mathbb{Z})/\mathfrak{a}_s \hookrightarrow \text{End}(A_s)$.

**Lemma 5.1.** Assume (F) and (A). Then we have $\widehat{A}_s^{\text{ord}} = \widehat{J}_s^{\text{ord}}[\varpi]_s$ and $\widehat{J}_s[\mathfrak{a}_s] = \widehat{A}_s$. The abelian variety $A_s \ (s > r)$ is the image of $A_r$ in $J_s$ under the morphism $\pi^* = \pi^*_s, r : J_r \rightarrow J_s$ induced by Picard functoriality from the projection $\pi = \pi_{s, r} : X_s \rightarrow X_r$ and is $\mathbb{Q}$-isogenous to $B_s$. The morphism $J_s \rightarrow B_s$ factors through $J_s \xrightarrow{\pi^*} J_r \rightarrow B_r$. In addition, the sequence

\[
0 \rightarrow \widehat{A}_s^{\text{ord}} \rightarrow \widehat{J}_s^{\text{ord}} \rightarrow \widehat{B}_s^{\text{ord}} \rightarrow 0
\]

is an exact sequence of fppf sheaves.

Passing to the limit, we get the following exact sequence of fppf sheaves:

\[
0 \rightarrow \widehat{A}_s^{\text{ord}} \rightarrow J_s^{\text{ord}} \rightarrow J_r^{\text{ord}} \rightarrow \widehat{B}_s^{\text{ord}} \rightarrow 0,
\]

where $J_s^{\text{ord}} = \lim_s \widehat{J}_s^{\text{ord}}$ and $\widehat{X}_s^{\text{ord}} = \lim_s \widehat{X}_s^{\text{ord}}$ for $X = A, B$.

**Proof.** Taking a finite set $G$ of generators of $\mathfrak{a}_s$ containing $\varpi$, we get an exact sequence $0 \rightarrow J_s[\mathfrak{a}_s] \rightarrow J_s \xrightarrow{x - \overline{g}(x)} \bigoplus_{g \in G} J_s$. Since $X \hookrightarrow \widehat{X}$ as in (S) is left exact, we have $\widehat{A}_s \subset \bigcap_{a \in \mathfrak{a}_s} \widehat{J}_s[a]$ with finite quotient. Applying further the idempotent, since $\mathfrak{a}_s = ((\varpi) + (1 - e)h_s(\mathbb{Z}_p)) \cap h_s(\mathbb{Z})$, we find

\[
\widehat{J}_s[\mathfrak{a}_s]^{\text{ord}} = \bigcap_{a \in \mathfrak{a}_s} \widehat{J}_s[a]^{\text{ord}} = \widehat{J}_s^{\text{ord}}[\varpi]_s.
\]

We have an exact sequence

\[
0 \rightarrow J_s[\mathfrak{a}_s][p^{\infty}]^{\text{ord}} \rightarrow J_s[p^{\infty}]^{\text{ord}} \rightarrow J_s[p^{\infty}]^{\text{ord}} \rightarrow \text{Coker}(\varpi) \rightarrow 0,
\]

and $\text{Coker}(\varpi)_s$ is $p$-divisible and is dual to $J_s[\mathfrak{a}_s][p^{\infty}]^{\text{ord}}$ under the $w_s$-twisted self Cartier duality of $J_s[p^{\infty}]^{\text{ord}}$ (over $\mathbb{Q}$; see [H14, §4]). This shows $\widehat{J}_s[\mathfrak{a}_s][p^{\infty}]^{\text{ord}}$ is $p$-divisible (so, $(\widehat{J}_s[\mathfrak{a}_s]/A_s)^{\text{ord}}$ has order prime to $p$), and hence $\widehat{A}_s^{\text{ord}} = \widehat{J}_s[\mathfrak{a}_s]^{\text{ord}}$.

Plainly by definition, $\pi^*(J_s[\mathfrak{a}_s]) \subset J_s[\mathfrak{a}_s]$. Since we have the following commutative diagram:

\[
\begin{array}{ccc}
J_s(\mathbb{Z}) & \xrightarrow{\varpi} & J_s(\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathbb{Z}(\mathbb{Z})/(\varpi, \mathbb{Z}) \oplus (1 - e)h_s(\mathbb{Z}_p) & \xrightarrow{\pi^*} & \mathbb{Z}(\mathbb{Z})/(\varpi, \mathbb{Z}) \oplus (1 - e)h_r(\mathbb{Z}_p)
\end{array}
\]

we have $\dim A_s = \text{rank}_\mathbb{Z} \mathbb{Z}(\mathbb{Z})/\mathfrak{a}_s = \text{rank}_\mathbb{Z} \mathbb{Z}(\mathbb{Z})/\mathfrak{a}_s = \dim \mathbb{A}_s$; so, $A_s = \pi^*(A_r)$.

The above commutative diagram also tells us that $\mathfrak{a}_s \supset b_s := \text{Ker}(h_s(\mathbb{Z}) \rightarrow h_r(\mathbb{Z}))$ in $h_s(\mathbb{Z})$. Thus the projection $J_s \rightarrow J_s/\mathfrak{a}_s J_s = B_s$ factors through $J_r = J_s/b_s J_s$. Indeed, the natural projection: $J_s/b_s J_s/\mathbb{Q} \rightarrow J_r/\mathbb{Q}$ has to be a finite morphism (as the tangent space at the origin of the two are isomorphic), and we conclude $J_s/b_s J_s = J_r$ by the universality of the categorical quotient $J_s/\mathfrak{a}_s J_s$ (cf., [NMD, page 219]).

Assuming $J_s(K) \neq \emptyset$ for a finite extension $K/\mathbb{Q}$, the dual sequence (over $K$) of the exact sequence of fppf sheaves: $0 \rightarrow J_s[\mathfrak{a}_s] \rightarrow J_s \xrightarrow{x - \overline{g}(x)} \bigoplus_{g \in G} J_s$ is

\[
\bigoplus_{g \in G} J_s \xrightarrow{x - \overline{\sum} g(x)} J_s \rightarrow B_s \rightarrow 0.
\]

Thus $A_s$ is isogenous to $B_s$ over $K$, and by Galois descent, $A_s$ is $\mathbb{Q}$-isogenous to $B_s$. Indeed, for the complementary abelian subvariety $A_s^\perp$ in $J_s$ of $A_s$, we have $J_s/A_s^\perp = B_s$, and the $\mathbb{Q}$-isogeny follows without taking duality. Here note that the quotient $J_s/A_s^\perp$ exists as an abelian variety and also as an fppf sheaves by [NMD, Theorem 8.2.12] (and [ARG, V.7]).
As explained just below (A), we have \( \text{Im}(\bigoplus_{g \in G} J_s \xrightarrow{x \mapsto \sum g(x)} J_s) = a_s J_s \) as fppf sheaves. Then applying the argument of [H15, Section 1] to the exact sequence

\[
0 \to J_s[a_s] \to \bigoplus_{g \in G} J_s \to aJ_s \to 0
\]

of fppf sheaves, we confirm the exactness of

\[
0 \to \hat{J}_s[a_s] \to \bigoplus_{g \in G} \hat{J}_s \to \hat{a}J_s \to 0
\]

as fppf sheaves. Thus applying the idempotent \( e \), we confirm

\[
\text{Im}(\bigoplus_{g \in G} \hat{J}_s \xrightarrow{x \mapsto \sum g(x)} \hat{J}_s) = a_s J_s[ord].
\]

Since the morphism \( \bigoplus_{g \in G} \hat{J}_s \xrightarrow{x \mapsto \sum g(x)} \hat{J}_s \) factors through \( \varpi_s(\hat{J}_s[ord]) \) as all \( g = \varpi_s x \) with \( x \in \mathfrak{h}_s \), noting \( \varpi \in G \), \( \varpi(\hat{J}_s[ord]) \hookrightarrow a_s(\hat{J}_s[ord]) \). Thus \( a_s(\hat{J}_s[ord]) = \varpi_s(\hat{J}_s[ord]) \) as fppf sheaves. This shows the exactness:

\[
0 \to \hat{A}_s[ord] \to \hat{J}_s[ord] \xrightarrow{\varpi_s} a_s \hat{J}_s = \varpi_s(\hat{J}_s[ord]) \to 0.
\]

Since \( B_s = J_s/a_s J_s \) as fppf sheaves, we see the exactness of

\[
0 \to \varpi_s(\hat{J}_s[ord]) \to \hat{J}_s[ord] \to \hat{B}_s[ord] \to 0
\]

as fppf sheaves. Combining the two exact sequences, we obtain the exactness of the last sequence in the lemma. \( \square \)

Assuming \( X_s(K) \neq \emptyset \), \( J_s \cong \text{Pic}^0_{J_s/K} \) via the polarization of the canonical divisor (e.g., [ARG, VII.6]). The Rosati involution \( h \mapsto h^* \) and \( T(n) \mapsto T^*(n) \) brings \( h_r(\mathbb{Z}) \) to \( h_r(\mathbb{Z}) \subset \text{End}(J_r/K) \). At the level of double coset operator \( [\Gamma \alpha \Gamma'] \), the involution has the following effect \( [\Gamma \alpha \Gamma']^* = [\Gamma \alpha \Gamma] \). Because of this fact, the involution \( h \mapsto h^* \) itself is well defined giving \( h_r(\mathbb{Z}) \cong h_r(\mathbb{Z})^* \) in \( \text{End}(J_r/Q) \) (even if \( X_r(Q) = \emptyset \)). Note that \( X_1(Np^r)(Q) \) contains the infinity cusp; so, for the standard tower, we have \( X_r(Q) \neq \emptyset \).

The Weil involution \( w_s = [\Gamma_s \left( \begin{smallmatrix} 0 & -1 \\ Np & 0 \end{smallmatrix} \right) \Gamma_s] \) has the effect that \( w_s[\Gamma_s \alpha \Gamma_s] = [\Gamma_s \alpha \Gamma_s]w_s \) as easily verified. Thus \( w_s \circ T^*(n) = T(n) \circ w_s \) for all \( n \) including \( T(l) = U(l) \) for \( l \mid Np \). We write \( \{X^r_s\}_{s,r} \) for the dual tower corresponding to \( \{(\hat{\Gamma}_s)^r = w_s \hat{\Gamma}_s w_s^{-1}\}_{s,r} \) with the main involution \( \iota \) given by \( x \cdot x = \det(x) \). Thus \( \{X^r_s\}_{s,r} \) corresponds to the triple \((-\alpha, -\delta, \xi')\) for \( \xi'(a,d) = \xi(d,a) \), and the \( l \)-component of \( (\hat{\Gamma}_s)^r \) for \( l \mid N \) is given by

\[
\{(a \ b \ c) \in \text{GL}_2(\mathbb{Z}_l) \mid c \in N\mathbb{Z}_l, a-1 \in N\mathbb{Z}_l\}.
\]

Then \( w_s \) gives an isomorphism \( w_r : X^*_s \to X^{*r}_s \) defined over \( \mathbb{Q} \). Note that the fixed isomorphism \( \mu_{p^r} \cong \mathbb{Z}/p^r\mathbb{Z} \) \((\zeta_{p^r} \mapsto 1)\) induces an isomorphism \( X^{*r}_s \cong X^*_s \) over \( \mathbb{Q}(\mu_{Np^r}) \). As an automorphism of \( X^r_s/Q[\mu_{p^r}] \), \( w_s \) satisfies \( w([z,q]) = [z] \circ w_s = w_s \circ [z]^{-1} \) for \( z \in \mathbb{Z}^\times \) (see Lemma 3.1).

Take a connected component \( \text{Spec}(\mathbb{T}) \) of \( \text{Spec}(h) \) and an irreducible component \( \text{Spec}(\mathbb{I}) \) of \( \text{Spec}(\mathbb{T}) \). Assume that \( \mathbb{I} \) is primitive in the sense of [H86a, Section 3]. For each arithmetic \( P \in \text{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p) \), the corresponding cusp form \( f_P \) is a \( p \)-stabilized Hecke eigenform of weight 2 new at each prime \( l \mid N \) if and only if \( \mathbb{I} \) is primitive.

We get directly from Lemma 5.1 the following proposition giving sufficient conditions for the validity of (A) for \( A_{f,s} \) when \( f = f_P \) is in a \( p \)-adic analytic family indexed by \( P \in \text{Spec}(\mathbb{I}) \).

**Proposition 5.2.** Let \( \text{Spec}(\mathbb{T}) \) be a connected component of \( \text{Spec}(h) \) and \( \text{Spec}(\mathbb{I}) \) be a primitive irreducible component of \( \text{Spec}(\mathbb{T}) \). Assume \( \Lambda \)-freeness of \( T \) (i.e., (F)). Then the condition (A) holds for the following choices of \( (\varpi, A_s, B_s) \):
(1) Suppose that an even line form \( f = f_P \) new at each prime \( p \) belongs to \( \text{Spec}(\mathbb{T}) \) and that \( \mathbb{T} = \mathbb{F} \) is regular (or more generally a unique factorization domain). Then writing the level of \( f_P \) as \( N \mathbb{P} \), the algebra homomorphism \( \lambda : \mathbb{T} \rightarrow \mathbb{Q} \) given by \( fT(l) = \lambda(T(l))f \) gives rise to the prime ideal \( P = \text{Ker}(\lambda) \). Since \( P \) is of height 1, it is principal generated by \( \mathfrak{w} \in \mathbb{T} \). This \( \mathfrak{w} \) has its image \( a_{\mathfrak{w}} \in \mathbb{T} = \mathbb{T} \otimes_{\mathbb{A}} \mathbb{A} \), for \( \mathbb{A} = \mathbb{A}(\mathbb{F}^p, -1) \). Write \( \mathfrak{h}_s = \mathfrak{h} \otimes_{\mathbb{A}} \mathbb{A}_s = \mathbb{T}_s \oplus 1, \mathfrak{h}_s \) as an algebra direct sum for an idempotent \( 1_s \). Then, the element \( \mathfrak{w}_s = a_{\mathfrak{w}} \oplus 1_s \in \mathfrak{h}_s \) for the identity \( 1_s \) of \( \mathbb{T}_s \) satisfies (A).

(2) Fix \( r > 0 \). Then \( \mathfrak{w} \in \mathfrak{m}_r \) for a factor \( \mathfrak{w}(\mathbb{F}^p, -1) \in \mathbb{A} \), satisfies (A).

Here is a criterion from [F02, Theorem 3.1] for regularity of \( \mathbb{T} \):

**Theorem 5.3.** Assume \( \Lambda \)-freeness of \( \mathfrak{h}_{\alpha, \delta, \xi} \). Let \( f \) be a Hecke eigenform of conductor \( N \), of weight 2 and with Neben character \( \chi \), and define \( a_{\mathfrak{w}} \in \mathbb{T} \) by \( fT(p) = a_{\mathfrak{w}}f \). Let \( p \) be a prime outside \( 6D_{2N} \mathbb{F}(N) \) (for \( \mathbb{F}(N) = \{ \mathbb{F}/(Z[N\mathbb{Z}]/\mathbb{Z}) \} \)). Suppose that for the prime ideal \( p \) of \( \mathbb{Z}[a_{\mathfrak{w}}] \) induced by \( i_p : \mathbb{T} \rightarrow \mathbb{T} \), \( (a_{\mathfrak{w}} \mod p) \) is different from 0 and \( \pm \sqrt{\chi(p)} \). Then for the connected component \( \text{Spec}(\mathbb{T}) \) of \( \text{Spec}( \mathfrak{h}_{\alpha, \delta, \xi} \mathbb{Z}/\mathbb{P} \) acting non-trivially on the \( p \)-stabilized Hecke eigenform corresponding to \( f \) in \( S_2(\Gamma_0(Np), \chi) \), \( \mathbb{T} \) is a regular integral domain isomorphic to \( W \otimes_{\mathbb{Z}} \mathbb{A} = \mathbb{W}[\mathbb{T}] \) for a complete discrete valuation ring \( W \) unramified at \( p \).

The result is valid always for \( p \geq 5 \) and for \( p = 3 \) under (F) (see Propositions 4.1 and 18.2). Here is a proof of this fact since [F02, Theorem 3.1] is slightly different from the above theorem.

**Proof.** Let \( c_0 := \lim_{n \rightarrow \infty} T(p) = h_2(\Gamma_0(N), \chi, A) \) for \( \mathbb{Z}[\chi] \)-algebra \( A \). Put \( h^\text{ord}_2(\Gamma_0(N), \chi, A) := c_0 h_2(\Gamma_0(N), \chi, A) \). Since \( U(p) \equiv T(p) \mod \mathbb{A} \), the natural algebra homomorphism:

\[
h^\text{ord}_2(\Gamma_0(Np), \chi, A) \rightarrow h^\text{ord}_2(\Gamma_0(N), \chi, A)
\]

sending \( U(p) \) to the unit root of \( X^2 - T(p)X + \chi(p)p \in h^\text{ord}_2(\Gamma_0(N), \chi, A)[X] \) and \( T(l) \) to \( T(l) \) for all primes \( l \neq p \) is a well defined surjective \( A \)-algebra homomorphism.

Since \( p \not| 6D_{2N} \mathbb{F}(N) \), we have \( p > 3 \) and \( p \not| \varphi(Np) \). Write \( h \) for \( \mathfrak{h}_{\alpha, \delta, \xi}(N) \). Then \( h \) is \( \Lambda \)-free by (F) and an exact control is valid (see Propositions 4.1 and 18.2). By the diamond operators \( z \) for \( z \in (\mathbb{Z}/N\mathbb{Z})^\times \), \( h \) is an algebra over \( \mathbb{Z}_p[[\mathbb{Z}/N\mathbb{Z}]^\times] \). We can decompose \( h = \oplus \psi(h) \) so that the diamond operator \( z \) acts by \( \psi(z) \) on \( h \), where \( \psi \) runs over all even characters of \( (\mathbb{Z}/N\mathbb{Z})^\times \). From the exact control \( h/Th \cong h \), \( (T = \gamma - 1 \in \Lambda) \), we thus get

\[
h(\chi)/Th(\chi) \cong h^\text{ord}_2(\Gamma_0(Np), \chi, \mathbb{Z}[\chi]) =: h
\]

for the character \( \chi \) of \( (\mathbb{Z}/N\mathbb{Z})^\times \), where

\[
h_2(\Gamma_0(Np), \chi, \mathbb{Z}[\chi]) = h_2(\Gamma_1(Np), \chi, \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]^\times} \mathbb{Z}_p[\chi]
\]

and \( \mathbb{Z}_p[\chi] \) is the \( \mathbb{Z}_p \)-subalgebra of \( \mathbb{Q}_p \) generated by the values of \( \chi \). Here the tensor product is with respect to the algebra homomorphism \( \mathbb{Z}_p[[\mathbb{Z}/N\mathbb{Z}]^\times] \rightarrow \mathbb{Z}_p[\chi] \) induced by \( \chi \). Writing \( \Sigma = \text{Hom}_{\text{alg}}(h(\chi), \mathbb{Q}_p) \), for each \( \lambda \in \Sigma \), \( \Sigma = \{ m_{\lambda} = \text{Ker}(\lambda) | \lambda \in \Sigma \} \) is the set of all maximal ideals of \( h(\chi) \). Thus we have compatible decompositions \( h(\chi) = \bigoplus_{m_{\lambda} \in \Sigma} h(\chi)_m \) and \( \mathbb{Z}_p[\chi] \) (see [BCM, III.4.6]). Here the subscript “\( m \)” indicates the localizations at the maximal ideal \( m \).

Identify \( \Sigma \) with \( \text{Hom}_{\text{alg}}(h, \mathbb{Q}_p) \). Write \( \mathfrak{A}^\circ \) for the subset of \( \Sigma = \text{Hom}_{\text{alg}}(h, \mathbb{Q}_p) \) made of \( \lambda \)'s such that there exists

\[
\lambda^\circ \in \text{Hom}_{\text{alg}}(h^\text{ord}_2(\Gamma_0(N), \chi, \mathbb{F}_p[\chi]), \mathbb{F}_p)
\]

with \( \lambda(T(l)) = \lambda^\circ(T(l)) \) for all primes \( l \neq p \). Here we put

\[
h^\text{ord}_2(\Gamma_0(N), \chi, \mathbb{F}_p[\chi]) := h^\text{ord}_2(\Gamma_0(N), \chi, \mathbb{Z}_p[\chi]) \otimes_{\mathbb{Z}} \mathbb{F}_p.
\]

Accordingly let \( \mathfrak{A}^\circ \) denote the set of maximal ideals corresponding to \( \lambda \in \mathfrak{A}^\circ \). Since \( p \)-new forms in \( S_2(\Gamma_0(Np), \chi) \) have \( U(p) \)-eigenvalues \( \pm \sqrt{\chi(p)} \) (see [MFM, Theorem 4.6.17]), by \( a_{\mathfrak{w}} \not| \pm \sqrt{\chi(p)} \), we have further decomposition \( h = h_N \oplus h' \) so that \( h_N \) is the direct sum of \( h_m \) for \( m \) running over \( \mathfrak{A}^\circ \). Since \( h(\chi)/Th(\chi) \cong h \), by Hensel’s lemma (e.g., [BCM, III.4.6]), we have a unique algebra decomposition \( h(\chi) = h_N \oplus h' \) so that \( h_N/Th = h_N \) and \( h'/Th = h' \).
Since \( T(p) \equiv U(p) \mod (p) \) in \( h_N \), we get \( h_N \cong h_N^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{Z}_p[\chi]) \). Since \( p \nmid D_\chi \), the reduction map modulo \( p \): \( \text{Hom}_{\text{alg}}(h, \overline{\mathbb{Q}}_p) \to \Sigma \) is a bijection. In particular, we have \( h = h^{\text{new}} \oplus h^{\text{old}} \) where \( h^{\text{new}} \) is the direct sum of \( h_{\lambda} \) for \( \lambda \) coming from the eigenvalues of \( N \)-primitive forms. Again by Hensel’s lemma, we have the algebra decomposition \( h_N = h^{\text{new}} \oplus h^{\text{old}} \) with \( h^*/T h^* = h^* \) for \( \ell = \text{new, old} \). Since \( h^{\text{new}} \) is reduced by the theory of new forms ([H86a, §6] and [MFM, §4.6]) and unramified over \( Z_p \) by \( p \nmid D_\chi \langle N \rangle \), we conclude \( h^{\text{new}} \cong \bigoplus W W \) for discrete valuation rings \( W \) finite unramified over \( Z_p \) (one of the direct summand \( \hat{W} \) acts on \( f \) non-trivially; i.e., \( W \), given by \( \mathbb{Z}_p[f] = \mathbb{Z}_p[n=1,2,\ldots] \subset \overline{\mathbb{Q}}_p \) for \( (n) \)-eigenvalues \( a_n \) of \( f \) ). Thus again by Hensel’s lemma, we have a unique algebra direct factor \( T \) of \( h^{\text{new}} \) such that \( T/T \hat{T} = \mathbb{Z}_p[f] = W \). Since \( W \) is unramified over \( Z_p \), by the theory of Witt vectors [BCM, IX.1], we have a unique section \( W \to T \) of \( T \to \mathbb{Z}_p[f] = W \). Then \( W[[T]] \subset T \) which induces a surjection after reducing modulo \( T \). Then by Nakayama’s lemma, we have \( \hat{T} = W[[T]] = W \otimes_{\mathbb{Z}_p} \Lambda \) as desired. \( \square \)

6. Limit abelian factors

We recall some elementary but useful facts (e.g. [H15, §6, after (6.6)]) with their proof. Let \( \iota : C_{r/q} \to J_{r/q} \) (resp. \( \pi : J_{r/q} \to D_{r/q} \)) be an abelian subvariety (resp. an abelian variety quotient) stable under Hecke operators (including \( U(l) \) for \( l \mid Np \)) and \( w_r \). Here the stability means that \( \text{Im}(\iota) \) and \( \text{Ker}(\pi) \) are stable under Hecke operators. Then \( \iota \) and \( \pi \) are Hecke equivariant. Let \( \iota_s : C_s := \pi_{s, r}^\ast(C) \subset J_s \) for \( s > r \) and \( D_s \) is the quotient abelian variety of \( \pi_s : J_s \to D_r \), where \( \pi_s^\ast = w_r \circ \pi_{s, r} \circ w_s \). The twisted projection \( \pi_{s, r}^\ast \) is rational over \( \mathbb{Q} \) as \( w_s[z, \Omega] = \langle z \rangle \circ w_s = w_s \circ \langle z \rangle^{-1} \) for \( z \in \hat{\mathbb{Z}}^\times \).

Since the two morphisms \( J_r \to J_s^\ast \to J_s[\gamma^{p^{s-r}} - 1] \) (Picard functoriality) are \( U(p) \)-isomorphism of fppf abelian sheaves by (u1) and Corollary 3.2, we get the following two isomorphisms of fppf abelian sheaves for \( s > r > 0 \):

\[
C_{r}[p^\infty]^{\text{ord}} \xrightarrow{\sim} C_s[p^\infty]^{\text{ord}} \quad \text{and} \quad \hat{C}_{r}^{\text{ord}} \xrightarrow{\sim} \hat{C}_s^{\text{ord}},
\]

since \( \hat{C}_{r}^{\text{ord}} \) is the isomorphic image of \( \hat{C}_r \subset J_r \) in \( J_s[\gamma^{p^{s-r}} - 1] \). By the \( w \)-twisted Cartier duality [H14, §4], we have

\[
D_s[p^\infty]^{\text{ord}} / \hat{C}_r \xrightarrow{\sim} D_r[p^\infty]^{\text{ord}}.
\]

This isomorphism (6.2) is over \( \mathbb{Q} \) not over the discrete valuation ring \( \mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q} \) as explained in [H16a, §11] (after the proof of Proposition 11.3 in [H16a]), but the isomorphism (6.1) is often valid over \( \mathbb{Z}_{(p)} \) (see the argument in Section 17). Thus by Kummer sequence, we have the following commutative diagram

\[
\begin{array}{ccc}
\hat{D}_{s}^{\text{ord}}(k) \otimes \mathbb{Z}/p^m\mathbb{Z} & \xrightarrow{\pi_{s}^{\ast}} & H^1(D_{s}[p^m]^{\text{ord}}) \\
\downarrow{\iota} & & \downarrow{\iota} \\
\hat{D}_{r}^{\text{ord}}(k) \otimes \mathbb{Z}/p^m\mathbb{Z} & \xrightarrow{\pi_{r}^{\ast}} & H^1(D_{r}[p^m]^{\text{ord}})
\end{array}
\]

This shows

\[
\hat{D}_{s}^{\text{ord}}(k) \otimes \mathbb{Z}/p^m\mathbb{Z} \cong \hat{D}_{r}^{\text{ord}}(k) \otimes \mathbb{Z}/p^m\mathbb{Z}.
\]

Passing to the projective/injective limit, we get

\[
\hat{D}_{s}^{\text{ord}} \xrightarrow{\sim} \hat{D}_{r}^{\text{ord}} \quad \text{and} \quad (D_s \otimes \mathbb{Z} T_p)^{\text{ord}} \xrightarrow{\sim} (D_r \otimes \mathbb{Z} T_p)^{\text{ord}}
\]
as fppf abelian sheaves. In short, we get

**Lemma 6.1.** Suppose that \( k \) is a field extension of finite type of either a number field or a finite extension of \( \mathbb{Q} \). Then we have the following isomorphism

\[
\hat{C}_{r}(k)^{\text{ord}} \xrightarrow{\sim} \hat{C}_s(k)^{\text{ord}} \quad \text{and} \quad \hat{D}_{s}(k)^{\text{ord}} \xrightarrow{\sim} \hat{D}_{r}(k)^{\text{ord}}
\]
for all \( s > r \) including \( s = \infty \).

Taking \( C_s \) to be \( A_s \) (and hence \( D_s = B_s \) by Lemma 5.1) and applying this lemma to the exact sequence (5.1), we get a new exact sequence (for \( \varpi \) in (A)):

\[
0 \to \hat{A}_r^{\text{ord}} \to J_{-r}^{\text{ord}} \xrightarrow{\pi_r} J_r^{\text{ord}} \to \hat{B}_r^{\text{ord}} \to 0,
\]

since \( \hat{A}_r^{\text{ord}} = \lim_{s \to r} \hat{A}_s^{\text{ord}} \cong \hat{A}_r^{\text{ord}} \) by the lemma.

We make \( \hat{B}_r^{\text{ord}} \) explicit. By computation, \( \pi_r^* \circ \pi^*_r = p^{s-r}U(p^{s-r}) \). To see this, as Hecke operators from \( \Gamma_s \)-coset operations, we have \( \pi^*_r = [\Gamma_s] \) (restriction with respect to \( \Gamma_r / \Gamma_s \)) and \( \pi^*_r \circ \pi^*_r = [\Gamma_r] \) (trace map with respect to \( \Gamma_r / \Gamma_s \)). Thus we have

\[
\pi_r^* \circ \pi^*_r = [\Gamma_s] \cdot [\Gamma_r] = [\Gamma_s \cdot [w_s w_r] \cdot [\Gamma_r] = [\Gamma_r : \Gamma_s] [\Gamma_r \left( \begin{smallmatrix} 1 & 0 \\ 0 & p^{s-r} \end{smallmatrix} \right) \Gamma_r] = p^{s-r}U(p^{s-r}).
\]

**Lemma 6.2.** We have the following two commutative diagrams for \( s' > s \)

\[
\begin{array}{ccc}
\hat{C}_s^{\text{ord}} & \xrightarrow{\sim} & \hat{C}_s^{\text{ord}} \\
\pi_r^* \downarrow & & \downarrow p^{s-r}U(p^{s-r}) \\
\hat{C}_s^{\text{ord}} & \xrightarrow{\sim} & \hat{C}_s^{\text{ord}}
\end{array}
\]

and

\[
\begin{array}{ccc}
\hat{D}_s^{\text{ord}} & \xrightarrow{\sim} & \hat{D}_s^{\text{ord}} \\
\pi_r^* \downarrow & & \downarrow p^{s-r}U(p^{s-r}) \\
\hat{D}_s^{\text{ord}} & \xrightarrow{\sim} & \hat{D}_s^{\text{ord}}
\end{array}
\]

In particular, we get \( \hat{D}_r^{\text{ord}} := \lim_{s \to \infty} \hat{D}_s^{\text{ord}} = \hat{D}_s^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \).

**Proof.** By \( \pi^*_r \) (resp. \( \pi^*_s \)), we identify \( \hat{C}_s^{\text{ord}} \) with \( \hat{C}_r^{\text{ord}} \) (resp. \( \hat{D}_s^{\text{ord}} \) with \( \hat{D}_r^{\text{ord}} \)) as in Lemma 6.1. Then the above two diagrams follow from (6.5).

For a free \( \mathbb{Z}_p \)-module \( F \) of finite rank, we suppose to have a commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{p^n} & F \\
\| & & \| \\
F & \xrightarrow{p^{-n}} & p^{-n}F
\end{array}
\]

Thus we have \( \lim_{n,x \to p^n x} F = \lim_{n,x \to p^n x} p^{-n}F \cong F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). If \( T \) is a torsion \( \mathbb{Z}_p \)-module with \( p^{B}T = 0 \) for \( B > 0 \), we have \( \lim_{n,x \to p^n x} T = 0 \). Thus for general \( M = F \oplus T \), we have \( \lim_{n,x \to p^n x} M \cong M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \).

Identifying \( \hat{D}_s^{\text{ord}} \) with \( \hat{D}_r^{\text{ord}} \) by \( \pi_r^* \) for all \( s \geq r \), the transition map of the inductive limit \( \lim_s \hat{D}_s^{\text{ord}} \) is given by

\[
\begin{array}{ccc}
\hat{D}_s^{\text{ord}} & \longrightarrow & \hat{D}_r^{\text{ord}} \\
\downarrow i & & \downarrow i \\
\hat{D}_r^{\text{ord}} & \longrightarrow & \hat{D}_r^{\text{ord}}
\end{array}
\]

Thus applying the above result for \( M = \hat{D}_s^{\text{ord}}(K) \), we find \( \lim_s \hat{D}_s^{\text{ord}}(K) = \hat{D}_r^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \).

Applying this lemma to \( D_s = B_s \), we get from (6.4), the following exact sequence:
Corollary 6.3. Assume $\Lambda$-freeness of $\mathfrak{h}, \delta$. Let $K$ be either a number field or a finite extension of $\mathbb{Q}_l$ for a prime $l$. For $(\varpi, A_r, B_r)$ satisfying (A), we get the following natural 4-term exact sequence of étale sheaves over $\text{Spec}(K)$:

$$0 \to \widehat{A}_r^{\text{ord}} \to J_{\infty}^{\text{ord}} \cong J_{\infty}^{\text{ord}} \to \widehat{B}_r^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$ 

In particular, for $K' = K^S$ if $K$ is a number field and $K' = K$ if $K$ is local, we have the following exact sequence of Galois modules:

$$0 \to \widehat{A}_r^{\text{ord}}(K') \to J_{\infty}^{\text{ord}}(K') \cong J_{\infty}^{\text{ord}}(K') \to \widehat{B}_r^{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$ 

Proof. Since a finite étale extension $R$ of $K$ is a product of finite field extensions of $K$, we may assume that $R$ is a field extension of $K$. Then by (S), $\widehat{B}_s(R)^{\text{ord}} \cong \widehat{B}_r(R)^{\text{ord}}$ is a $\mathbb{Z}_p$-module of finite type. Then by the above lemma Lemma 6.2, taking $D_s$ to be $B_s$, we find that $\lim_{s \to s} \widehat{B}_s(R)^{\text{ord}} = \widehat{B}_r^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Since passing to injective limit is an exact functor, this proves the first exact sequence:

$$0 \to \widehat{A}_r^{\text{ord}} \to J_{\infty}^{\text{ord}} \cong J_{\infty}^{\text{ord}} \to \widehat{B}_r^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$ 

Since $\widehat{X}^{\text{ord}}(K') = \varprojlim_{K'/F} \widehat{X}^{\text{ord}}(F)$ for $F$ running a finite extension of $K$, we get the exactness of $0 \to \widehat{A}_r^{\text{ord}}(K') \to J_{\infty}^{\text{ord}}(K') \cong J_{\infty}^{\text{ord}}(K')$. 

Since

$$0 \to \widehat{A}_r^{\text{ord}}(K') \to J_{\infty}^{\text{ord}}(K') \cong J_{\infty}^{\text{ord}}(K') \to \widehat{B}_r^{\text{ord}}(K') \to 0$$

is an exact sequence of Galois modules, passing to the limit, we still have the exactness of

$$0 \to \widehat{A}_r^{\text{ord}}(K') \to J_{\infty}^{\text{ord}}(K') \cong J_{\infty}^{\text{ord}}(K') \to \widehat{B}_r^{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$ 

Note here that $\widehat{B}_r^{\text{ord}}(K')$ is $p$-divisible, and hence

$$\widehat{B}_r^{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \widehat{B}_r^{\text{ord}}(K') / \widehat{B}_r^{\text{ord}}[p^{\infty}](K') \cong \widehat{A}_r^{\text{ord}}(K') / \widehat{A}_r^{\text{ord}}[p^{\infty}](K'),$$

and we have the sheaf identity: $\widehat{B}_r^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \widehat{B}_r^{\text{ord}} / \widehat{B}_r^{\text{ord}}[p^{\infty}] \cong \widehat{A}_r^{\text{ord}}(K') / \widehat{A}_r^{\text{ord}}[p^{\infty}].$ 

Corollary 6.4. We have a sheaf isomorphism $i: (\widehat{A}_r^{\text{ord}} \oplus \varpi(J_r^{\text{ord}})) / \widehat{A}_r^{\text{ord}}[p^{\infty}] \cong J_r^{\text{ord}}$ with $\widehat{A}_r^{\text{ord}} \cong \widehat{A}_r^{\text{ord}}$, where $i(a + x) = a + x$ for $a \in \widehat{A}_r^{\text{ord}}$ and $x \in \varpi(J_r^{\text{ord}})$ and $a \in \widehat{A}_r^{\text{ord}}[p^{\infty}]$ is sent to $(a + -a) \in \widehat{A}_r^{\text{ord}} \oplus \varpi(J_r^{\text{ord}}).$ In particular, for a finite field extension $k$ of $\mathbb{Q}$ or $\mathbb{Q}_l$, the $p$-torsion module $\text{Coker}(J_{\infty}^{\text{ord}}(k) \to B_r(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is isogenous to $A^{\text{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \varpi$. 

Proof. Consider the composite $f_s$ of the inclusion $\widehat{A}_r^{\text{ord}} \hookrightarrow J_r^{\text{ord}}$ and the projection: $\widehat{J}_r^{\text{ord}} \twoheadrightarrow \widehat{B}_r^{\text{ord}}$. Since $\widehat{A}_r^{\text{ord}} \cap \varpi(J_r^{\text{ord}}) \cong \text{Ker}(\widehat{A}_r^{\text{ord}} \to \widehat{B}_r^{\text{ord}})$, we may think $\text{Ker}(f_s)$ also as a group subscheme of $\varpi(J_r^{\text{ord}})$. Since we have a commutative diagram:

\[
\begin{array}{ccc}
\widehat{A}_r^{\text{ord}} & \xrightarrow{f_s} & \widehat{B}_r^{\text{ord}} \\
\uparrow{\pi^*_s \circ f_r} & & \downarrow{\pi^*_r} \\
\widehat{A}_r^{\text{ord}} & \xrightarrow{f_r} & \widehat{B}_r^{\text{ord}},
\end{array}
\]

we have $\text{Ker}(f_s) \hookrightarrow \widehat{A}_r^{\text{ord}}[p^{s-r}] \cong \widehat{A}_r^{\text{ord}}[p^{s-r}]$ (from $\pi^*_s \circ \pi^*_r = p^{s-r}U(p)^{s-r}$ whose cokernel is bounded by $\text{Ker}(f_r)$. Passing to the limit, we find

\begin{equation}
\widehat{A}_r^{\text{ord}}[p^{\infty}] \cong \widehat{A}_r^{\text{ord}}[p^{\infty}] = \varprojlim_s \text{Ker}(f_s) = \widehat{A}_r^{\text{ord}} \cap \varpi(J_r^{\text{ord}})
\end{equation}

inside $J_r^{\text{ord}}$. Since we have the sheaf identity $\widehat{A}_r^{\text{ord}} / \widehat{A}_r^{\text{ord}}[p^{\infty}] = \widehat{A}_r^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \widehat{B}_r^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $J_r^{\text{ord}} / \varpi(J_r^{\text{ord}}) = \widehat{B}_r^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, the morphism of sheaves $\widehat{A}_r^{\text{ord}} \oplus \varpi(J_r^{\text{ord}}) \supseteq (a \oplus x) \mapsto a + x$ in $J_r^{\infty}$ is an epimorphism of sheaves. This shows the first assertion.
As for the last assertion, we consider the commutative diagram of $\mathbb{Z}_p$-modules with exact rows:

$$
\begin{array}{ccc}
\tilde{A}_r^{\text{ord}}(k) & \xrightarrow{\varphi_s} & \tilde{J}_s^{\text{ord}}(k) & \xrightarrow{\varphi_s} & \text{Coker}(i_s) \\
\downarrow f_r & & \downarrow \rho_s & & \downarrow \rho_s' \\
\tilde{B}_r^{\text{ord}}(k) & \xrightarrow{p^{s-r}U(p^{s-r})} & \tilde{B}_r^{\text{ord}}(k) & \xrightarrow{p^{s-r}U(p^{s-r})} & \tilde{B}_r^{\text{ord}}(k)/p^{s-r}\tilde{B}_r^{\text{ord}}(k),
\end{array}
$$

where $\rho_s$ is the projection map $\tilde{J}_s^{\text{ord}} \rightarrow \tilde{B}_r^{\text{ord}} \cong \tilde{B}_r^{\text{ord}}$. The commutativity of the left square follows from Lemma 6.2, and the top sequence is exact as $0 \rightarrow \tilde{A}_r^{\text{ord}} \rightarrow \tilde{J}_s^{\text{ord}} \rightarrow \varpi(\tilde{J}_s^{\text{ord}}) \rightarrow 0$ is sheaf exact. Since $0 \rightarrow \varpi(\tilde{J}_s^{\text{ord}}) \rightarrow \tilde{J}_s^{\text{ord}} \rightarrow \tilde{B}_r^{\text{ord}} \rightarrow 0$ is sheaf exact, we have $\text{Ker}(\rho_s) \supseteq \varpi(\tilde{J}_s^{\text{ord}})(k)$ for $\rho_s$ in (6.7); hence, $\rho_s'$ is the zero map as $\text{Coker}(i_s) \subset \varpi(\tilde{J}_s^{\text{ord}})(k)$. Since $\text{Ker}(\tilde{B}_r^{\text{ord}}(k) \xrightarrow{p^{s-r}U(p^{s-r})} \tilde{B}_r^{\text{ord}}(k))$ is contained in $\tilde{B}_r^{\text{ord}}(k)$ which has bounded order independent of $s$ (see Lemma 10.1), from the exact sequence $\text{Coker}(f_r) \rightarrow \text{Coker}(\rho_s) \rightarrow \text{Coker}(\rho_s') = \tilde{B}_r^{\text{ord}}(k)/p^{s-r}\tilde{B}_r^{\text{ord}}(k) \rightarrow 0$, we conclude $\text{Coker}(\rho_s) \rightarrow \tilde{B}_r^{\text{ord}}(k)/p^{s-r}\tilde{B}_r^{\text{ord}}(k)$ has kernel whose order is bounded by $|\text{Coker}(f_r)|$ independent of $s$. Passing to the limit,

$$\text{Coker}(\rho_s) = \text{Coker}(J^{\text{ord}}_s(k) \xrightarrow{p^{s-r}} \tilde{B}_r^{\text{ord}}(k)) \xrightarrow{\text{Corollary 6.3}} \text{Coker}(J^{\text{ord}}_{\infty}(k) \rightarrow \tilde{B}_r^{\text{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)
$$

is isogenous to $\lim_{\longrightarrow} \tilde{B}_r^{\text{ord}}(k)/p^{s-r}\tilde{B}_r^{\text{ord}}(k) = \tilde{B}_r^{\text{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Since $A_r$ is isogenous to $B_r$, $\tilde{B}_r^{\text{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ is isogenous to $\tilde{A}_r^{\text{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. \hfill $\Box$

Let $\mathcal{G}_s = J_s[p^{\infty}]^{\text{ord}}$. Then we have an exact sequence $A_r[p^{\infty}]^{\text{ord}} \rightarrow \mathcal{G}_s \xrightarrow{\varphi_s} \mathcal{G}_s \rightarrow B_s[p^{\infty}]^{\text{ord}}$ of fppf sheaves. In this case, $\lim_{\longrightarrow} B_s[p^{\infty}]^{\text{ord}} = 0$ as $B_s[p^{\infty}]^{\text{ord}}$ is $p$-torsion. Passing to the limit, we recover the following fact proven in [H14, §3, (DV)]:

**Corollary 6.5.** We have an exact sequence of fppf sheaves over $\mathbb{Q}$:

$$0 \rightarrow A_r[p^{\infty}]^{\text{ord}} \rightarrow \mathcal{G}_\infty \xrightarrow{\varphi_s} \mathcal{G}_\infty \rightarrow 0.$$

Actually, this sequence is exact even as fppf sheaves over $\mathbb{Z}_p[\mu_\infty]$ as shown in [H14].

### 7. Generality of Galois cohomology

We prove some general result on Galois cohomology for our later use. Let $S$ be a set of places of a number field $K$. Suppose that $S$ contains all archimedean places and $p$-adic places of $K$ (and primes for bad reduction of the abelian varieties we deal with abelian varieties). Let $K^S$ be the maximal extension unramified outside $S$.

**Lemma 7.1.** Let $\{M_n\}_n$ be a projective system of finite $\mathbb{Z}_p[\text{Gal}(K^S/K)]$-modules $M_n$. Write $M_\infty := \lim_{\longrightarrow} M_n$ and $M_\infty^\vee := \lim_{\longrightarrow} M_n^\vee$ for the Pontryagin dual $M_n^\vee$ of $M_n$. Write $G$ (resp. $G_v$ for a place $v$ of $K$) for the (point by point) stabilizer of $M_\infty^\vee$ in $\text{Gal}(K^S/K)$ (resp. $\text{Gal}(K_v/K_v)$) and $G = \text{Gal}(K^S/K)/G$ (resp. $G_v := \text{Gal}(K_v/K_v)/G_v$). Then, we have

1. $\Pi^1(K^S/K, M_\infty) = \lim_{\longrightarrow} \Pi^1(K^S/K, M_n)$, and if $S$ is a finite set, we have $\Pi^1(K^S/K, M_n^\vee) = \lim_{\longrightarrow} \Pi^1(K^S/K, M_n^\vee)$.

2. $\Pi^2(K^S/K, M_\infty) = \lim_{\longrightarrow} \Pi^2(K^S/K, M_n^\vee)$, and if $S$ is a finite set, we have $H^2(K^S/K, M_\infty) = \lim_{\longrightarrow} H^2(K^S/K, M_n)$ and $\Pi^2(K^S/K, M_\infty) = \lim_{\longrightarrow} \Pi^2(K^S/K, M_n)$.

**Proof.** We first prove the assertion in (1) for projective limit. Since $H^0(\cdot, M_n)$ for $\cdot = K^S/K$ and $K_v$ is finite for all $n$, we have $\lim_{\longrightarrow} H^1(\cdot, M_n) = H^1(\cdot, M_\infty)$ for $\cdot = K^S/K$ and $K_v$. Then $\Pi^1$ is defined using Corollary 2.7.6. By definition, we have an exact sequence:

$$0 \rightarrow \Pi^1(K^S/K, M_n) \rightarrow H^1(K^S/K, M_n) \rightarrow \prod_{v \in S} H^1(K_v, M_n).$$

We have a projective limit $\Pi^1(K^S/K, M_n) \rightarrow \Pi^1(K^S/K, M_\infty)$ because each $\Pi^1(K^S/K, M_n)$ is a direct product of local Galois cohomology groups. This isomorphism is natural under the identification $\Pi^1(K^S/K, M_\infty) = \text{Gal}(K^S/K)\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Similarly, we can obtain the second isomorphism $\Pi^2(K^S/K, M_\infty) = \Pi^2(K^S/K, M_\infty)$.
Since any continuous cochain with values in $\lim_{n} M_n$ is a projective limit of continuous cochains with values in $M_n$, we have a natural map $H^1(?, \lim_{n} M_n) \to H^1(?, M_n)$ for $? = K^S/K$ and $K_v$. Passing to the limit, we get the following commutative diagram with exact rows

\[ \begin{array}{cccc}
\varprojlim (K^S/K, \lim_{n} M_n) & \xrightarrow{i} & H^1(K^S/K, \lim_{n} M_n) & \longrightarrow \prod_{v \in S} H^1(K_v, \lim_{n} M_n) \\
\downarrow & & \downarrow & 1 \\
\lim_{n} \varprojlim (K^S/K, M_n) & \xrightarrow{i} & \lim_{n} H^1(K^S/K, M_n) & \longrightarrow \prod_{v \in S} \lim_{n} H^1(K_v, M_n).
\end{array} \]

This shows

\[ \varprojlim (K^S/K, \lim_{n} M_n) = \lim_{n} \varprojlim (K^S/K, M_n) \]

as desired. As for the injective limit, we first note that the cohomology functor commutes with the limit. However it may not commute with infinite product; so, we need to assume that $S$ is finite (this fact is pointed out to the author by D. Harari).

As for (2), since cohomology functor commutes with injective limit, the assertion (2) for injective limits follows from the same argument as in the case of (1), noting that by local Tate duality, the theorem (e.g., [CNF, 2.3.4] or [CGP, (0.8)]).

Let $A$ be an abelian variety over a field $K$. Since the Galois group $\text{Gal}(\overline{K}/K)$ and $\text{Gal}(K^S/K)$ is profinite and $A(\overline{K})$ and $A(K^S)$ are discrete modules, for $q > 0$, the continuous cohomology group $H^q(K^S/K, A)$ for a number field $K$ and $H^q(K, A)$ for a local field $K$ are torsion discrete modules (see [MFG, Corollary 4.26]).

**Lemma 7.2.** If $K$ is either a number field or a local field of characteristic 0, we have a canonical isomorphism for $0 < q \in \mathbb{Z}$:

\[ H^q(\hat{A}) \cong H^q(A)[p^\infty], \]

where $H^q(?) = H^q(K^S/K, ?)$ if $K$ is a number field, and $H^q(?) = H^q(K, ?)$ if $K$ is local.

**Proof.** By (S), if $K$ is a number field, we have

\[ H^q(K^S/K, \hat{A}) \cong H^q(K^S/K, A \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{(*)} \cong H^q(K^S/K, A) \otimes_{\mathbb{Z}} \mathbb{Z}_p = H^q(K^S/K, A)_p, \]

as $H^q(K^S/K, A)$ is a torsion module. Here the identity $(*)$ follows from the universal coefficient theorem (e.g., [CNF, 2.3.4] or [CGP, (0.8)]).

Now suppose that $K$ is an $l$-adic or archimedean local field with $l \neq p$. Then $\hat{A} = A[p^\infty]$, and we have a natural inclusion $0 \to \hat{A}(\overline{K}) \to A(\overline{K}) \to Q \to 0$ for the quotient Galois module $Q$. Thus $Q$ is $p$-torsion-free and $p$-divisible; i.e., the multiplication by $p$ is invertible on $Q$. Therefore $H^q(K, Q)_p := H^q(K, Q) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a $\mathbb{Q}_p$-vector space for $q \geq 0$ (so, $H^q(K, Q)_p = 0$ though we do not need this vanishing). By the exact sequence $H^{q-1}(K, Q)_p \to H^q(K, \hat{A})_p \to H^q(K, A)_p \to H^q(K, Q)_p$, we conclude $H^q(K, \hat{A})_p \cong H^q(K, A)_p$ as the two modules are $p$-torsion.

If $l = p$, we have $A(\overline{K}) = \hat{A}(\overline{K}) \oplus A^{(p)}(\overline{K})$ under the notation of (S), and hence $H^q(K, A)_p = H^q(A)_p \oplus H^q(K, A^{(p)})_p$. Since $\hat{A}(\overline{K})$ is a union of $p$-profinite group, we have $H^q(\hat{A})_p = H^q(A)$. Since $A^{(p)}(\overline{K})$ is prime-to-$p$ torsion, we have $H^q(K, A^{(p)})_p = 0$. Thus we get

\[ H^q(K, A)_p = H^q(K, A) \otimes_{\mathbb{Z}} \mathbb{Z}_p = H^q(K, \hat{A}) \oplus H^q(K, A^{(p)})_p = H^q(K, \hat{A}) \]

as desired. \qed
8. Diagrams of Selmer groups and Tate–Shafarevich groups

We describe a commutative diagram involving different cohomology groups and Tate–Shafarevich groups, which lays a base of the proof of the control result in later sections. We assume $p > 2$ for simplicity.

Recall the definition of the $p$-part of the Selmer group and the Tate–Shafarevich group for an abelian variety $A$ defined over a number field $K$:

$$\Pi_K(A)_p = \ker(H^1(K^S/K, A)_p) \bigoplus \prod_{v \notin S} H^1(K_v, A)_p,$$

$$\text{Sel}_K(A)_p = \ker(H^1(K^S/K, A[p^\infty])) \bigoplus \prod_{v \notin S} H^1(K_v, A)_p.$$

As long as $S$ is sufficiently large containing all bad places for $A$ in addition to all archimedean and $p$-adic places, these groups are independent of $S$ (see [ADT, I.6.6]) and are $p$-torsion modules.

**Lemma 8.1.** We can replace $A$ in the above definition by $\hat{A}$, and we get

$$\Pi_K(A)_p = \Pi_K(\hat{A}) = \ker(H^1(K^S/K, \hat{A}) \bigoplus \prod_{v \notin S} H^1(K_v, \hat{A})),
$$

(8.1)

$$\text{Sel}_K(A)_p = \text{Sel}_K(\hat{A}) = \ker(H^1(K^S/K, A[p^\infty]) \bigoplus \prod_{v \notin S} H^1(K_v, \hat{A})).$$

**Proof.** It is known that image of global cohomology classes lands in the direct sum $\bigoplus_{v \notin S} H^1(K_v, \hat{A})$ in the product $\prod_{v \notin S} H^1(K_v, \hat{A})$ (see [ADT, I.6.3]).

By Lemma 7.2, we have $\Pi_K(\hat{A}) = \Pi_K(A)_p = \Pi_K(A) \otimes \mathbb{Z}_p$. Thus we may replace the $p$-primary part of the traditional III-functor $A \mapsto \Pi_K(A)_p$ by the completed one $A \mapsto \Pi_K(\hat{A})$. \hfill $\square$

Since $A \mapsto \Pi_K(\hat{A})$ is a covariant functor from abelian varieties defined over a number field $K$ to an abelian groups, from Lemma 3.3 (and Remark 3.4), we get the commutative diagram for $X = \Pi$ and Sel:

$$X_K(\hat{J}_r) \xrightarrow{\pi^*} X_K(\hat{J}_r')$$

$$\downarrow u \quad \uparrow u' \quad \downarrow u''$$

$$X_K(\hat{J}_s) \xrightarrow{\pi^*} X_K(\hat{J}_s').$$

Similarly to the diagram as above, from Corollary 3.2, we get the following commutative diagram:

$$X_K(\hat{J}_s) \xrightarrow{\pi^*} X_K(\hat{J}_s[\gamma^{p^{-s}} - 1])$$

$$\downarrow u \quad \uparrow e_s \quad \downarrow u''$$

$$X_K(\hat{J}_r) \xrightarrow{\pi^*} X_K(\hat{J}_r[\gamma^{p^{-s}} - 1]).$$

These diagrams provide us the following canonical isomorphisms

(8.4) $X_K(\hat{J}_r)^{\text{ord}} \cong X_K(\hat{J}_s[\gamma^{p^{-s}} - 1])^{\text{ord}}$ for $X = \Pi$ and Sel.

For any group subvariety $A/Q$ of $J_s$ proper over $Q$ or any abelian variety quotient $A/Q$ of $J_s$ stable under $U(p)$, we have $\hat{A} = \hat{A}^{\text{ord}} \oplus (1 - e)\hat{A}$, and hence $H^q(?, \hat{A}) = H^q(?, \hat{A}^{\text{ord}}) \oplus H^q(?, (1 - e)\hat{A})$ for $? = \overline{K}$ and $K^S$. This shows $H^q(?, \hat{A})^{\text{ord}} = H^q(?, \hat{A}^{\text{ord}})$, and hence $X^q_k(\hat{A}^{\text{ord}}) = X^q_k(\hat{A})^{\text{ord}} = X^q_k(A)^{\text{ord}}$ for $X = \Pi$ and Sel. Thus hereafter, we attach the superscript “ord” inside the cohomology/Tate–Shafarevich group if the coefficient is $p$-adically completed in the sense of (S).

We define the ind A-TS group and the ind A-Selmer group by

$$\Pi_K(J_s)^{\text{ord}} := \Pi_K(J_s^{\text{ord}}) = \lim_r \Pi_K(J_r^{\text{ord}}) = \lim_r \Pi_K(J_r)_p^{\text{ord}},$$

(8.5)

$$\text{Sel}_K(J_s)^{\text{ord}} := \text{Sel}_K(J_s^{\text{ord}}) = \lim_r \text{Sel}_K(J_r^{\text{ord}}) = \lim_r \text{Sel}_K(J_r)_p^{\text{ord}}$$

which are naturally $h$-modules.
Write $H^q_S(M) = \bigoplus_{v \in S} H^q(K_v, M)$ and $H^q(M) = H^q(K_S/K, M)$ for a $\text{Gal}(K_S/K)$-module $M$. By [ADT, I.6.6], $\Pi(K_S/K, A) = \Pi_S(A)$ for an abelian variety $A/K$ as long as $S$ contains all bad places of $A$ and all archimedean and $p$-adic places. Consider a triple $(\varpi, A_s = J_s[A_s], B_s = J_s/A_s)$ satisfying the condition (A) of Section 5 and (F) in Section 4. Note that $J_s[\varpi] = A_s$ (see Lemma 5.1), we have $H^q(\varpi, J_s[\varpi]) = H^q(\varpi, A_s)$. This implies $\Pi_S(J_s[\varpi]) = \Pi_S(A_s) \cong \Pi_S(\hat{A}_s)$, where the last identity follows from [ADT, I.6.6]. Recall the following exact sequence from Corollary 6.3:

$$0 \to \hat{A}_s^\text{ord}(K') \to J_s^\text{ord}(K') \xrightarrow{\varpi} J_s^\text{ord}(K') \to \hat{B}_s^\text{ord}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0,$$

where $J_s^\text{ord} = \varinjlim_s J_s^\text{ord}$ and $K' = K_S^\text{ord}$ and $\mathbb{K}_s$. We separate it into two short exact sequences:

$$0 \to \hat{A}_s^\text{ord}(K') \to J_s^\text{ord}(K') \xrightarrow{\varpi} (J_s^\text{ord})(K') \to 0,$$

$$0 \to \varpi(J_s^\text{ord})(K') \to J_s^\text{ord}(K') \to \hat{B}_s^\text{ord}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$

Define

$$\text{Sel}_K(\varpi(J_s^\text{ord})) := \ker(i : H^1(K_S^\text{ord}/K, \varpi(J_s^\text{ord}))[\mathbb{Z}^\infty]) \to H^1_S(\varpi(J_s^\text{ord})),$$

$$\Pi_K(\varpi(J_s^\text{ord})) := \ker(i : H^1(K_S^\text{ord}/K, \varpi(J_s^\text{ord}))) \to H^1_S(\varpi(J_s^\text{ord}))),$$

$$E^*_\text{Sel}(K_v) := \text{coker}(J_s^\text{ord}(K_v) \to \hat{B}_s^\text{ord}(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

Here we have written $H^1_S(K_v, X) := \prod_{v \in S} H^1(K_v, X)$.

Look into the following commutative diagram of sheaves with exact rows:

$$A_s[p^\infty]^\text{ord} \xrightarrow{i} J_s^\text{ord}[p^\infty] \xrightarrow{\varpi[p^\infty]} J_s^\text{ord}[p^\infty] \xrightarrow{i} 0$$

Since $\hat{B}_s^\text{ord} \otimes \mathbb{Q}_p$ is a sheaf of $\mathbb{Q}_p$-vector spaces and $J_s^\text{ord}[p^\infty]$ is $p$-torsion, the inclusion map $i$ factors through the image $\text{Im}(\varpi) = \varpi(J_s^\text{ord})$, so,

$$\varpi(J_s^\text{ord})[p^\infty] = J_s^\text{ord}[p^\infty].$$

From the exact sequence, $\varpi(J_s^\text{ord}) \xrightarrow{i} J_s^\text{ord} \to \hat{B}_s^\text{ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, taking its cohomology sequence, we get the bottom sequence of the following commutative diagram with exact rows:

$$0 \to \hat{B}_s^\text{ord}[\varpi(J_s^\text{ord})[p^\infty]] \xrightarrow{i} H^1(\varpi(J_s^\text{ord}[p^\infty])) \to H^1(\varpi(J_s^\text{ord}[p^\infty]))$$

By the snake lemma, we get an exact sequence

$$0 \to \text{Sel}_K(\varpi(J_s^\text{ord})) \to \text{Sel}_K(J_s^\text{ord}) \to \prod_{v \mid p} E^*_\text{Sel}(K_v),$$

since $\hat{B}_s^\text{ord}(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = E^*_\text{Sel}(K_v) = 0$ if $v \nmid p$ by (S) in the introduction.

Define terms

$$E^*_\text{Sel}(F) := \frac{J_s^\text{ord}[p^\infty](F)}{\varpi(J_s^\text{ord}[p^\infty](F))} = \lim_{\xrightarrow{s}} \frac{\varpi(\hat{J}_s^\text{ord}[p^\infty](F))}{\varpi(\hat{J}_s^\text{ord}[p^\infty](F))}, \ E^\infty(F) := \frac{\varpi(J_s^\text{ord}(F))}{\varpi(J_s^\text{ord}(F))} = \lim_{\xrightarrow{s}} \frac{\varpi(J_s^\text{ord}(F))}{\varpi(J_s^\text{ord}(F))}$$

for $F = K, K_v$, and put $E^*_\text{Sel}(K) = \prod_{v \in S} E^*_\text{Sel}(K_v)$. If $A_r = A_P$ for an arithmetic point $P$, we often write $E^*_\text{Sel}(F)$ for $E^\infty(F)$ as it depends on $P$. Noting $\varpi \rightarrow \mathcal{G}$ is epimorphism of sheaves for
we have an exact sequence

\[ 0 \longrightarrow A \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/p^n \rightarrow 0 \]

By Corollary 17.4, we then get the following commutative diagram with two bottom exact rows and columns:

\[
\begin{array}{cccc}
\text{Ker}(\iota_{\text{Sel},*}) & \rightarrow & \text{Sel}_K(\hat{A}_r^{\text{ord}}) & \rightarrow & \text{Sel}_K(J_\infty^{\text{ord}}) & \rightarrow & \text{Sel}_K(\pi(J_\infty^{\text{ord}})) \\
\cap & & \cap & & \cap & & \cap \\
E_{\text{Sel}}(K) & \rightarrow & H^1(\hat{A}_r^{\text{ord}}[p^{\infty}]) & \rightarrow & H^1(J_\infty^{\text{ord}}[p^{\infty}]) & \rightarrow & H^1(J_\infty^{\text{ord}}[p]) \\
\cap & & \cap & & \cap & & \cap \\
E_{\text{Sel}}^\infty(K) & \rightarrow & H^1_1(\hat{A}_r^{\text{ord}}) & \rightarrow & H^1_1(J_\infty^{\text{ord}}) & \rightarrow & H^1_1(J_\infty^{\text{ord}}) \\
\end{array}
\]

Here the last map \(\pi_{S,*}\) could have 2-torsion finite cokernel if \(p = 2\).

We look into \(\Lambda\)-TS groups. Let \(\pi \in \mathfrak{h}\) coming from \(\pi_r \in \text{End}(J_r/\mathbb{Q})\) and suppose that \((\pi) = \pi \mathfrak{h} \subset (\pi^{p-\epsilon} - 1)\). The long exact sequence obtained from (8.7) produces the following commutative diagram with exact columns and bottom two exact rows:

\[
\begin{array}{cccc}
\text{Ker}(\iota_{\text{m},*}) & \rightarrow & \text{III}_K(\hat{A}_r^{\text{ord}}) & \rightarrow & \text{III}_K(\hat{A}_r^{\text{ord}}) & \rightarrow & \text{III}_K(\pi(J_\infty^{\text{ord}})) \\
\cap & & \cap & & \cap & & \cap \\
E^\infty(K) & \rightarrow & H^1(\hat{A}_r^{\text{ord}}) & \rightarrow & H^1(J_\infty^{\text{ord}}) & \rightarrow & H^1(J_\infty^{\text{ord}}) \\
\cap & & \cap & & \cap & & \cap \\
E_{\infty}^S(K) & \rightarrow & H^1_1(\hat{A}_r^{\text{ord}}) & \rightarrow & H^1_1(J_\infty^{\text{ord}}) & \rightarrow & H^1_1(J_\infty^{\text{ord}}) \\
\end{array}
\]

By the vanishing of \(H^1_2(\hat{A}_r)\) ([ADT, Theorem I.3.2] and Lemma 7.2), \(\pi_{S,*}\) are surjective. In each term of the diagram (8.14), we can bring the superscript "ord" inside the functor III and \(H^1\) outside the functor as the ordinary projector acts on \(\hat{J}_r\), \(J_\infty\), and \(\hat{A}_r\) and gives direct factor of the sheaf. The diagram "ord" inside is the one obtained directly from the short exact sequence of Corollary 6.3.

9. VANISHING OF THE ERROR TERM FOR \(l\)-ADIC FIELDS WITH \(l \neq p\).

In this section, we prove vanishing of the error term \(E^\infty(K)\) for local fields of residual characteristic \(l \neq p\), which combined with a similar (but more difficult) result for \(p\)-adic fields given in Section 17 will be used in the following sections to prove the control result up to finite error of the limit Selmer group, the finite Mordell–Weil group and the limit Tate–Shaferievich group.

More generally, for the moment, we denote by \(K\) either a number field or an \(l\)-adic field (the prime \(l\) can be \(p\) unless we mention that \(l \neq p\)).

**Lemma 9.1.** Let \(K\) either a number field or an \(l\)-adic field. Then the Pontryagin dual \(E^\infty(K)^{\vee}\) of \(E^\infty(K)\) is a finite \(\mathbb{Z}_p\)-module of finite type (i.e., \(E^\infty(K)\) is \(p\)-torsion of finite corank).

**Proof.** Let \(K' = \overline{K}\) if \(K\) is local and \(K = K^S\) if \(K\) is global. We have an exact sequence

\[ 0 \rightarrow E^\infty(K) \rightarrow H^1(K'/K, \hat{A}_r^{\text{ord}}) \rightarrow H^1(K'/K, \hat{J}_\infty^{\text{ord}}). \]

By [ADT, I.3.4], if \(K\) is local, \(H^1(K'/K, \hat{A}_r^{\text{ord}}) \cong \text{Pic}_{A/K}(K)^{\vee}\); so, we get the desired result. If \(K\) is global, \(\hat{A}_r^{\text{ord}}(K) \otimes \mathbb{F} \rightarrow H^1(K'/K, \hat{A}_r^{\text{ord}}[p]) \rightarrow H^1(K'/K, \hat{A}_r^{\text{ord}})[p]\) is exact, and the middle term is finite by Tate’s computation of the global cohomology (taking \(S\) to be finite); so, \(H^1(K'/K, \hat{A}_r^{\text{ord}})\) has Pontryagin dual finite type over \(\mathbb{Z}_p\). This finishes the proof.

As before, we write \(H^0(M)\) for \(H^0(K, M)\) (resp. \(H^0(K^S/K, M)\)) if \(K\) is local (resp. global). For each \(\mathbb{Z}_p\)-module \(M\), we write \(T_p M := \text{Hom}((\mathbb{Q}_p/\mathbb{Z}_p, M) = \lim_{\leftarrow n} M[p^n]\). For any abelian variety \(A/K\), we have an exact sequence \(A(K) \otimes \mathbb{Z}/p^n \mathbb{Z} \rightarrow H^1(A[p^n]) \rightarrow H^1(A)[p^n]\) by Kummer theory. If \(K\) is a number field, by Mordell–Weil theorem, \(A(K) \otimes \mathbb{Z}/p^n \mathbb{Z}\) satisfies the Mittag–Leffler condition.
If $K$ is a finite extension of $\mathbb{Q}$, $A(K) \otimes \mathbb{Z}/p^n\mathbb{Z}$ is finite, as $A(K)$ is an $l$-adic Lie group; so, again \(\{A(K) \otimes \mathbb{Z}/p^n\mathbb{Z}\}_n\) satisfies the Mittag–Leffler condition. Thus passing to the projective limit, by Lemma 7.1 (1), \(\hat{A}(K) \rightarrow H^1(T_{el}A) \rightarrow T_pH^1(\hat{A})\) is exact (e.g., [H15, Lemma 2.2]). Let $A = J_s, A_s$ and $B_s$. Then we can apply $e$ and get the following exact sequence: \(\hat{A}^\text{ord}(K) \rightarrow H^1(T_p\hat{A}^\text{ord}) \rightarrow T_pH^1(\hat{A}^\text{ord})\). Let $K' = K^\otimes$ if $K$ is a number field and $K' = K$ if $K$ is an $l$-adic field. Writing $\varpi(\hat{J}^\text{ord}_s)$ for the étale sheaf quotient of $\hat{J}^\text{ord}_s$ by $\text{Ker}(\varpi: \hat{J}^\text{ord}_s \rightarrow \hat{J}^\text{ord}_s)$, we have an exact sequence $0 \rightarrow \varpi(\hat{J}^\text{ord}_s)[p^n](K') \rightarrow \varpi(\hat{J}^\text{ord}_s)(K') \rightarrow \varpi(\hat{J}^\text{ord}_s)(K') 
rightarrow 0$. Via the associated long exact sequence, we get another exact sequence:

\[
0 \rightarrow \varpi(\hat{J}^\text{ord}_s)(K) \otimes \mathbb{Z}/p^n\mathbb{Z} \rightarrow H^1(\varpi(\hat{J}^\text{ord}_s)[p^n]) \rightarrow H^1(\varpi(\hat{J}^\text{ord}_s))[p^n] \rightarrow 0.
\]

By the same reasoning as above, passing to the limit with respect to $n$, by [H15, Lemma 2.1], we find the exactness of $0 \rightarrow \varpi(\hat{J}^\text{ord}_s)(K) \rightarrow H^1(T_p\varpi(\hat{J}^\text{ord}_s)) \rightarrow T_pH^1(\varpi(\hat{J}^\text{ord}_s)) \rightarrow 0$. Thus we have the following commutative diagram in which the left two columns and the top three rows are exact for the above reason:

\[
\begin{array}{cccccc}
\hat{A}^\text{ord}_r(K) & \rightarrow & H^1(T_p\hat{A}^\text{ord}_r) & \rightarrow & T_pH^1(\hat{A}^\text{ord}_r) \\
\cap \downarrow i & & a \downarrow & & b_s \\
\hat{J}^\text{ord}_s(K) & \rightarrow & H^1(T_p\hat{J}^\text{ord}_s) & \rightarrow & T_pH^1(\hat{J}^\text{ord}_s) \\
\varpi \downarrow f & & j \downarrow & & h_s \\
\varpi(\hat{J}^\text{ord}_s)(K) & \rightarrow & H^1(T_p\varpi(\hat{J}^\text{ord}_s)) & \rightarrow & T_pH^1(\varpi(\hat{J}^\text{ord}_s)) \\
\end{array}
\]

(9.1)

Recall that $E^s(K)$ is defined to be the $\text{Coker}(\hat{J}^\text{ord}_s(K) \rightarrow \varpi(\hat{J}^\text{ord}_s)(K))$. The last column:

\[
T_pH^1(\hat{A}^\text{ord}_r) \xrightarrow{b_s} T_pH^1(\hat{J}^\text{ord}_s) \xrightarrow{h_s} T_pH^1(\varpi(\hat{J}^\text{ord}_s)) \xrightarrow{g} E^s(K)
\]

may not be exact.

**Lemma 9.2.** The map $b_s$ is injective, and $\text{Ker}(h_s)/\text{Im}(b_s)$ canonically injects into $E^s(K)$ whose cokernel has bounded order independent of $s$.

**Proof.** From the exact sequence: $0 \rightarrow E^s(K) \rightarrow H^1(\hat{A}^\text{ord}_r) \rightarrow \text{Im}(b) \rightarrow 0$ for $b: H^1(\hat{A}^\text{ord}_r) \rightarrow H^1(\hat{J}^\text{ord}_s)$, we have the following new exact sequence:

\[
0 \rightarrow T_pE^s(K) \rightarrow T_pH^1(\hat{A}^\text{ord}_r) \rightarrow T_p\text{Im}(b) \rightarrow \text{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, E^2(K)) \rightarrow \text{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, H^1(\hat{A}^\text{ord}_r)).
\]

Note $\text{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, E^s(K)) \cong E^s(K)$ (as $E^s(K)$ is finite $p$-torsion) and

\[
\text{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, H^1(\hat{A}^\text{ord}_r)) \cong H^1(\hat{A}^\text{ord}_r)^{\vee}[p^\infty]
\]

which is finite. Here $M \mapsto M^\vee$ is the Pontryagin duality. Since the natural map $i: T_p\text{Im}(b) \rightarrow T_pH^1(\hat{J}^\text{ord}_s)$ is injective with $\text{Im}(i) = \text{Ker}(h_s)$ (as $M \mapsto T_pM$ is left exact), we have an exact sequence $0 \rightarrow \text{Ker}(h_s) \rightarrow \text{Im}(b_s) \rightarrow E^s(K) \rightarrow H^1(\hat{A}^\text{ord}_r)^{\vee}[p^\infty]$ whose extreme right term has finite order independent of $s$. Since $T_pE^s(K) = 0$ as $E^s(K)$ is finite, we find $b_s$ is injective from the left exactness of the functor $T_p$. \(\square\)

To see the existence of the map $e_s$, we suppose that $x = \varpi_s(y) \in \varpi_s(\hat{J}^\text{ord}_s(K))$. Then we have $\varpi_s(\beta(x)) = \varpi_s(\beta(\varpi_s(y))) = \varpi_s(j(f(y))) = 0$. If $b \equiv b' \mod \varpi_s(\hat{J}^\text{ord}_s(K))$ for $b, b' \in \hat{B}^\text{ord}_s(K)$, we have $\varpi_s(\beta(b)) = \varpi_s(\beta(b'))$. In other words, $\pi(b) \mapsto \varpi_s(\beta(b))$ is a well-defined homomorphism from $E^s(K) \cong \varpi_s(\hat{J}^\text{ord}_s(K))/\varpi_s(\hat{J}^\text{ord}_s(K))$ into $\text{Im}(\varpi_s) \cong \text{Coker}(j) \subset H^2(T_p\hat{A}^\text{ord}_r)$, which we have written as $e_s$. 


We have the following fact.

**Lemma 9.3.** Let $K$ be either a number field or a local field over $\mathbb{Q}_l$ for a prime $l$. If $A/K$ is an abelian variety defined over $K$ with its $p$-adic Tate module $T_pA$, then we have $H^0(K, T_pA) = 0$. In particular, the map $a$ in (9.1) is injective.

This lemma applied to $T_p\varpi(J_{\text{ord}}) \subset T_pJ_s$ tells us $H^0(K, T_p\varpi(J_{\text{ord}})) \subset H^0(K, T_pJ_s) = 0$, and therefore the the map $a$ in (9.1) is injective.

**Proof.** Note that $H^0(K, T_pA_{\text{ord}}) = 0$ (from $H^0(K, T_pA) = 0$) which surjects to $\ker(a)$, and hence $a$ is injective.

Assume now that $K$ is local. After extending scalars, we may assume that $A$ extends to a semi-abelian scheme over the integer ring $W$ of $K$ (see [NMD, §7.4]). Since $T_pA$ can be embedded into the product of the Tate modules of simple factors of $A$, we may assume that $A$ is absolutely simple. By extending scalars, we may assume that $A$ has semi-stable reduction over $W$. Changing $A$ by an isogeny, we may assume that the abelian factor of its reduction modulo $\mathfrak{m}_F$ has complex multiplication by the maximal order of a CM semi-simple algebra. Writing $\hat{A} := \text{Pic}^0(A/K)$ and $\hat{A}$ for the formal group of $A$ with its toric part $\hat{A}_{\text{tor}}$. Then we have a unique dual étale quotient $A[p^{\infty}]_{\text{ét}}$ of $\hat{A}_{\text{tor}}[p^{\infty}]$ (by Cartier duality), which is actually a direct factor of the Barsotti–Tate group $A[p^{\infty}]_{\text{ét}}$. This gives rise to an extension

$$0 \to T_pA_{\text{tor}} \to T_{\text{tor}}A_{\text{tor}} \to T_pA[p^{\infty}]_{\text{ét}} \to 0.$$  

Then we conclude from the theory of degeneration of abelian varieties [DAV] this extension of the toric part

$$0 \to T_{\text{tor}}A_{\text{tor}} \to T_{\text{tor}}A[p^{\infty}]_{\text{tor}} \to T_pA[p^{\infty}]_{\text{ét}} \to 0$$

does not have any split factor as the module of the inertia subgroup $I_l$ of $\text{Gal}(\overline{K}/K)$. Thus we conclude $H^0(K, T_{\text{tor}}A_{\text{tor}}) = 0$. On the other hand, the quotient $A[p^{\infty}]_{\text{tor}}/A[p^{\infty}]_{\text{tor}}$ is a Barsotti–Tate group over $W$, which is étale if $l \neq p$. On $A[p^{\infty}]_{\text{tor}}/A[p^{\infty}]_{\text{tor}}$, the Frobenius action $\phi$ has eigenvalues given by $\text{Weil } l\text{-number of weight } f > 0$ if the residue field of $W$ has order $l^f$. This shows that $H^0(K, T_{\text{tor}}A_{\text{tor}}/T_pA[p^{\infty}]_{\text{tor}}) = 0$. Combined with the vanishing of the toric part, we conclude $H^0(K, T_pA) = 0$ if $l \neq p$.

If $l = p$, $A[p^{\infty}]_{\text{tor}}/A[p^{\infty}]_{\text{tor}}$ is an extension of a product $A_{\text{LT}}$ of Lubin–Tate formal groups over $W$ associated with the $p$-Frobenius eigenvalues $\hat{\varphi}$ on which the Frobenius has eigenvalues given by Weil $p\text{-number of weight } f$. Then we see $H^0(K, T_p(A[p^{\infty}]/A[p^{\infty}]_{\text{tor}})) = 0$, and again we conclude $H^0(K, T_pA) = 0$. 

**Remark 9.4.** Just to know $H^0(K, T_p\varpi(J_{\text{ord}})) = 0$ which we really need, this follows from [H15, Corollary 4.4] directly without using the result in Lemma 9.3 for a general abelian variety.

We have a similar lemma for $H^2$.

**Lemma 9.5.** Let $K$ be a local field over $\mathbb{Q}_l$ for a prime $l$. If $A/K$ is an abelian variety defined over $K$ with its Tate module $T_pA$. Then we have $H^2(K, T_pA) \cong A[p^{\infty}]/\mathbb{Z}_p$, which is a finite module. Here $A[p^{\infty}]/\mathbb{Z}_p \cong A[p^{\infty}]/\mathbb{Z}_p$ for the Galois module $\mathbb{Z}_p(-1) := \text{Hom}_{\mathbb{Z}_p}(\mu_{p^{\infty}}(\overline{K}), \mathbb{Q}_p/\mathbb{Z}_p)$.

**Proof.** By Lemma 7.1 (2), we have $H^2(K, T_pA) = \lim_n H^2(K, A[p^n])$. By Tate duality (e.g., [MFG, Theorem 4.43]), we have $H^2(K, A[p^n]) \cong A^d[p^n]((K)^\vee$. Thus we have

$$H^2(K, T_pA) = \lim_n A^d[p^n]((K)^\vee \cong \lim_n A^d[p^n](K)^\vee = A^d[p^{\infty}](K)^\vee.$$ 

By Lemma 9.3, $A^d[p^{\infty}](K)$ is a finite module. Since we have a canonical pairing $A[p^n] \times A^d[p^n] \to \mu_{p^n}$, we have $A[d][p^{\infty}](K)^\vee \cong A[p^n](-1)(K)$. Thus we get the desired assertion. 

\[\square\]
Lemma 9.6. Assume (A) and (F). Let $K$ be a finite extension of $\mathbb{Q}$ for a prime $l \neq p$. Then we have $\text{Ker}(h_s) = \text{Im}(h_s) = 0$.

Proof. By [ADT, I.3.4], if $K$ is local, for an abelian variety $A$ over $K$, we have $H^1(K, A) = A^t(K) \otimes$ for $A^t = \text{Pic}^0_{A/K}$. Then if $l \neq p$, we find $T_p H^1(K, \hat{A}) = 0$ as $A^t(K) \cong W^{\dim A} \times \Delta$ for a finite group $\Delta$ and the $l$-adic integer ring $W$ of $K$. Thus $h_s$ and $h_s$ are zero maps.

Proposition 9.7. Assume (A) and (F). Let $K$ be a finite extension of $\mathbb{Q}$ for a prime $l$. Suppose that the complex (9.2) is exact. Then we have $E^s(K) = 0$. In particular, $E^s(K) = 0$ if $l \neq p$. If $l = p$, writing $X' = \text{Pic}^0_{X/K}$ for an abelian variety $X/K$, we have

$$E^s(K) \cong \text{Coker}(\hat{J}^s(K) \rightarrow \hat{A}^t(K)) \otimes \text{co-ord}.$$  

We will prove the finiteness and boundedness of $E^s(K)$ when $l = p$ later in Section 17 under some extra assumptions (see Theorem 17.2).

Proof. By the exactness of (9.2), we may apply the snake lemma to the middle two exact rows of (9.1), and we find an exact sequence

$$0 \rightarrow E^s(K) \rightarrow \text{Im}(\varpi_s) \rightarrow \text{Coker}(h_s) \rightarrow 0.$$  

This implies $E^s(K) \hookrightarrow \text{Im}(\varpi_s) \subset H^2(T_p A_r^{\text{ord}})$. By Lemma 9.5, we have

$$H^2(K, T_p A_r^{\text{ord}}) \cong A_r[\mathbb{p}^{\infty}] \cong J_s[\mathbb{p}^{\infty}] \cong H^2(K, T_p J_s^{\text{ord}}),$$  

which is injective as $A_r \cong J_s$. We have an exact sequence

$$H^1(K, T_p \varpi_s(\hat{J}^s)) \rightarrow H^2(K, T_p A_r^{\text{ord}}) \rightarrow H^2(K, T_p J_s^{\text{ord}}).$$  

Since $a_2$ is injective, we find $\text{Im}(\varpi_s) = 0$; so, $E^s(K) = 0$. If $l \neq p$, (9.2) is exact by Lemmas 9.2 and 9.6, and hence $E^s(K) = 0$.

Suppose $l = p$. We have an exact sequence

$$0 \rightarrow E^s(K) := \text{Coker}(\varpi_s : \hat{J}^s(\hat{J}^s)(K) \rightarrow \varpi(\hat{J}^s)(K)) \rightarrow H^1(K, \hat{A}_r^{\text{ord}}) \rightarrow H^1(K, \hat{J}^s).$$  

By [ADT, I.3.4], if $K$ is local, for an abelian variety $A$ over $K$, we have $H^1(K, A) = A^t(K) \otimes$ for $A^t = \text{Pic}^0_{A/K}$. This shows

$$E^s(K) = \text{Ker}(\hat{A}_r^{\text{co-ord}}(K) \otimes \hat{J}^s(\hat{J}^s)(K)) \rightarrow H^1(K, \hat{J}^s).$$  

as desired.

10. CONTROL OF $\Lambda$-SELMER GROUPS

We start with a lemma.

Lemma 10.1. For a number field or an $l$-adic field $K$ and $\mathcal{G} = J^{\text{ord}}_\infty[p^\infty]$, the Pontryagin dual $\mathcal{G}(K)^\vee$ is a $\Lambda$-torsion module of finite type. For any arithmetic prime $P$, $\mathcal{G}(K)^\vee \otimes_\mathfrak{h} P^n$, $\mathcal{G}(K) \otimes_\mathfrak{h} P^n$ and $\mathcal{G}(K)[P^n]$ are all finite for any positive integer $n$.

Proof. We give a detailed argument when $K$ is a number field and briefly touch an $l$-adic field as the argument is essentially the same. Let $P \in \omega_h$, and suppose $K$ is a number field. Suppose that the Galois representation $\rho_P$ associated with $P$ contains an open subgroup $G$ of $SL_2(\mathbb{Z}_p)$. Let $L$ be the Pontryagin dual module of $\mathcal{G}(\mathbb{Q})$. If the cusp form $f_\rho$ associated to $P$ has conductor divisible by $N$, the localization $L_P$ is free of rank 2 over the valuation ring $V = \mathfrak{h}_P$ finite over $\Lambda_P$ (e.g., [HMI, Proposition 3.78]). If $N$, by the theory of new form (e.g. [H86a, §3.3]), $L_P$ is free of rank 2 over a local ring of the form $V[X_1, . . . , X_m]/(X_1^n, . . . , X_m^n) = \mathfrak{h}_P$ with nilradical coming from old forms (e.g. [H13a, Corollary 1.2]). The contragredient $\rho_P = \iota \rho_P^{-1}$ of $\rho_P$ is realized by $L_P/PL_P$. Then $G$ is also contained in $\text{Im}(\rho_P)$, and $H_0(K, L_P/PL_P) \cong H(K, L_P)/PH_0(K, L_P)$ is a surjective image of $H_0(G, L_P/PL_P)$, which vanishes. Thus $H_0(K, L_P/PL_P) = 0$, which implies $H_0(K, L_P) = 0$ by Nakayama’s lemma. In particular, $H_0(K, L)$ is a $\Lambda$-torsion module whose support is outside $P$. 


If \( \rho_p \) does not contain an open subgroup of \( SL_2(\mathbb{Z}_p) \), by Ribet \cite{R85} (see also \cite[Theorem 4.3.18]{GME}), there exists an imaginary quadratic field \( M \) such that \( \tilde{\rho}_P = \text{Ind}_M^G \phi \) for an infinite order Hecke character \( \phi \) of \( \text{Gal}(\overline{\mathbb{Q}}, M) \). Then it is easy to show that \( H_0(K, L_P / PL_P) = 0 \), and in the same way as above, we find \( H_0(K, L_P) = 0 \) and hence \( G(\mathbb{K})^\vee = H_0(K, L) \) is \( \Lambda \)-torsion with support outside \( P \). Thus for any arithmetic prime \( p \in \omega_h, L_p = 0 \) and hence \( G(\mathbb{K})^\vee = H_0(K, L) \) is \( \Lambda \)-torsion with support outside \( P \). Thus in any case, \( G(\mathbb{K})^\vee \otimes_b \mathbb{H}^n = H_0(K, L) \otimes_b \mathbb{H}^n \) is finite for all \( n \). Thus for any arithmetic prime \( p \in \omega_h, L_p = 0 \) and hence \( G(\mathbb{K})^\vee = H_0(K, L) \) is \( \Lambda \)-torsion with support outside \( P \). Thus in any case, \( G(\mathbb{K})^\vee \otimes_b \mathbb{H}^n = H_0(K, L) \otimes_b \mathbb{H}^n \) is finite for all \( n \) as \( \mathbb{H} \) is a semi-local ring of dimension 2 finite torsion-free over \( \Lambda \). The module \( G(\mathbb{K}) / \mathbb{P}^n \) is just the dual of \( H_0(K, L) \otimes_b \mathbb{H}^n \) and hence is finite. Then \( (G(\mathbb{K}) \otimes_b \mathbb{H}^n)^\vee = H_0(K, L)[\mathbb{P}^n] \), which is finite by the above fact that \( H_0(K, L) \) is \( \Lambda \)-torsion with support outside \( P \).

If \( K \) is \( l \)-adic, replacing \( K \) by its finite extension, we may assume that \( A_P \) has split semi-stable reduction. Write \( F \) for the residue field of \( P \). Then either \( \tilde{\rho}_P(Frob) \) for a Frobenius element \( Frob \) of \( \text{Gal}(\overline{\mathbb{Q}}_l / K) \) has infinite order without eigenvalue 1 or the space \( V(\tilde{\rho}_P) \) fits into a non-split extension \( F \hookrightarrow V \rightarrow F(-1) \) for the Tate twist \( F(-1) \) (by the degeneration theory of Mumford–Tate; cf., \cite[Appendix]{DAV}). Because of this description \( H_0(K, L_P / PL_P) = 0 \), and by the same argument above, the results follows.

Since \( (\varpi) \) is supported by finitely many arithmetic primes, \( E_{\text{Sel}}(\mathbb{K})^\vee := (G(\mathbb{K}) \otimes_b \mathbb{H}/(\varpi))^\vee \cong G(\mathbb{K})^\vee(\varpi) \) is finite by the above lemma; so, we get

**Corollary 10.2.** Assume (A). If \( K \) is a number field, then \( E_{\text{Sel}}(\mathbb{K}) \) is finite.

Let \( T \) be the local ring such \( \varpi \in \mathfrak{m}_T \). We define \( \Omega_T \) to be the set of points \( P \in \text{Spec}(\mathbb{T})\langle \overline{\mathbb{Q}}_p \rangle \) such that

\[
(10.1) \quad P \cap \Lambda \text{ contains } \mathfrak{m}_P^{s} - 1 \text{ for some } 0 < s \in \mathbb{Z} \text{ and } P \text{ is principal as a prime ideal.}
\]

So, \( \Omega_T \) is a subset of \( \text{Spec}(\mathbb{T}) \langle \omega_h \rangle \) made of principal ideals. We see \( E_{\text{Sel}}(\mathbb{K})_T = \text{Coker}(\varpi : G(\mathbb{K})_T \rightarrow (G(\mathbb{K})_T)_\varpi) \), where \( M_T = M \otimes_b \mathbb{T} \) for an \( h \)-module \( M \). Thus for the Galois representation \( \rho_T \) acting on \( T_G \mathfrak{m}_T = \text{Ind}_{\mathfrak{m}_T}^{G(\mathbb{K})} (\gamma \mathfrak{m}_P^{s} - 1) \), if \( \rho_T \) modulo \( \mathfrak{m}_T \) is absolutely irreducible over \( \text{Gal}(\overline{\mathbb{Q}} / K) \), we conclude \( E_{\text{Sel}}(\mathbb{K}) = 0 \). Here \( \varpi_T = (\rho_T \text{ mod } \mathfrak{m}_T) \) is the semi-simple two dimensional representation whose trace is given by \( \text{Tr}(\rho_T) \text{ mod } \mathfrak{m}_T \). Indeed, the Galois module \( G[\mathfrak{m}_T] \) has Jordan-Hölder sequence whose sub-quotients are all isomorphic to \( \varpi_T \); so, by Nakayama’s lemma, \( G(\mathbb{K}) = 0 \).

Write \( \varpi_T \langle \text{Gal}(\overline{\mathbb{Q}}_p / Q_p) \rangle \cong \left( \begin{smallmatrix} \varpi_T & * \\ 0 & \varpi_T \end{smallmatrix} \right) \) mod \( \mathfrak{m}_T \) with the nearly ordinary character \( \varpi \) (i.e., \( \varpi([p, Q_p]) \) is equal to the image modulo \( \mathfrak{m}_T \) of \( U(p) \)). Here \( \varpi_T = \nu_p \text{ mod } p \). Then it is plain that \( G(K_{\nu}) = 0 \) for all place \( v|p \) of \( K \) if \( \varpi_T \) and \( \varpi \) are both non-trivial over \( \text{Gal}(\overline{\mathbb{Q}}_p / K_v) \) for all \( v|p \). We record this fact as

**Corollary 10.3.** Let \( p > 2 \), and suppose one of the following two conditions:

1. \( \varpi_T \) is irreducible over \( \text{Gal}(\overline{\mathbb{Q}} / K) \);
2. \( \varpi_T \psi \) and \( \varpi \) are both non-trivial over \( \text{Gal}(\overline{\mathbb{Q}}_p / K_v) \) for all \( p \)-adic places \( v|p \) of \( K \).

Then we have \( E_{\text{Sel}}(\mathbb{K}) = 0 \).

If \( \xi(a, d) = \alpha(a) \beta(d) \), we have \( \varpi([z, Q_p]) = \alpha(z) \text{ mod } p \) and \( \varpi([z, Q_p]) = \beta(z) \text{ mod } p \) for \( z \in Q_p^* \), where \( [z, Q_p] \) is the local Artin symbol. Let \( \rho_A \) be the Galois representation realized on \( T_P \mathfrak{A}^{\text{tor}}_P \) into \( GL_2(W) \) for a finite flat extension \( W \) of \( \mathbb{Z}_p \). Write \( \rho_A \mid \text{Gal}(\overline{\mathbb{Q}}_p / Q_p) \cong \left( \begin{smallmatrix} \nu_p \psi * \varphi \\ 0 \end{smallmatrix} \right) \). Supposing \( p > 2 \), for each \( p \)-adic place \( v|p \) of \( K \), let \( \sigma_v \) (resp. \( \text{Frob}_v \)) be a topological generator of \( \text{Gal}(K_v[\mu_{p^\infty}] / K_v) \) (resp. \( \text{Gal}(K_v^{nr}[\mu_{p^\infty}] / K_v) \) inducing on \( K_v[\mu_{p^\infty}] \) a power of the local Artin symbol \( [p, Q_p] \)), where \( K_v^{nr} \) is the maximal unramified extension of \( K_v \) in \( \mathbb{Q}_p \). Recall \( M_T = M \otimes_b \mathbb{T} \) for an \( h \)-module \( M \).

**Theorem 10.4.** Suppose \( p > 2 \), (A) and (F). Let \( K \) be a number field and \( \text{Spec}(\mathbb{T}) \) be the connected component such that \( \varpi \in \mathfrak{m}_T \).

1. (e1) \( E(\mathbb{K}_{v})^\vee \) is finite for all \( v|p \).
2. (e2) \( A_v \) does not have split multiplicative reduction modulo \( p \) at all primes \( p|v \) of \( K \).

Then the following sequence

\[
0 \rightarrow E_{\text{Sel}}(\mathbb{K}^{\text{nr}}) \rightarrow E_{\text{Sel}}(\mathbb{K}^{\text{nr}}) \cong E_{\text{Sel}}(\mathbb{K}^{\text{nr}}).
\]
is exact up to finite error.

2. Assume one of the following two conditions:
   (E1) \( E_{Sel}(K) = E^\infty(K) = 0 \) for all \( v \mid p \),
   (E2) \( A_p \) has good reduction at \( v \) and
   \( |\varphi(Frob_v)| - 1 |p = 1 \) for all \( v \mid p \).

   Here \( Frob_v \) is a Frobenius element in \( \text{Gal}(\overline{K}/K) \) acting
   trivially on \( K[\mu_p^{\infty}] \).

   Then the sequence (10.2) is exact, and if in addition \( E^*_p(K) = 0 \) for all \( v \mid p \), we have
   \( \text{Sel}_K(\varpi(J^\text{ord}))(\vartheta) \cong \text{Sel}_K(J^\text{ord}) \).

   By (8.11), we have an exact sequence:
   \[
   0 \rightarrow \text{Sel}_K(\varpi(J^\text{ord})) \rightarrow \text{Sel}_K(J^\text{ord}) \rightarrow \prod_{v \mid p} E^*_p(K) \rightarrow 0.
   \]

   We conclude from (e1) that
   \( \text{Sel}_K(\varpi(J^\text{ord})) \cong \text{Sel}_K(J^\text{ord}) \).

   Recall the following commutative diagram with two bottom exact rows and three right exact
   columns from (8.13) (tensored with \( \mathbf{h} \) over \( \mathbf{T} \)):

   \[
   \begin{array}{cccc}
   \text{Ker}(\iota_{\text{Sel}}) & \xrightarrow{\iota_{\text{Sel}}} & \text{Sel}_K(A^\text{ord}) & \xrightarrow{\iota_{\text{Sel}}} & \text{Sel}_K(J^\text{ord}) & \xrightarrow{\iota_{\text{Sel}}} & \text{Sel}_K(J^\text{ord})_T & \xrightarrow{\iota_{\text{Sel}}}
   \\
   & i & \cap & s & i & \cap & s & i
   \\
   \text{E}_{\text{Sel}}(K) & \xrightarrow{e} & H^1(A^\text{ord}_T[p^{\infty}]) & \xrightarrow{\iota} & H^1(J^\text{ord}_T[p^{\infty}]) & \xrightarrow{\iota} & H^1(J^\text{ord}_T[p^{\infty}]) & \xrightarrow{\iota}
   \\
   & e & \cap & s & e & \cap & s & e
   \\
   & \prod_{v \mid p} E^\infty(K_v) & \xrightarrow{e_0} & H^1_J(A^\text{ord}) & \xrightarrow{\iota_0} & H^1_J(J^\text{ord}) & \xrightarrow{\iota_0} & H^1_J(J^\text{ord})_T & \xrightarrow{\iota_0}
   \end{array}
   \]

   Since the middle two columns are exact, the left column is exact with injection \( i \) (e.g., [BCM, I.1.4.2
   (1)]). Since the bottom row is exact with injection \( e_0 \), the map \( i_0 \) is injective and \( \text{Im}(i_0) = \text{Ker}(\iota_{\text{Sel}}) \).

   Suppose (E1). Then all the terms of the left column vanish. So \( \text{Ker}(\iota_{\text{Sel}}) = 0 \) and the sequence:

   \[
   0 \rightarrow \text{Sel}_K(A^\text{ord}) \rightarrow \text{Sel}_K(J^\text{ord}) \rightarrow \text{Sel}_K(\varpi(J^\text{ord})) \xrightarrow{(8.11)} \text{Sel}_K(J^\text{ord})
   \]

   is exact. The cokernel \( \text{Coker}(\text{Sel}_K(J^\text{ord})) \cong \text{Sel}_K(\varpi(J^\text{ord})) \) is global in nature and seems difficult to
determine, although \( \text{Coker}(\text{Sel}_K(\varpi(J^\text{ord}))) \rightarrow \text{Sel}_K(J^\text{ord}) \) is local as in (8.11), and if \( E_{\text{Sel}}(K_v) = 0 \) for all \( v \mid p \), it vanishes.

   Now we assume (E1). We need to prove the sequence (10.4) is exact up to finite error. By
Corollary 10.2, \( E_{\text{Sel}}(K) \) is finite. Since we know \( E^\infty(K) = 0 \) for \( v \) prime to \( p \) by Proposition
9.7, we conclude from (e1) that \( E^\infty(K) \) is finite. Then the diagram (8.13) has two bottom rows exact
up to finite error. Since the Pontryagin dual of all the modules in the above diagram are \( \Lambda \)-modules of
finite type, we can work with the category of \( \Lambda \)-modules of finite type up to finite error (e.g., [BCM,
VII.4.5]). Then in this new category, the bottom two rows are exact and the extreme left terms a
pseudo-null. Thus the dual sequence of the theorem is exact up to finite error, and by taking dual
sequence in the theorem is exact up to finite error.

\( \square \)

Corollary 10.5. Assume (F) and \( p > 2 \). Then we have

1. The Pontryagin dual \( \text{Sel}_K(J^\text{ord})^\vee \) of \( \text{Sel}_K(J^\text{ord}) \) is a \( \Lambda \)-module of finite type.
2. If further \( \text{Sel}_K(A^\text{ord}) = 0 \) for a single element \( \varpi \in \mathfrak{m}^\infty \) satisfying (A) and (E1), then
   \( \text{Sel}_K(J^\text{ord}) = 0 \) and \( \text{Sel}_K(A^\text{ord}) = 0 \) for every \( \varpi \in \mathfrak{m}^\infty \) satisfying (A) and (E1).
3. Suppose that \( \mathbf{T} \) is an integral domain. If \( \text{Sel}_K(A^\text{ord}) \) is finite for some \( \varpi \) satisfying (A)
   and (e1), then \( \text{Sel}_K(J^\text{ord})^\vee \) is a torsion \( \mathbf{T} \)-module of finite type. Thus if \( \mathbf{T} \) is a unique
   factorization domain, for almost all \( p \in \mathbf{T} \), \( \text{Sel}_K(A_p^\text{ord}) \) is finite.

Proof. The condition (A) and (e2) is satisfied by any non-trivial factor \( \varpi \) of \( (\gamma^{p^r} - 1)/(\gamma - 1) \).

Thus \( \text{Sel}_K(J^\text{ord})^\vee / \varpi \cdot \text{Sel}_K(J^\text{ord})^\vee \) pseudo isomorphic to \( \text{Sel}_K(A_p^\text{ord})^\vee \) which is \( \mathbb{Z}_p \)-module of finite type;
so, by the topological Nakayama’s lemma, we conclude that $\text{Sel}_K(J_{\text{ord}}^\infty)$ is a $\Lambda$-module of finite type. The last two assertions can be proven similarly. 

Suppose that every prime factor of $p$ in $K_1/Q$ has residual degree 1 and that $T$ is a unique factorization domain (so, (A) holds for every $P \in \Omega_T$). Then if $A_P$ for every $P \in \Omega_T$ has potential good reduction and $A_P$ for some $P$ has good reduction at $p$, writing $f_p(U(p)) = a_p f_P$, we have $|a_p - 1|_p = 1$ as otherwise $f_p$ has level raising congruence, and therefore $A_Q$ for some $Q \in \Omega_T$ is potentially multiplicative. Therefore, (E1) is satisfied by Theorem 17.2, and taking $K = Q$, the assertion (4) of Theorem A follows from Corollary 10.5.

From the short exact sequence $(J_{\text{ord}}^\infty(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \text{Sel}_K(J_{\text{ord}}^\infty(K)) \to \text{III}_K((J_{\text{ord}}^\infty)^\vee)$, we get

**Corollary 10.6.** Assume (F). The limit Mordell–Weil group $\text{I}_K(J_{\text{ord}}^\infty(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ and the limit Tate–Shafarevich group $\text{III}_K((J_{\text{ord}}^\infty)^\vee)$ are $\Lambda$-module of finite type.

## 11. Control of ind $\Lambda$-MW groups

**Theorem 11.1.** Assume (A) and (F). Let $p > 2$ and $K$ be either a number field or an $l$-adic field. Put $E_{MW}(K) := \text{Coker}(\varphi(J_{\text{ord}}^\infty)(K) \to J_{\text{ord}}^\infty(K))$. The Pontryagin dual of the following sequence

$$0 \to \widehat{\Lambda}_{\text{ord}}^\infty(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to J_{\text{ord}}^\infty(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to J_{\text{ord}}^\infty(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to E_{MW}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p 	o 0$$

is exact up to $\Lambda$-torsion error of finite type. More precisely, we have

1. **Suppose $K$ is a number field. Then except possibly for $\text{Ker}((p))$ of $\text{I}_K$ which is pseudo isomorphic to $E_{\infty}(K)$, (11.1) is exact up to finite error; so, if $E_{\infty}(K)$ is finite (in particular, if $\text{III}_K(\widehat{A}_{\text{ord}}^\infty)$ is finite), the entire sequence (11.1) is exact up to finite error.**

2. **If $K$ is $l$-adic and either $l \neq p$ or $A_r$ does not have split multiplicative reduction over $W_{\infty}$ for the $p$-adic integer ring $W_{\infty}$ of $K[\mu_p^\infty]$, the sequence (11.1) is exact up to finite error.**

**Proof.** Since $\varphi(J_{\text{ord}}^\infty(K)) = J_{\text{ord}}^\infty(K)/\widehat{\Lambda}_{\text{ord}}^\infty(K)$, we have the following three exact sequences:

$$0 \to \varphi(J_{\text{ord}}^\infty(K)) \xrightarrow{\varphi} J_{\text{ord}}^\infty(K) \to E_{\infty}(K) \to 0,$$

$$0 \to \widehat{A}_{\text{ord}}^\infty(K) \to J_{\text{ord}}^\infty(K) \to \varphi(J_{\text{ord}}^\infty(K)) \to 0,$$

$$0 \to \varphi(J_{\text{ord}}^\infty(K)) \to J_{\text{ord}}^\infty(K) \to E_{MW}(K) \to 0.$$
12. Control of $\Lambda$-BT groups and its cohomology

Recall $G := G_{\alpha,\delta,\xi} = J^\text{ord}_{\infty}[p^\infty]$ which is a $\Lambda$-BT group in the sense of [H14]. Here the set $S$ is supposed to be finite. We study the control of the Tate–Shafarevich group of $G$.

**Theorem 12.1.** Let $K$ be a number field. Suppose $|S| < \infty$, (F) and (A) for $\varpi$. Then the sequence $0 \to \text{III}(K^S/K, \hat{A}_r^\text{ord}[p^\infty]) \to \text{III}(K^S/K, G) \xrightarrow{\varpi} \text{III}(K^S/K, G)$ is exact up to finite error.

**Proof.** From the exact sequence $0 \to \hat{A}_r^\text{ord}[p^\infty] \to G \xrightarrow{\varpi} G \to 0$ of Corollary 17.4, we get a commutative diagram with exact bottom two rows and exact columns:

\[
\begin{array}{cccccccc}
\text{Ker}(\iota_{\text{III}*}) & \longrightarrow & \text{III}(K^S/K, \hat{A}_r^\text{ord}[p^\infty]) & \xrightarrow{\iota_{\text{III}*}} & \text{III}(K^S/K, G) & \xrightarrow{\varpi_{\text{III}*}} & \text{III}(K^S/K, G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E^\infty_{\text{BT}}(K) & \longrightarrow & H^1(\hat{A}_r^\text{ord}[p^\infty]) & \xrightarrow{\iota_*} & H^1(G) & \xrightarrow{\varpi_*} & H^1(G) \\
\prod_{v \in S} E^\infty_{\text{BT}}(K_v) & \longrightarrow & H^1(\hat{A}_r^\text{ord}[p^\infty], S) & \xrightarrow{\iota_{S,*}} & H^1_S(G) & \xrightarrow{\varpi_{S,*}} & H^1_S(G),
\end{array}
\]

where $E^\infty_{\text{BT}}(k) = \text{Coker}(\varpi : G(k) \to G(k))$.

By Lemma 10.1, $E^\infty_{\text{BT}}(K)$ and $E^\infty_{\text{BT}}(K_v)$ are finite. Thus as long as $S$ is finite, $\prod_{v \in S} E^\infty_{\text{BT}}(K_v)$ is finite. Then the above diagram proves the desired exactness. \qed

**Corollary 12.2.** Let the notation and the assumption be as in the theorem. Assume that $\mathbb{T}$ is an integral domain and that $\text{III}(K^S/K, \hat{A}_r^\text{ord}[p^\infty])$ is finite for a principal arithmetic prime $P_0 \in \text{Spec}(\mathbb{T})$. Then $\text{III}(K^S/K, G_{\mathbb{T}})$ is a torsion $\mathbb{T}$-module of finite type. In particular, for almost all principal arithmetic points $P \in \text{Spec}(\mathbb{T})$, $\text{III}(K^S/K, \hat{A}_r^\text{ord}[p^\infty])$ is finite.

13. Control of $\Lambda$-TS groups

We now study control of the limit Tate–Shafarevich group from which Theorem A (2) and (3) in the introduction follows.

**Proposition 13.1.** Suppose that $\mathbb{T}$ is an integral domain flat over $\Lambda$. Let $K$ be a number field and pick an arithmetic point $P \in \text{Spec}(\mathbb{T})$. Assume $|S| < \infty$ and that $P$ is principal with $P = (\varpi)$. Let $\text{Ker}_P^M$ be the kernel of the natural diagonal map: $\hat{A}_P^\text{ord}(K) \otimes_{\mathbb{Z}[\varpi]} \mathbb{Q}_p / \mathbb{Z}_p \to \prod_{v \in S} \hat{A}_P^\text{ord}(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p$. Then we have the following exact sequence

\[
0 \to \text{Ker}_P^M \to \text{III}(K^S/K, \hat{A}_P^\text{ord}[p^\infty]) \xrightarrow{\Pi_P} \text{III}_K(\hat{A}_P^\text{ord}).
\]

In particular, if $\text{III}(K^S/K, \hat{A}_P^\text{ord}[p^\infty])$ vanishes (resp. is finite), the error term $\text{Ker}_P^M$ vanishes (resp. is finite).

By principality of $P$, the conditions (A) is satisfied for $(\varpi, P, A_P, \mathbb{T})$, and as we remarked after stating the condition (F) in Section 4, we actually need in this section is flatness of $\mathbb{T}$ over $\Lambda$ (though we could have supposed the stronger condition (F)).

**Proof.** For $K' = K^S$ and $\overline{K}_v, \hat{A}_v(K')$ (and hence $\hat{A}_v^\text{ord}(K')$) is $p$-divisible $\mathbb{Z}_p$-modules; so, the $\mathbb{Z}_p$-module $\hat{A}_v^\text{ord}(K') / \hat{A}_v^\text{ord}[p^\infty](K')$ is a $\mathbb{Q}_p$-vector space (i.e., it is isomorphic to $\hat{A}_v^\text{ord}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$). From the short exact sequence $\hat{A}_v^\text{ord}[p^\infty](K') \hookrightarrow \hat{A}_v^\text{ord}(K') \to \hat{A}_v^\text{ord}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p (K' = K^S, \overline{K}_v)$ of Galois
modules, we get the following commutative diagram with the bottom two exact rows:

\[
\begin{array}{cccccc}
\text{Ker}^\text{MW}_p & \rightarrow & \Pi(K^S/K, \hat{A}^\text{ord}_r[p^\infty]) & \text{tll.} & \rightarrow & \Pi(K(\hat{A}^\text{ord}_r)) \\
\downarrow & & \downarrow & & \downarrow & \\
\hat{A}^\text{ord}_r(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H^1(\hat{A}^\text{ord}_r[p^\infty]) & \text{t} & H^1(\hat{A}^\text{ord}_r) & \rightarrow 0 \\
\delta & \downarrow & \text{Res}[p^\infty] & \downarrow & \text{Res} & \\
\prod_{v|p} (\hat{A}^\text{ord}_r(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) & \rightarrow & H^1_S(\hat{A}^\text{ord}_r[p^\infty]) & \text{i} & H^1_S(\hat{A}^\text{ord}_r) & \rightarrow 0.
\end{array}
\]

(13.1)

The injectivity of \( I_S \) and exactness of the bottom row prove the exact sequence in the proposition by [BCM, I.1.4.2 (1)]. □

**Proposition 13.2.** Suppose that \( T \) is an integral domain flat over \( \Lambda \). Let \( K \) be a number field and pick an arithmetic point \( P \in \text{Spec}(T) \). Assume \( |S| < \infty \) and that \( P = (\varpi) \) is principal. If \( \text{Ker}^\text{MW}_P \) is finite \( (\Leftrightarrow [\Pi(K^S/K, \hat{A}^\text{ord}_r[p^\infty]) < \infty) \), then the map \( \Pi(K(\varpi(J^\infty_\infty))) \rightarrow \Pi(K(J^\infty_\infty)) \) induced from the inclusion \( \varpi(J^\infty_\infty) \hookrightarrow J^\infty_\infty \) has finite kernel.

**Proof.** Let \( k \) be finite field extension of \( \mathbb{Q} \) or \( \mathbb{Q}_p \). From Corollary 6.4,

\[
C_k := \text{Coker}(J^\text{ord}_\infty(k)/\varpi(J^\text{ord}_\infty)(k) \rightarrow \hat{B}^\text{ord}_P(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \text{Coker}(J^\text{ord}_\infty(k) \rightarrow \hat{B}^\text{ord}_P(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)
\]

is isogenous to \( \hat{A}^\text{ord}_P(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \). If

\[
\text{Ker}^\text{MW}_P := \text{Ker}(\hat{A}^\text{ord}_P(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \prod_{v|p} \hat{A}^\text{ord}_P(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)
\]

is finite, we conclude that \( \text{Ker}(C_K \rightarrow \prod_{v|p} C_K_v) \) is finite. From the short exact sequence \( \varpi(J^\text{ord}_\infty) \hookrightarrow J^\text{ord}_\infty \rightarrow \hat{B}^\text{ord}_P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \), we have the following commutative diagram with exact columns and the bottom two exact rows:

\[
\begin{array}{cccccc}
\text{Ker}(C_K \rightarrow \prod_{v|p} C_K_v) & \rightarrow & \Pi(K(\varpi(J^\text{ord}_\infty))) & \rightarrow & \Pi(K(J^\text{ord}_\infty)) \\
\downarrow & & \downarrow & & \downarrow & \\
C_K & \rightarrow & H^1(\varpi(J^\text{ord}_\infty)) & \rightarrow & H^1(J^\text{ord}_\infty) & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\prod_{v|p} C_K_v & \rightarrow & H^1_S(\varpi(J^\text{ord}_\infty)) & \rightarrow & H^1_S(J^\text{ord}_\infty) & \rightarrow 0.
\end{array}
\]

This shows that the first row is exact by [BCM, I.1.4.2 (1)], and we get the desired assertion. □

**Remark 13.3.** For any arithmetic point \( P \), we write \( H_P \) for the field in \( \text{End}(A_P/\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} \) generated by Hecke operators; so, \( [H_P : \mathbb{Q}] = \dim A_P \). Since \( A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \) is an \( H_P \) vector space, we have \( \dim_\mathbb{Q} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = [H_P : \mathbb{Q}] \cdot \dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \). Similarly, \( \dim_\mathbb{Q} \text{Ker}^\text{MW}_P \otimes_{\mathbb{Z}} \mathbb{Q} \) is a multiple of \( [H_P : \mathbb{Q}] \). Thus \( \dim_\mathbb{Q} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \dim A_P \iff \dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1 \Rightarrow \text{Ker}^\text{MW}_P \) is finite (as the image of \( A_P(K) \) in \( \prod_{v|p} A_P(K_v) \) span at least rank 1 submodule if \( \dim_\mathbb{Q} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} > 0 \)).

Here is an argument telling us that the proof of the finiteness result of the Tate–Shafarevich group stated in Theorem A can be divided into the two cases (1) and (2) in the introduction: Suppose that \( T \) is an integral domain flat over \( \Lambda \) with \( |\Omega_T| = \infty \) (for \( \Omega_T \) in (10.1)). Plainly we have the following four possibilities for such an arithmetic point \( P \in \Omega_T \):

1. \(|\Pi(K(\hat{A}^\text{ord}_P))| = \infty \) and \(|\hat{A}^\text{ord}_P(K)| < \infty \),
2. \(|\Pi(K(\hat{A}^\text{ord}_P))| = \infty \) and \(|\hat{A}^\text{ord}_P(K)| = \infty \),
3. \(|\Pi(K(\hat{A}^\text{ord}_P))| < \infty \) and \(|\hat{A}^\text{ord}_P(K)| < \infty \),
4. \(|\Pi(K(\hat{A}^\text{ord}_P))| < \infty \) and \(|\hat{A}^\text{ord}_P(K)| = \infty \).
Assume that $\dim_{\mathcal{H}_p}(A_{P_0}(K) \otimes \mathbb{Q}) \leq 1$ and $|\mathbf{III}_K(\hat{A}_{P_0}^{\text{ord}})| < \infty$ for a principal arithmetic point $P_0 \in \text{Spec}(\mathbb{T})$ satisfying (e1). Because of (e2)$\Rightarrow$(e1) (by Theorem 17.2), for almost all principal arithmetic points $P \in \text{Spec}(\mathbb{T})$, the condition (e1) is satisfied. Then Case (3) for one $P$ implies finiteness of $\text{Sel}_K(\hat{A}_{P_0}^{\text{ord}})$ for almost all $P \in \Omega_T$ (by Corollary 10.5) which implies $|\mathbf{III}_K(\hat{A}_{P_0}^{\text{ord}})| < \infty$ for almost all principal arithmetic $P \in \text{Spec}(\mathbb{T})$. Thus the case (3) is done. Since we want to prove finiteness of $|\mathbf{III}_K(\hat{A}_{P_0}^{\text{ord}})| < \infty$ in this section, the goal is achieved in the case (4). Therefore we may concentrate on the two cases (1) and (2).

Since there are infinitely many principal arithmetic points $P \in \text{Spec}(\mathbb{T})$, we may assume that one of the cases (1) and (2) (or both) occurs for infinitely many arithmetic principal $P$. We now treat Cases (1) and (2) separately. We first show that Case (2) cannot occur for infinitely many $P$.

**Theorem 13.4.** Suppose that $\mathbb{T}$ is a unique factorization domain flat over $\Lambda$ (so, $|\Omega_T| = \infty$).

Suppose that $\dim_{\mathcal{H}_p}(A_{P_0}(K) \otimes \mathbb{Q}) \leq 1$ and $|\mathbf{III}_K(\hat{A}_{P_0}^{\text{ord}})| < \infty$ for an arithmetic point $P_0 = (\pi_0) \in \text{Spec}(\mathbb{T})$ satisfying (e1). If $|A_{P}(K)| = \infty$ for infinitely many $P \in \Omega_T$, we have $|\mathbf{III}_K(\hat{A}_{P}^{\text{ord}})| < \infty$ and $\dim_{\mathcal{H}_p}(A_{P}(K) \otimes \mathbb{Q}) = 1$ for almost all arithmetic points $P \in \Omega_T$.

Write for simplicity $J := J_{1,\mathbb{T}}^{\text{ord}}(K)$ and $J := J_{1,\mathbb{T}}^{\text{ord}}(K) = \varinjlim_n J_{1,\mathbb{T}}^{\text{ord}}(K)/p^n J_{1,\mathbb{T}}^{\text{ord}}(K)$. The control of the $\mathbb{T}$-module $J$ (resp. the Banach $\mathbb{Z}_p$-dual module $J^* = \text{Hom}_{\mathbb{Z}_p}(J, \mathbb{Z}_p)$) is studied in [H15, Proposition 6.4] (resp. [H15, Theorem 6.6]) for the standard tower, but the argument there is plainly valid for general towers without modification, and the control results of Proposition 6.4 and Theorem 6.6 in [H15] still holds for the exotic towers. In particular, the $\mathbb{T}$-module $J^*$ is pseudo-isomorphic to $\mathbb{T}^\ast \oplus X$ for a torsion $\mathbb{T}$-module $X$ of finite type, and $\text{Supp}_p(X) \cap \Omega_T$ is a finite set (for $\Omega_T \subseteq \text{Spec}(\mathbb{T})$ defined in (10.1)).

**Proof.** By definition, the image of $J$ in $J$ is $p$-adically dense in $J$. By [H15, Proposition 6.4] applied to $A_{r_0} = J_{r_0,\mathbb{T}}^{\text{ord}}$ and taking $r = s$, we find that $\text{Ker}(J_{r_0,\mathbb{T}}^{\text{ord}}(K) \rightarrow J)$ is contained in $G_0(K)$ for all $s$; so, we have $K := \text{Ker}(J \rightarrow J) = \bigcap_n p^n J \subseteq G_0(K)$, which is a $p$-torsion module. Thus by Lemma 10.1, the Pontryagin dual $K^\vee$ is a torsion $\Lambda$-module of finite type. Put $\mathcal{J} := J/K = \text{Im}(J \rightarrow J)$. By [H15, Lemma 6.5] (and its proof), the Pontryagin dual $K^\vee$ is a torsion $\Lambda$-module of finite type, $p^n K = 0$ for some finite $0 < B \in \mathbb{Z}$, and $|K| \leq |G_0(K)|$ if $G_0(K)$ is finite. Tensoring $\mathbb{Q}_p/\mathbb{Z}_p$ with the short exact sequence $0 \rightarrow K \rightarrow J \rightarrow \mathcal{J} \rightarrow 0$, we get an isomorphism

$$J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \cong \mathcal{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p. \quad (13.2)$$

So we have $K := \text{Ker}(\mathcal{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \cong \text{Ker}(J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p).

By $p$-adic density of $J$ in $J$, we have a natural surjection $J/p^n J \rightarrow J/p^n J$ for all $n > 0$. Passing to the limit, we still have a surjection $\mathcal{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p = \varinjlim_n \mathcal{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J/p^n J = J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$.

Put $C := \text{Coker}(\mathcal{J} \rightarrow J)$. We apply the snake lemma to the commutative diagram with exact rows:

$$\begin{array}{ccc}
\mathcal{J} & \longrightarrow & J \\
p^n & \longrightarrow & p^n \\
\mathcal{J} & \longrightarrow & J \
\end{array}$$

From the above surjectivity: $\mathcal{J}/p^n \mathcal{J} \rightarrow J/p^n J$, we conclude that $C$ and hence $C[p^\infty]$ are $p$-divisible and an exact sequence

$$0 \rightarrow \mathcal{J}[p^\infty] \rightarrow J[p^\infty] \rightarrow C[p^\infty] \rightarrow \mathcal{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0. \quad (13.3)$$

Since $C[p^\infty]$ is $p$-divisible, its surjective image $K$ is $p$-divisible. Taking $P_0$-torsion of the sequence

$$J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathcal{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p,$$
we get the following commutative diagram

\[
\begin{array}{ccc}
J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[P_0] & \xrightarrow{\alpha} & \mathcal{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[P_0] \quad \xrightarrow{\beta} \quad J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[P_0] \\
\uparrow{a} & & \uparrow{b} \\
\tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\tilde{\alpha}} & \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \\
& & \uparrow{c} \\
& & \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p
\end{array}
\]

The map \(\alpha\) is an isomorphism by (13.2). The vertical arrow \(a\) is an isogeny by Theorem 11.1 (1) as \(\Pi_{K}(\tilde{A}^{\text{ord}}_{\mathcal{P}_0})\) is finite. If the map \(c\) is also an isogeny (i.e., has finite kernel and cokernel) which we show later in this proof, the map \(\beta\) is an isogeny, and hence Ker(\(\beta\)) is finite. Therefore \(\mathbb{K}[P_0] = \text{Ker}(\beta)\) is finite; so, the Pontryagin dual \(\mathbb{K}^\vee\) is a torsion \(T\)-module and \(P_0 \not\in \text{Supp}_T(\mathbb{K}^\vee)\). Thus we conclude an exact sequence with \(T\)-torsion \(\mathbb{K}^\vee\) of finite type:

\[
(13.4) \quad 0 \to \mathbb{K} \xrightarrow{\mathcal{J}} J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0.
\]

We now show the map \(c\) is an isogeny. Take a generator \(\varpi_0\) of \(P_0\). By [H15, Proposition 6.4], we have the following exact sequence:

\[
0 \to \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \to J \xrightarrow{\varpi_0} J \to \text{Coker}(\varpi_0) \to 0
\]

with \(\mathbb{Z}_p\)-free module \(\text{Coker}(\varpi_0)\) of finite rank and a surjection \(\tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \to \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K)\) with finite kernel. By \(\mathbb{Z}_p\)-flatness of \(\text{Coker}(\varpi_0)\), the sequence

\[
0 \to \varpi_0(J) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \text{Coker}(\varpi_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0
\]

is exact. Tensoring \(\mathbb{Q}_p/\mathbb{Z}_p\) with the short exact sequence \(\tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \xrightarrow{\varpi_0} \varpi_0(J)\), we get an exact sequence

\[
\text{Tor}^1_{\mathbb{Z}_p}(\varpi_0(J), \mathbb{Q}_p/\mathbb{Z}_p) \to \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \varpi_0(J) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0
\]

with \(\text{Tor}^1_{\mathbb{Z}_p}(\varpi_0(J), \mathbb{Q}_p/\mathbb{Z}_p) = \varpi_0(J)[p^\infty]\) which is killed by \(p^{10}\) again by [H15, Lemma 6.5] as already remarked. Thus \(0 \to \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi_0} J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \) is exact up to finite error (as \(\tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K)\) is a \(\mathbb{Z}_p\)-module of finite type). In other words, \(\tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\tilde{\alpha}} J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p\) is an isogeny, and hence \(\mathbb{K}^\vee\) is a \(T\)-torsion of finite type as we already argued.

The \(\mathbb{Z}_p\)-dual module \(J^* = \text{Hom}_{\mathbb{Z}_p}(J, \mathbb{Z}_p)\) is well controlled [H15, Theorem 6.6 (1)], is a \(T\)-module of finite type with \(J^*/\varpi^r J^* (P_0 = (\varpi_0))\) isomorphic to \(\tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K)\) up to finite error. Recall the pseudo-isomorphism: \(J^* \sim T^r \otimes X\) for a \(T\)-torsion \(X\). Thus \(r \leq 1\) by Nakayama’s lemma, but \(r > 0\) as \(A_{\mathcal{P}}(K)\) is infinite for infinitely many \(P\). Applying [H15, Theorem 6.6] to \(A_{\mathcal{P}} = J_{\mathcal{P}}\), we have \(J^r_{\mathbb{Z}_p}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\); so, \(J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\), if \(P \not\in \text{Supp}_T(X)\) is an arithmetic point in \(\Omega_T\), we have \(\dim_{H} A_{\mathcal{P}}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1\) and hence \(|\text{Ker}^{MW}| < \infty\). Since \(\Omega_T \cap \text{Supp}_{\Omega_T}(X)\) is finite, we conclude \(\dim_{H} A_{\mathcal{P}}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1\) for almost all \(P \in \Omega_T\).

Since \(P \not\in \text{Supp}_T(X)\), \(0 \to J^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\varpi_0} J^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K)^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0\) is exact, and dualizing back, \(0 \to \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\varpi_0} J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0\) is exact. By [H15, Proposition 6.4], the sequence \(0 \to \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \to J \xrightarrow{\varpi_0} J \to J/\varpi J \to 0\) is exact up to finite error with \(\mathbb{Z}_p\)-flat quotient \(J/\varpi J\) of finite type. Combined with these two exact sequences, we conclude that

\[
0 \to \tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \to J \xrightarrow{\varpi_0} J \to 0
\]

is exact up to finite error for almost all \(P \in \Omega_T\). Tensoring \(\mathbb{Q}_p/\mathbb{Z}_p\), we get, for almost all \(P \in \Omega_T\), the following exact sequence up to finite error:

\[
\tilde{A}^{\text{ord}}_{\mathcal{P}_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi_0} J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0.
\]
By (13.4), we have the following commutative diagram with exact rows up to finite error:

\[
\begin{array}{cccc}
\mathbb{K} & \rightarrow & J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{K} & \rightarrow & J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \\
on & \downarrow & \downarrow & \downarrow & \\
\mathbb{K}/\mathbb{Q}_p & \rightarrow & J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & 0.
\end{array}
\]

If \( P \notin \text{Supp}(X) \cup \text{Supp}(\mathbb{K}') \), \( \mathbb{K}/\mathbb{Q}_p \) is finite, and we get the following exact sequence up to finite error:

\[ \widehat{A}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathbb{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0.\]

By Theorem 11.1 (1), \( \text{Ker}(\nu) \) is finite. Thus we get a short exact sequence up to finite error:

\[ 0 \rightarrow \widehat{A}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathbb{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0. \]

Thus we have the following commutative diagram with exact top two rows up to finite error and exact columns:

\[
\begin{array}{cccc}
\widehat{A}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow \ \mathbb{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{A}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow \ \mathbb{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \\
on & \downarrow & \downarrow & \downarrow & \\
\text{Sel}_K(\widehat{A}^{\text{ord}}(K)) & \rightarrow & \text{Sel}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) & \rightarrow & \text{Sel}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \rightarrow \ \\
\downarrow & & \downarrow & & \downarrow \\
\text{III}_K(\widehat{A}^{\text{ord}}(K)) & \rightarrow & \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) & \rightarrow & \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \\
\end{array}
\]

Since \( J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \) has finite cokernel, by the snake lemma applied to the top two rows (and \([\text{BCM}, \text{I.4.2 (2)}]\)), we conclude the exactness of

\[ 0 \rightarrow \text{III}_K(\widehat{A}^{\text{ord}}(K)) \rightarrow \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \rightarrow \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \rightarrow \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})), \]

up to finite error.

Since \( \text{III}_K(\widehat{A}^{\text{ord}}(K)) \) is finite, by (8.14), \( E^{\text{ord}}_p(K) \) for \( P_0 \) is also finite. By (8.14), the sequence

\[ 0 \rightarrow \text{III}_K(\widehat{A}^{\text{ord}}(K)) \rightarrow \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \rightarrow \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})), \]

is exact up to finite error. By Proposition 13.2, we may replace \( \text{III}_K(\varpi_0(J^{\text{ord}}(\mathbb{T}, \mathbb{T}))) \) at the right end of the above sequence by \( \text{III}_K(J^{\widehat{\text{ord}}(\mathbb{T}, \mathbb{T})}) \) keeping exactness. Thus finiteness of \( \text{III}_K(\widehat{A}^{\text{ord}}(K)) \) tells us that the Pontryagin dual \( \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \) is a torsion \( \mathbb{T} \)-module of finite type. Thus by (13.7), \( \text{III}_K(\widehat{A}^{\text{ord}}(K)) \) is finite for almost all \( P \).

Instead of using Proposition 13.2, we could use Theorem 11.1. Here is the alternative argument. Since \( E^{\text{ord}}_p(K) \) for \( P_0 \) is finite as already remarked, by Theorem 11.1 (1), we have an exact sequence up to finite error:

\[ 0 \rightarrow \widehat{A}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathbb{J} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow E \rightarrow 0 \]

with \( E := J/\mathbb{Z}_p(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \) \( \mathbb{Q}_p/\mathbb{Z}_p \). In particular, \( P_0 \) is not in \( \text{Supp}(\mathbb{K}') \). We have either (a) \( E = 0 \) or (b) \( E \cong (\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbb{Q}_p}(f_{P_0}) \) as \( J/\mathbb{Z}_p(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \) \( \cong \widehat{A}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{Q}_p(f_{P_0}) \) for the \( p \)-adic Hecke field \( \mathbb{Q}_p(f_{P_0}) \) generated by Hecke eigenvalues of \( f_{P_0} \).

Suppose we are in Case (a). Then applying the commutative diagram (13.6) to \( P = P_0 \) with top two exact rows and exact columns, we get an exact sequence

\[ 0 \rightarrow \text{III}_K(\widehat{A}^{\text{ord}}(K)) \rightarrow \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})) \rightarrow \text{III}_K(J^{\text{ord}}(\mathbb{T}, \mathbb{T})). \]
up to finite error. Then by $|\Pi_K(\hat{A}_P)| < \infty$, we conclude $\Pi_K(J_{\infty, T}^{\text{ord}}) = Z$ is a torsion $T$-module of finite type, and hence by (13.7), we conclude $|\Pi_K(\hat{A}_P)| < \infty$ for almost all $P \in \Omega_T$.

We now assume that we are in Case (b); then, we have $\hat{B}_P(K) \otimes_{Z_p} \mathbb{Q}_p \neq 0$. Thus $\hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p) \neq \hat{A}_P(K)$ is isogenous to $\hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p) \neq \hat{A}_P(K)$ is finite (by $|\Pi_K(\hat{A}_P)| < \infty$).

If $\hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p) \neq \hat{A}_P(K)$ is finite (by $|\Pi_K(\hat{A}_P)| < \infty$), then $\hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p) \neq \hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p)$. If $\hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p) \neq \hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p)$, then $\hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p) \neq \hat{A}_P(K) = Z_p(M \otimes \mathbb{Q}_p)$.

Thus we conclude $\Pi_K(\hat{A}_P) = Z_p(M \otimes \mathbb{Q}_p)$ is finite only for finitely many principal arithmetic places containing archimedean places.

Remark 13.5. If rank $E(\mathbb{Q}) \geq 2$ for a rational elliptic curve $A_P = E/\mathbb{Q}$, $\text{Ker}_E$ has divisible part of corank $E(\mathbb{Q}) - 1$ as rank$_{\mathbb{Q}} E(\mathbb{Q}_p) = \dim E = 1$. In particular, choosing a finite set $S$ of places containing archimedean places, $p$-adic places and all bad places for $E$, $\Pi_1(Q_S/\mathbb{Q}, T_E) = \lim \Pi_1(Q_S/\mathbb{Q}, T_E(1/p^n)) = \lim \Pi_1(Q_S/\mathbb{Q}, T_E(1/p^n))$ contains $T_E(1/p^n)$ for infinitely many arithmetic places.

Thus by the finiteness of $\Pi_K(\hat{A}_P)$, the Pontryagin dual $\Pi_K(J_{\infty, T}^{\text{ord}}) = Z_p(M \otimes \mathbb{Q}_p)$ is finite only for finitely many principal arithmetic places containing archimedean places.

Theorem 13.6. Suppose that $T$ is an integral domain flat over $\Lambda$ with infinitely many principal arithmetic $P \in \text{Spec}(T)$. Suppose further that $\dim_H P(\hat{A}_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}) \leq 1$ and $|\Pi_K(\hat{A}_P)| < \infty$ for a principal arithmetic point $P \in \text{Spec}(T)$ satisfying (e1). If $|\text{Ker}_E| < \infty$ for infinitely many principal arithmetic points $P \in \text{Spec}(T)$, then we have $|\text{Ker}_E| < \infty$ and $|\Pi_K(\hat{A}_P)| < \infty$ for almost all principal arithmetic $P \in \text{Spec}(T)$.

Proof. If $|\Pi_K(\hat{A}_P)| < \infty$ for a single principal arithmetic $P$ with $|\text{Ker}_E| < \infty$ (satisfying (e1)), there is nothing to prove (i.e., Case (3); see Corollary 10.5). Thus we may assume that $|\Pi_K(\hat{A}_P)| = \infty$ for almost all principal arithmetic $P$ with $|\text{Ker}_E| < \infty$ (satisfying (e1)), and try to get absurdity. Because of our assumption $\dim_H P(\hat{A}_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}) \leq 1$ and $|\Pi_K(\hat{A}_P)| < \infty$, by the exact sequence of Proposition 13.1, we get $|\Pi_K(K^S/K, \hat{A}_P[p^n])| < \infty$. By Corollary 12.2, we have $|\Pi_K(K^S/K, \hat{A}_P[p^n])| < \infty$

for almost all principal $P \in \text{Spec}(T)$. Thus we conclude $|\Pi_K(\hat{A}_P)| < \infty$ for those $P$.

By our assumption of the theorem, we have finiteness: $|\text{Ker}_E| < \infty$ for infinitely many arithmetic $P$. Then by the control [H15, Theorem 6.6] of the $\mathbb{Z}_p$-dual of the limit Mordell–Weil group $J$, we conclude

(13.8) $\hat{A}_P(K)$ is finite only for finitely many principal arithmetic $P \in \text{Spec}(T)$. 
If \(|A_P(K)| < \infty|, \hat{B}_P^{\text{ord}}(K)| < \infty\) as \(A_P\) is isogenous to \(B_P\). Then from the exact sequence 
\[
0 \rightarrow \varpi(J_\infty) \rightarrow J_\infty^{\text{ord}} \rightarrow B_P^{\text{ord}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p 
\]
we get an identity \(\varpi(J_\infty^{\text{ord}})(K) = J_\infty^{\text{ord}}(K); \) so, \(E_{MW}(K) = 0\) for the error term \(E_{MW}(K)\) in Theorem 11.1. Since \(J_{K, g}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \bigoplus_{p | \ell^s - 1} J_{s, g}^{\text{ord}}(K)[P] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\) for \(P\) running over arithmetic prime factors of \(t^s - 1\), we have

\[
J_{K, g}^{\text{ord}}(K)(P) = \lim_{\substack{s \to \infty \atop s \neq \ell^s - 1}} J_{s, g}^{\text{ord}}(K)[P] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\]

Since \(J_{K, g}^{\text{ord}}(K)[P] = \hat{A}_P^{\text{ord}}(K)\) by (5.1), by (13.8), the right-hand-side of (13.9) is a finite sum; so, \(K = \text{Ker}(\hat{J} \otimes_{\mathbb{Q}_p} \hat{Q}_p / J \otimes_{\mathbb{Q}_p} \hat{Q}_p)\) is a torsion \(\mathbb{T}\)-module. Since \(G_{\hat{T}}(K)\) is the torsion-part of \(J_{K, g}^{\text{ord}}(K)\), by Lemma 10.1 (or by [H15, §4]), \(G_{\hat{T}}(K)^{\vee}\) is a torsion \(\Lambda\)-module of finite type. From the exact sequence

\[
0 \rightarrow G_{\hat{T}}(K) \rightarrow J_{K, g}^{\text{ord}}(K) \rightarrow J_{K, g}^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,
\]

the support of \(J_\infty^{\text{ord}}(K)\) is a closed subscheme \(\text{Supp}(J_\infty^{\text{ord}}(K))\) of \(\text{Spec}(\mathbb{T})\) of positive codimension. Thus if an arithmetic point \(P = (\varpi)\) is outside \(\text{Supp}(J_\infty^{\text{ord}}(K))\), \(J_\infty^{\text{ord}}(K) / \varpi(J_\infty^{\text{ord}}(K))\) is finite, and \(E_\infty^{\text{ord}}(K) = E_\infty^P(K) := \varpi(J_\infty^{\text{ord}}(K)) / \varpi(J_\infty^{\text{ord}}(K)) \subset J_\infty^{\text{ord}}(K) / \varpi(J_\infty^{\text{ord}}(K))\) is a finite module. Thus for almost all \(P\), by Theorem 11.1 (1), the sequence

\[
0 \rightarrow \hat{A}_P^{\text{ord}}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p / Z_p \rightarrow J_\infty^{\text{ord}}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p / Z_p \rightarrow J_\infty^{\text{ord}}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p / Z_p 
\]

is exact up to finite error. Thus we have the following commutative diagram with exact top two rows up to finite error and exact columns:

\[
\begin{array}{ccc}
\hat{A}_P^{\text{ord}}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p / Z_p & \longrightarrow & J_\infty^{\text{ord}}(K) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p / Z_p \\
\n & \simeq & 0 \\
\nSel_K(\hat{A}_P^{\text{ord}}) & \longrightarrow & Sel_K(J_\infty^{\text{ord}}) \\
\nonto & \longrightarrow & onto \\
\n\Pi_K(\hat{A}_P^{\text{ord}}) & \longrightarrow & \Pi_K(J_\infty^{\text{ord}}) \\
\end{array}
\]

Since \(J_\infty^{\text{ord}}(K) \cong J_\infty^{\text{ord}}(K)\) has finite cokernel, by the snake lemma applied to the top two rows (and [BCM, I.4.2 (2)]), we conclude the exactness of

\[
0 \rightarrow \Pi_K(\hat{A}_P^{\text{ord}}) \rightarrow \Pi_K(J_\infty^{\text{ord}}) \rightarrow \Pi_K(J_\infty^{\text{ord}})
\]

up to finite error.

Since \(\Pi_K(\hat{A}_P^{\text{ord}})\) is finite, by (8.14), \(E_\infty^P(K)\) for \(P_0\) is also finite, and by (8.14), writing \(P_0 = (\varpi_0)\), we have an exact sequence

\[
0 \rightarrow \Pi_K(\hat{A}_P^{\text{ord}}) \rightarrow \Pi_K(J_\infty^{\text{ord}}) \rightarrow \Pi_K(J_\infty^{\text{ord}})
\]

up to finite error. By Proposition 13.2, we can replace the last term \(\Pi_K(J_\infty(\varpi_0))\) by \(\Pi_K(J_\infty^{\text{ord}})\), and the sequence

\[
0 \rightarrow \Pi_K(\hat{A}_P^{\text{ord}}) \rightarrow \Pi_K(J_\infty^{\text{ord}}) \rightarrow \Pi_K(J_\infty^{\text{ord}})
\]

is exact up to finite error. Since \(\Pi_K(J_\infty^{\text{ord}})^{g}\) is a torsion \(\mathbb{T}\)-module of finite type. Then for almost all \(P\), by the exactness of (13.11), \(\Pi_K(\hat{A}_P^{\text{ord}})\) is finite, as desired. \(\square\)

Here we have the following result supplementing Theorem 11.1 which was shown in the above proof of Theorems 13.4 and 13.6:
Corollary 13.7. Suppose that \( \mathcal{T} \) is a unique factorization domain flat over \( \Lambda \). Suppose further that \( \dim_{H_0}(\hat{A}_P(K) \otimes \mathbb{Q}) \leq 1 \) and \( |\Pi_K(\hat{A}_P^{\text{ord}})| < \infty \) for an arithmetic point \( P_0 \in \Omega_{\mathcal{T}} \) satisfying (e1). Then the following two sequences are exact up to finite error

\[
0 \to \hat{A}_P^{\text{ord}}(K) \to J^{\text{ord}}_{\infty, \mathcal{T}}(K) \xrightarrow{\varphi} J^{\text{ord}}_{\infty, \mathcal{T}}(K) \to 0 \quad \text{and} \\
0 \to \hat{A}_P^{\text{ord}}(K) \otimes \mathbb{Z}_p \otimes \mathbb{Q}_p / \mathbb{Z}_p \to J^{\text{ord}}_{\infty, \mathcal{T}}(K) \otimes \mathbb{Z}_p \otimes \mathbb{Q}_p / \mathbb{Z}_p \to 0
\]

for almost all \( P \in \Omega_{\mathcal{T}} \).

Now we record the control result of the limit Tate–Shafarevich group we have proven in the proof of Theorems 13.4 and 13.6:

Theorem 13.8. Suppose that \( \mathcal{T} \) is a unique factorization domain flat over \( \Lambda \). Let \( K \) be a number field. Assume \( |S| < \infty \), that \( \dim_{H_0}(\hat{A}_P(K) \otimes \mathbb{Q}) \leq 1 \) and \( |\Pi_K(\hat{A}_P^{\text{ord}})| < \infty \) for an arithmetic point \( P_0 \in \text{Spec}(\mathcal{T}) \) satisfying (e1). Then for almost all \( P = (\varpi) \in \Omega_{\mathcal{T}} \), we have an exact sequence

\[
0 \to \Pi_K(\hat{A}_P)^{\text{ord}} \to \Pi_K(J_\mathcal{T})^{\text{ord}} \xrightarrow{\varphi} \Pi_K(J^T_{\mathcal{T}}) \quad \text{with finite error, and the Pontryagin dual } \Pi_K(J^T_{\mathcal{T}})^{\vee} \text{ is a } \mathcal{T}\text{-torsion module of finite type.}
\]

Only Case (3) or Case (4) occurs for infinitely many \( P \in \Omega_{\mathcal{T}} \) as proven above, and Case (4) of the above theorem follows from the proof of Theorems 13.4 and 13.6. Case (3) follows from Theorem 10.4.

14. Parameterization of congruent abelian varieties

Let \( B_{/\mathbb{Q}} \) be a \( \mathbb{Q} \)-simple abelian variety of \( GL(2) \)-type (as in the introduction). We assume that \( O_B = \text{End}(B_{/\mathbb{Q}}) \cap H_B \) is the integer ring of its quotient field \( H_B \). Then the compatible system of two dimensional Galois representations \( \rho_B = \{ \rho_B, \lambda \} \) realized on the Tate module of \( B \) has its L-function \( L(s, B) \) equal to \( L(s, f) \) for a primitive form \( f \in S_2(\Pi_1(C)) \) for the conductor \( C = C_B \) of \( \rho_B \) (see [KW09, Theorem 10.1]). Thus \( B \) is isogenous to \( A_f \) over \( \mathbb{Q} \) (by a theorem of Faltings). The abelian variety \( A_f \) is known to be \( \mathbb{Q} \)-simple as \( H_{A_f} \) is generated by \( \text{Tr}(\rho_B(Frob_l)) \) for primes \( l \) outside \( Np \). Let \( \pi_f \) be the automorphic representation of \( GL_2(A) \) associated to \( f \).

Fix a connected component \( \text{Spec}(\mathcal{T}) \) of \( \text{Spec}(h_{\alpha, \delta, \xi}) \cdot (\alpha, \delta, \xi) \neq (0,1,\omega_d) \), for \( P \in \Omega_{\mathcal{T}} \), the minimal (nearly ordinary) form \( f := f_c \) (in the sense of [H09, (L1–3)]) in \( \pi_f \) may not be primitive. Assume that \( P \) is principal (i.e. (A)) and \( f_P \) is on \( \hat{T} \). Then we define \( A_f = J_\mathcal{T}[a_c]^o \) as in (A). If \( H_{A_f} = H_{A_f} = H_B, A_f \) is \( \mathbb{Q} \)-simple and is isogenous to \( A_f \).

Lemma 14.1. Let the notation be as above. If the conductor of \( f \) is divisible by \( Np \), the abelian variety \( A_f \) is isogenous to \( B \) and \( H_{A_f} = H_f = H_B \). If the conductor of \( f \) is equal to \( N \) prime to \( p \) and \( f(U(p)) = \varphi(p)f \), \( A_f \) is isogenous to \( B \otimes_{O_B} B_{/\mathbb{Q}}[\varphi(p)] \) as abelian varieties of \( GL(2) \)-type, which is in turn isogenous to \( B \times B \) just as abelian varieties.

Proof. Since \( a := \text{Tr}(\rho_B(Frob_l)) \in H_{A_f} \) for all \( a \mid Np \), we have \( H_{B_f} \subset H_{A_f} \). Write \( \pi_f = \hat{\otimes}_v \pi_v \) and \( \pi_p = \pi(\varphi, \beta) \) or \( \pi(\varphi, \beta) \) with p-adic unit \( i_p(\varphi(p)) \). Note that the \( f \) is characterized by

\[
f \in H^0(\hat{T}, \mathcal{F}_1(Np^r)), \quad \pi \subset H^1(\mathcal{F}_1(Np^r)), \quad f(T(l)) = a_lf \quad \text{for all } a \mid Np, \quad f(U(p)) = \varphi(p)f
\]

and \( \pi((0_{\delta, \xi}))f = \varphi(d)\beta(a)f \) for \( a, d \in \mathbb{Z}_p^* \), writing \( T(l) \) for \( U(l) \) if \( l \mid N \) (see [H89, §2]). Moreover for the member \( \rho_f \) of \( \rho_B \) associated to the place \( p_A \) induced by \( i_p : \mathcal{T} \leftarrow \mathcal{T}_p \), we have (cf. [H89, §2])

\[
\rho_f|_{I_{p}} \cong \left( \begin{array}{cc}
v & 0 \\ \psi & \varphi \end{array} \right)
\]

with \( \beta = \left| i_p^{-1} \circ \psi \right| \) (\( \psi \) has finite order over \( I_p \))

for the inertia subgroup \( I_p \subset \text{Gal}(\mathcal{T}_p / \mathcal{T}) \), regarding \( \varphi, \psi \) as characters of \( I_p \) by local class field theory. Then \( \sigma \in \text{Gal}(\mathcal{T}/H_B) \Leftrightarrow \rho_B^{(\sigma)} \cong \rho_B \Leftrightarrow (\pi(\infty))^{\sigma} \cong \pi(\infty) \), where \( \pi(\infty) = \hat{\otimes}_{l \mid \infty} \pi_l \). This shows the minimal field of definition of \( \pi(\infty) \) is \( H_B \) (a result of Waldspurger), and by (14.2), \( H_B \) contains the values of \( \varphi|_{I_p} \). Thus \( H_{A_f} = H_B(\varphi) \) generated over \( H_B \) by the values of \( \varphi \), as the central
character \( \psi P \) of \( \pi \) has values in \( H_B \) over \( A^{(\infty)} \) (which follows from the fact that \( \det \rho_B = \psi P \nu \) for the compatible system \( \nu \) of the cyclotomic characters). Let \( \sigma \in \text{Gal}(\overline{Q}/Q) \). If \( \varphi \) or \( \beta \) is non-trivial over \( \mathbb{Z}_p^\times \) or \( A_F \) is potentially multiplicative at \( p \) (i.e., the conductor of \( f \) is divisible by \( p \)), the nearly ordinary vector \( \mathbf{f} \) is characterized by the above properties (14.1) without \( \mathbf{f} / \mathbf{U}(p) = \varphi(p) \mathbf{f} \). Thus in this case, \( \mathbf{f}' \in \pi'^{\infty} \cong \pi \) for \( \sigma \in \text{Gal}(\overline{Q}/H_B) \) implies \( \mathbf{f}' = \mathbf{f} \). In particular, \( H_{A_F} = H_B \) as desired. If \( f \) has conductor \( N \), \( \mathbf{f} \) is \( p \)-stabilized (i.e., \( f(z) = f(z - \beta(p)f(pz)) \)), then \( H_{A_F} = H_B(\varphi(p)) \). Since \( \varphi(p) \) satisfies \( X^2 - a_p X + \psi P(p) = 0 \) for the \( T(p) \) eigenvalue \( a_p \) of \( f \), we have \( |H_{A_F} : H_B| \leq 2 \), and \( A_F \) is isogenous to \( B \otimes_{O_B} O_B[\varphi(p)] \) (as an abelian variety of GL(2)-type).

If the central character \( \psi P \) is trivial, \( H_B \) is totally real, and \( H_B(\varphi(p)) \) is totally imaginary; so, \( A_F \) is isogenous to \( B \times X \) if the conductor of \( B \) is prime to \( p \). Even if the central character is not trivial, choosing a square root \( \zeta := \sqrt{\psi P(p)} \), \( T(p) \zeta^{-1} \) is self adjoint on \( S_2(\Gamma_0(N), \psi P) \) (e.g., [MFM, Theorem 4.5.4]), and hence \( a_p \zeta^{-1} \) is totally real, but for the root \( \varphi(p) \zeta^{-1} \) of \( X^2 - a_p \zeta^{-1} X + p \), \( Q(\varphi(p) \zeta^{-1}) \) is totally imaginary as with \( |a_p| \leq 2 \sqrt{p} \) combined with \( |\beta(p)|_p < |\varphi(p)|_p = 1 \). This shows that \( H_{A_F} \) is a quadratic extension of \( H_B \), and hence \( A_F \) is isogenous to \( B \times X \). \( \square \)

Let \( A \) be another \( Q \)-simple abelian variety of GL(2)-type. Thus \( A \) is isogenous to \( A_0 \) for a primitive form \( g \in S_2(\Gamma_1(C_A)) \) of conductor \( C_A \). Let \( \pi_0 \) be the automorphic representation of \( g \), and write \( g \) for the minimal nearly \( p \)-ordinary form in \( \pi_0 \). Without losing generality, we may (and do) assume that \( O_A = \text{End}(A_0) \cap H_A \). Note that \( H_B \cong Q(f) \subset \overline{Q} \) and \( H_A \cong Q(g) \). Suppose \( A \) is congruent to \( B \) modulo \( p \) with \( (B[p_1] \otimes_{\mathbb{Z}[p_1]} \mathbb{F}_p)^{ss} \cong (A[p_1] \otimes_{\mathbb{Z}[p_1]} \mathbb{F}_p)^{ss} \) as \( \text{Gal}(\overline{Q}/Q) \)-modules. Here, for any ring \( R \) and a prime ideal \( p \), \( \kappa(p) \) is the residue field of \( p \).

Write \( O_{p_A} \) for the \( p_A \)-adic completion of \( O_A \), and let \( \mathbb{T}_{p_A} = \lim_{\leftarrow} A[p_A] / \mathbb{Q} \) (the \( p_A \)-adic Tate module of \( A \)). We call that \( A \) is of \( p_A \)-type (\( \alpha, \delta, \xi \)) if we have an exact sequence of \( I_p \)-modules \( 0 \to V(\nu \epsilon^\delta, \epsilon^{-1}) \to T_{p_A} A \to V(\epsilon^\alpha, \xi^{-1}) \to 0 \) with \( V(\nu \epsilon^\delta, \epsilon^{-1}) \cong V(\epsilon^\alpha, \xi^{-1}) \cong O_{p_A} \) as \( O_{p_A} \)-modules, where \( \epsilon \) is a character of \( \text{Gal}(\mathbb{Q}_p)[\mu_{p_{\infty}}] / \mathbb{Q}_p \cong \mathbb{Z}_p / O_{p_A} \) with values in \( \mu_{p_{\infty}} \). Write \( \mu_{p_{\infty}} \) for \( \zeta_{p_{\infty}} \) for \( \zeta \in \mu \) acts on \( V(\nu \epsilon^\delta, \epsilon^{-1}) \) by \( u(1) \cdot (\epsilon^{-1}(1)) \). Here \( [x, \mathbb{Q}_p] \) is the local Artin symbol. If \( \xi(\zeta, \zeta') = \xi(\zeta) \) for \( (\zeta, \zeta') \in \mu_2 \), then \( \zeta = 0 \), this is just a \( p_A \)-ordinariness.

Choosing \( g \) (resp. \( f \)) well in the Galois conjugacy class of \( g \) (resp. \( f \)), we may assume that \( A \) and \( B \) are both induced by the embedding \( i_p: \overline{Q}_p \hookrightarrow \overline{Q} \).

**Lemma 14.2.** Let the notation be as above. Suppose that \( C_A / C_B \in \mathbb{Z}[1/2] \times \mathbb{Z} \) and that \( B \) (resp. \( A \)) is of \( p_B \)-type (resp. \( p_A \)-type) \( (\alpha, \delta, \xi) \). Write \( C_B = NP \). Then there exists a connected component Spec(\( \mathcal{T} \)) of Spec(\( \mathcal{S}_{\alpha, \beta}(N) \)) such that for some primes \( P, Q \in \text{Spec}(\mathcal{T}) \), \( f = f_P \) and \( g = f_Q \).

**Proof.** Let \( \varphi \) be the two-dimensional Galois representation into GL(2)(\( \mathbb{F} \)) realized on \( B[p_B] / \mathbb{F} \). Write \( N \) for the prime-to-\( p \)-part of \( C_B \) (and hence of \( C_A \)). Replacing \( \varphi \) by its semi-simplification, we may assume that \( \varphi \) is semi-simple. Since \( (B[p_B] \otimes_{\mathbb{Z}[p_B]} \mathbb{F}_p)^{ss} \cong (A[p_A] \otimes_{\mathbb{Z}[p_A]} \mathbb{F}_p)^{ss} \), \( L(s, A) = L(s, g) \) and \( L(s, B) = L(s, f) \) imply \( f \) mod \( p_B = g \) mod \( p_B \). Since \( f_B := f \) is nearly \( p \)-ordinary with nearly ordinary character given by \( u(\zeta, \mathbb{Q}_p) \mapsto \epsilon_B(u)^\alpha \xi^{-1}(1, \zeta) \) for all \( \zeta \in \mu \), \( \zeta \in \mu_{p_{\infty}} \) for each character \( \epsilon_B : Z^\times \to \mu_{p_{\infty}} \) has central character \( \zeta \mapsto \epsilon_B(z)^{\alpha} \xi^{-1}(z, \zeta) \) for \( z \in \mathbb{T} \), \( f_B \) generates an automorphic representation whose \( p \)-component \( \pi_p \) is given by the principal series \( \pi(\phi, \varphi) \) (or the Steinberg representation \( \sigma(\phi, \varphi) \)) with \( \varphi(\zeta) = \epsilon(\zeta)^{\alpha} \xi^{-1}(1, \zeta) \) and \( \phi(\zeta) = |\zeta| \). Moreover, \( f_B U(p) = \varphi(p) f_B \) with \( \ord(p)(\varphi(p)) = 0 \). See [H89, §2] for these facts (in particular, the \( p \)-component of \( f_B \) is proportional to the \( p \)-ordinary character \( v \) in \( p \), fixed by the \( p \)-component of \( \Gamma H_{\varphi} \) characterized by \( \pi_p((\zeta, \zeta')) v = \phi(\delta \varphi(\delta) v \) for \( a, d \in \mathbb{Q}_p \) and \( U(p)(v) = \varphi(p) v \).

The form \( f_A := g \) associated to \( A \) has similar property whose \( p \)-component is given by \( \pi(\phi', \varphi') \) (or the Steinberg representation \( \sigma(\phi', \varphi') \)) with \( \phi' \) mod \( p_A = \varphi \) mod \( p_B \) and \( \phi' \) mod \( p_B = \phi \) mod \( p_B \). More precisely, we have \( \varphi(\zeta) = \epsilon A(\zeta)^{\alpha} \xi^{-1}(1, \zeta) \) and \( \phi'(\zeta) = |\zeta| \phi A(\zeta)^{\alpha} \xi^{-1}(1, \zeta) \) for a character \( \epsilon A : Z^\times \to \mu_{p_{\infty}} \). Thus \( g = f_A \) (resp. \( f = f_B \)) is lifted to a \( p \)-adic analytic family (of type \( (\alpha, \delta, \xi) \)) parameterized by an irreducible component Spec(\( I \)) (resp. Spec(\( J \))).
Spec(h_{0,\delta,\xi}(N)). Since f \mod p_B = g \mod p_B, the algebra homomorphisms \( \lambda_f : h_{0,\delta,\xi}(N) \to \overline{\mathbb{Q}}_p \)
realized as \( f(T(n)) = \lambda_f(T(n))f \) and \( g(T(n)) = \lambda_g(T(n))g \) satisfy \( \lambda_f \equiv \lambda_g \mod m \) for a maximal ideal \( m \) of \( h_{0,\delta,\xi}(N) \). Then, \( P = \text{Ker}(\lambda_f) \) and \( Q = \text{Ker}(\lambda_g) \) belong to the connected component Spec(\( \mathbb{T} \)) given by \( \mathbb{T} = h_{0,\delta,\xi}(N)_m \), since the local rings of \( h_{0,\delta,\xi}(N) \) corresponds one-to-one to the maximal congruence classes modulo \( \mathfrak{P} \) (\( \mathfrak{P} := \{x \in \overline{\mathbb{Q}}_p : |x|_p < 1\} \)) of Hecke eigenforms of prime-to-\( p \) level \( N \) (and of type \( (\alpha, \delta, \xi) \)) just because the set of maximal ideals \( \Sigma \) of \( h_{0,\delta,\xi}(N) \) is made of \( \ker(\lambda) \) for \( \lambda \in \Sigma = \text{Hom}_{\text{alg}}(h_{0,\delta,\xi}(N), \mathfrak{P}_p) \). The maximal ideal \( m \) is given by
\[
\ker(\lambda_f \mod \mathfrak{P}) = \ker(\lambda_g \mod \mathfrak{P})
\]
for \( \mathfrak{P} = \{x \in \overline{\mathbb{Q}}_p : |x|_p < 1\} \). \( \square \)

The following result is just a combination of the above Lemma 14.2 and Theorem 5.3.

**Corollary 14.3.** Let the notation and the assumptions be as in Lemma 14.2 and Theorem 5.3 (in particular, we assume (F)). Assume that the abelian variety \( B \) has conductor \( N \) prime to \( p \). Let \( f \in S_2(\Gamma_0(N), \chi) \) be the primitive form with conductor \( N \) prime to \( p \) (so, \( \xi = 1 \)) whose \( L \)-function gives \( L(s, B) \). Write \( \chi \cdot |x|^1 \) for the central character of the automorphic representation generated by \( f \). Write \( f(T(p)) = a_p f \). If \( p \nmid 6D_{\chi}N\varphi(N) \) and \( (a_p \mod p_B) \not\in \Omega_{B,p} := \{0, 1, \pm \sqrt{\chi(p)}\} \), then \( \mathbb{T} \) is a regular integral domain and \( f \) and \( g \) belongs to Spec(\( \mathbb{T} \)).

Again we can replace the condition: \( p \nmid 6D_{\chi}N\varphi(N) \) by \( p \nmid 2D_{\chi}N\varphi(N) \) in the case where \( h_{0,\delta,\xi}(N) \) is \( \Lambda \)-free (see Proposition 18.2 for such cases).

**15. A Generalized version of Theorem B including exotic towers**

Let \( B/Q \) be a \( Q \)-simple abelian variety of GL(2)-type of conductor \( N \) such that \( O_B = \text{End}(B/Q) \cap H_B \) is the integer ring of its quotient field \( H_B \). Let \( \rho_B = \{\rho_{B,1}\} \) be the two dimensional compatible system of Galois representations associated to \( B \). Then \( \rho_B \) comes from a Hecke eigenform \( f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N), \chi) \) by [KW09, Theorem 1.10.1]: so, \( L(s, B) = L(s, \rho_B) = L(s, f) \). Fix an embedding \( O_B \hookrightarrow \overline{\mathbb{Q}} \) and write \( p_B \) for the prime ideal of \( O_B \) induced by \( i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \). Then we realize the Hecke algebra \( h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) \) inside \( \text{End}_{\mathbb{C}}(S_2(\Gamma_0(N), \chi)) \) which is generated over \( \mathbb{Z}[\chi] \) by all Hecke operators \( T(n) \) and \( U(l) \). Then this Hecke algebra is free of finite rank over \( \mathbb{Z} \), and hence its reduced part (modulo the nilradical) has a well defined discriminant \( D_{\chi} \) over \( \mathbb{Z} \).

**Definition 15.1.** Let \( S = S_B \) be the set of prime factors of \( 6D_{\chi}N\varphi(N) \) for the conductor \( N \) of \( \rho_B \), where \( D_{\chi} \) is the discriminant of the reduced part of \( h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) \).

We could include \( p = 3 \) defining \( S = S_B \) to be the set of prime factors of \( 2D_{\chi}N\varphi(N) \) if \( h_{0,\delta,1} \) is \( \Lambda \)-free (see remarks after Proposition 4.1 and see also Proposition 18.2).

The prime \( p \) is admissible for \( B \) if \( B \) has good reduction modulo \( p \) (so, \( p \nmid N \)) and \( (a_p \mod p_B) \not\in \Omega_{B,p} := \{0, 1, \pm \sqrt{\chi(p)}\} \) (so, \( B \) has potential partially \( p_B \)-ordinary reduction modulo \( p \)) and write \( p_B \)-type of \( B \) as \( (\alpha, \delta, 1) \). Since \( B \) has conductor prime to \( p \), \( \rho_B \) is unramified at \( p \), and \( \xi \) has to be the identity character \( 1 \) of \( \mu \times \mu \) (on the other hand, \( (\alpha, \delta) \) can be freely chosen). We prove the following result more general than Theorem B including abelian varieties \( A \) of \( p_A \)-type \( (\alpha, \delta, 1) \) (not just those \( p_A \)-ordinary ones):

**Theorem 15.2.** Assume (F) for \( (\alpha, \delta, 1) \), and let \( K \) be a number field. Let \( p \not\in S_B \) be a prime admissible for \( B \) and \( N \) be the conductor of \( B \). Suppose that \( B \) is isogenous to \( A_p \) and \( \text{dim}_{H_B} B(K) \otimes \mathbb{Q} \leq 1 \). Consider the set \( A_{B,p} \) made up of all \( Q \)-isogeny classes of \( Q \)-simple abelian varieties \( A/Q \) of \( p_A \)-type \( (\alpha, \delta, 1) \) congruent to \( B \) modulo \( p \) over \( Q \) with prime-to-\( p \) conductor \( N \).

Then, almost all members \( A \in A_{B,p} \) have finite \( \text{dim}_{K}(A)_{\mathbb{Q}_{p}} \) and \( \text{dim}_{H_A, \mathbb{Q}} (A) \otimes \mathbb{Q} \) is a constant independent of \( A \) given by \( 0 \) or \( 1 \). If further \( \mathbb{B}^{\text{ord}} \cong \mathbb{A}^{\text{ord}} \) for \( p_0 \in \Omega_{\mathbb{Q}} \) with \( \text{Sel}_{K}(B)_{p_0} = 0 \) and all prime factors of \( p \) in \( K \) has residual degree \( 1 \), then \( \text{Sel}_{K}(A)_{\mathbb{Q}_{p}} \) is finite for all \( A \in A_{B,p} \) without exception.

As is well known, there are density one (partially) ordinary admissible primes in \( O_B \) if \( B \) does not have complex multiplication (e.g., [H13b, Section 7]).
Proof. Suppose that $p$ is outside $S_B$, by Theorem 5.3.7, $T$ is a regular integral domain $\mathcal{I}$. Thus for any $P \in \Omega_T$, we have $P = (\omega)$ for $\omega \in \mathcal{I}$ and $(\omega, A_P)$ satisfies (A).

Since $B[p_B^\infty]$ is an ordinary Barsotti–Tate group by our assumption, $A[p_B^\infty]$ is potentially ordinary by the congruence modulo $p$ between $A$ and $B$. Here we say $A[p_B^\infty]$ "potentially ordinary" if $H_0(k, A[p_B^\infty])$ has non-trivial $p$-divisible rank and $A[p_B^\infty]$ over $\mathbb{Q}_p$ extends to a Barsotti–Tate group with non-trivial étale quotient over the integer ring of a finite extension $k$ of $\mathbb{Q}_p$. Choosing the embedding $O_A \hookrightarrow \mathbb{Q}$ well, we may assume that $p_\lambda$ is induced by $i_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$. Then by Lemma 14.2, $A$ is isogenous to a modular abelian variety $A_P$ for $P \in \Omega_T$ of a connected component $\text{Spec}(\mathbb{T})$ of $\text{Spec}(\mathfrak{m}_{a,1}(N))$ for the big $p$-adic Hecke algebra $\mathfrak{m}_{a,1}(N)$. Since $B$ is of $\text{GL}(2)$-type, we have $B \sim A_{P_0}$ (an isogeny) for $P_0 \in \Omega_T$ with $P_0 = (\omega_0)$. Thus we conclude, up to isogeny,

$$A_{B,P} = \{ A_Q | Q \in \Omega_T \}$$

by the theorem of Khare–Wintenberger [KW09, Theorem 10.1.1] (combined with the proof of the Tate conjecture for abelian varieties by Faltings). Since $O_A$ is the integer ring of $H_A$, we can factor $O_{A,p} = O_A \otimes \mathbb{Z}_p$ into the product $O_{A,p} = O_{A,p}^\text{ord} \otimes O_{A,p}^{ss}$ so that for the idempotent $e$ of the factor $O_{A,p}^\text{ord}$, $eA[p_B^\infty]$ is the maximal $p$-ordinary Barsotti–Tate group which becomes étale and multiplicative after étale extension. Since $O_{A,p}$ acts on $\hat{A}$, we can define $\hat{A}^\text{ord} := e(\hat{A})$. Since $A$ is isogenous to $A_P$, $\hat{A}^\text{ord}$ is isogenous to $\hat{A}_P^\text{ord}$; so, $\prod_k(\hat{A}_P^\text{ord})$ is isogenous to $\prod_k(\hat{A}_P^\text{ord})$. Then if $\prod_k(\hat{A}_P^\text{ord})$ is finite, $\prod_k(\hat{A}_P^\text{ord})$ is finite as it is isogenous to the finite $\prod_k(\hat{A}_P^\text{ord})$. Since $\prod_k(\hat{A}_P^\text{ord})$ is finite and $\prod_k(\hat{A}_P^\text{ord})$ is finite, we conclude finiteness of $\prod_k(\hat{A}_P^\text{ord})$ for almost all $\lambda \in A_{B,P} \cong \Omega_T$. The assertion for the Mordell–Weil rank also follows from Theorems 13.4 and 13.6.

Suppose $\text{Sel}_K(\hat{A}_P^\text{ord}) = 0$ and $\mathcal{K}$ for all $v|p$ has residue field $\mathbb{F}_p$. Then $|\varphi(\text{Frob}_v)|^{-1} = |a_p - 1|_p = 1$ as $p \notin \Omega_{B,P}$. Thus by Schneider [Sc83, Proposition 2, Lemma 3] (see also [Sc82, Proposition 2]), we have, for all $v|p$,

$$|H^1(K_v[\mu_p^\infty]/K_v, \hat{A}_P^\text{ord}(K_v[\mu_p^\infty]))| = |\hat{A}_P^\text{ord}(\mathbb{F}_p)|^2 = |\hat{A}_P(\mathbb{F}_p)|^2.$$  

(15.1)

Note that $|\hat{A}_P(\mathbb{F}_p)|^2 = 1$ by our assumption. Strictly speaking, Schneider assumes in [Sc83, §7] that $A_v$ has ordinary good reduction, but his argument works well without change replacing $(A_v(p) := A_v[p^{\infty}], A_v)$ there by $(A_v[p^{\text{ord}}, A_v])$. Indeed, he later takes care of the general case of formal Lie groups in [Sc87, Theorem 1] (including the case of the ordinary part of the formal group of $A_v$). So, $E^\infty(K_v)^T = E\text{Sel}(K_v)^T = 0$ for all $v|p$ (see Theorem 17.2 and Corollary 10.3 for more details of this fact). Then by the same argument as above, using Corollary 10.5 (2) in place of Theorems 13.4 and 13.6, we conclude $\text{Sel}_K(\hat{A}_P)|_{P_0}$ is finite for all $P \in \Omega_T$. \hfill $\square$

Remark 15.3. If we start with an elliptic curve $E$ as in Theorem B, by its modularity, we find a modular factor $B \subset J_1(Np')$ isogenous to $E$. Choose $(\alpha, \delta, 1) = (0, 1, 1)$. The finiteness of $\prod_k(\hat{A}_P^\text{ord})$ implies the finiteness of $\prod_k(\hat{A}_P^\text{ord})$; so, the above theorem implies the statements of Corollary in Theorem 13.4. The rational elliptic curves listed in Corollary C give examples for such curves with Mordell–Weil $\mathbb{Q}$-rank 0 and finite Tate–Shaferievich group, and the elliptic curve factor of $J_0(37)$ with root number $-1$ give a Mordell–Weil $\mathbb{Q}$-rank 1 example with finite $\prod_k(E)$. 

Here is a conjecture:

Conjecture 15.4. Suppose and $\xi(\alpha, \alpha) = 1$ for all $\alpha \in \mathbb{Z}_p$. Fix a totally real field $K$. Let $\text{Spec}(\mathcal{I})$ be a primitive irreducible component of $\text{Spec}(\mathfrak{m})$. If $\alpha/\delta = 1$, we assume that the root number is $\epsilon := \pm 1$ for $K$. Then,

1. if $\alpha/\delta = 1$, we have $\dim_{H_P} A_P(K) \otimes \mathbb{Q} = \frac{\dim K}{2}$ for almost all $P \in \Omega_2$,

2. if $\alpha/\delta \neq 1$, we have $\dim_{H_P} A_P(K) \otimes \mathbb{Q} = 0$ for almost all $P \in \Omega_2$.

As we remarked after stating Theorem A, assuming that $T$ is a unique factorization domain, thanks to the proof of the parity conjecture [N07], Conjecture 15.4 (1) holds once we find a good point $P_0$ with $A_{P_0}$ satisfying the assumptions of Theorem 15.2.
16. $p$-LOCAL COHOMOLOGY OF FORMAL LIE GROUPS

We prove a technical lemma on Galois cohomology for proving vanishing of the error terms when $l = p$ in Theorem 17.2. Just for finiteness of the error term, as will be explained in the proof of the theorem, it follows from the computation of the universal norm by P. Schneider in [Sc83, Proposition 2 and Lemma 3, §7] and [Sc87, Theorem 1], and therefore, perhaps, for the first reading, the reader may want to skip this section.

Let $K$ be a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. Write $K_s = K[\mu_p]$ and $X^{ur}$ for the maximal unramified extension of $X = K, K_s$ and $\overline{X}^{ur}$ is the completion of $X^{ur}$. Let $A$ be an abelian variety defined over $K$. Suppose that $\text{End}(A/K)$ contains a reduced commutative algebra $\mathfrak{A}$. Assume

(A1) $A/K$ has semi-simple reduction over the integer ring $W_r$ of $K$;

(A2) The formal Lie group of the Néron model of $A$ over $W_r$ has a maximal multiplicative factor $\mathfrak{A}$ (see [Sc87, §1] for the maximal multiplicative factor);

(A3) We now study the Galois action on the group functor $\text{Gal}(\mathbb{Q}/K)$ of the theorem, it follows from the computation of the universal norm by P. Schneider in [Sc83, Theorem 1], and therefore, perhaps, for the first reading, the reader may want to skip this section.

Let $\text{Res}_K$ be a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. Write $K_s = K[\mu_p]$ and $X^{ur}$ for the maximal unramified extension of $X = K, K_s$ and $\overline{X}^{ur}$ is the completion of $X^{ur}$. Let $A$ be an abelian variety defined over $K$. Suppose that $\text{End}(A/K)$ contains a reduced commutative algebra $\mathfrak{A}$. Assume

(A1) $A/K$ has semi-simple reduction over the integer ring $W_r$ of $K$;

(A2) The formal Lie group of the Néron model of $A$ over $W_r$ has a maximal multiplicative factor $\mathfrak{A}$ (see [Sc87, §1] for the maximal multiplicative factor);

(A3) Writing $\mathfrak{A}$ for the p-adic closure of the image of $A$ End($A/W_r$, we have $\mathfrak{A} \cong \hat{\mathfrak{A}} \otimes_{\mathbb{Z}_p} \mathfrak{A}$ over $\hat{W}_r$ as formal $\mathcal{O}_{\mathfrak{A}}$-modules, where $\mathfrak{A}$ is an $\mathcal{O}_{\mathfrak{A}}$-lattice in $\mathcal{O}_{\mathfrak{A}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.)

We now study the Galois action on the formal $\mathcal{O}_{\mathfrak{A}}$-algebra $\mathfrak{A}$.

We now study the Galois action on the group functor $R \mapsto A(R)$ so that

$$
\sigma \cdot x := \psi^{-1}(\sigma_{\mathfrak{A}}(\mu_p)) \sigma(x)
$$

for $\mathfrak{A}[\mu_p]$-algebras $R$, where $\sigma \in \text{Aut}(R/\mathfrak{A})$. Since $\psi(\sigma)^{-1} \in \text{Aut}(R/\mathfrak{A})$ gives a descent datum (see [GME, §1.11.3, (DS2)], we can twist $A$ by this cocycle, and get another abelian variety $A_{\mu/\mathfrak{A}}$.)

Similarly, on $\text{Res}_K[A[p^\infty]^\text{et}]$, $\text{Gal}(K^{ur}/K^{ur}) \cong \text{Gal}(\mathbb{Q}_p[\mu_p]/\mathbb{Q}_p)$ acts by a character $\varphi$ with values in $O_{\mathfrak{A}}$. Identifying $\mathfrak{A}$ with the corresponding character of $\text{Gal}(\mathbb{Q}_p[\mu_p]/\mathbb{Q}_p)$, we get another abelian variety $A_{\mu/\mathfrak{A}}$. Thus the Galois action on $A_{\mu/\mathfrak{A}}$ is unramified over $\mathbb{Q}_p$.

For a scheme $X$ over $S'$, and finite flat morphism $S' \rightarrow S$, we write $\text{Res}_{S'/S}X$ for the Weil restriction of scalars; so, $\text{Res}_{S'/S}X$ is a scheme over $S$ such that $\text{Res}_{S'/S}X(T) = X(S' \times S T)$. We describe the twisted abelian variety $A_{\tau}$ ($= \mu, et$) as a factor of $\text{Res}_{K_s/K}A$. Here is a known facts from [NMD, §6]:

(Res1) If $S'/S$ is finite flat, $\text{Res}_{S'/S}X$ exists [NMD, Theorem 4],

(Res2) If $X$ is a separated scheme over $S$, the natural map $X \rightarrow \text{Res}_{S'/S}(X \times S S')$ corresponding to the projection $T \times S S' \rightarrow T$ is a closed immersion [NMD, page 197],

(Res3) If $X \rightarrow S$ is a closed immersion, then $\text{Res}_{S'/S}X \rightarrow \text{Res}_{S'/S}X$ is a closed immersion,

(Res4) Let $k'/k$ be a finite extension of fields. If $X/k'$, for a field $k'$, is a closed immersion of $A_{\tau}$, then $X_{k'/k}$ is a closed immersion.

Let $\text{Res}_{K_s/K}A$ be the restriction of scalars. Since $A_{\mu} \cong A \cong A_{et}$ over $W_r$, we find $\text{Res}_{K_s/K}A \cong \text{Res}_{K_s/K}A_{et}$.

Thus $\text{Res}_{K_s/K}A(R) = A(R \otimes_K K_s)$ for each $K$-algebra $R$, the inclusion $R \hookrightarrow R \otimes_K K_s$ gives by $x \mapsto x \otimes 1$ produces a monomorphism of covariant functors $A(R) \rightarrow \text{Res}_{K_s/K}A(R)$; so, we have a morphism of schemes (by Yoneda’s lemma), $A \rightarrow \text{Res}_{K_s/K}A$. Since $A$ and $\text{Res}_{K_s/K}A$ are projective, we find that $A \rightarrow \text{Res}_{K_s/K}A$ is a closed immersion. In the same way, we have another closed immersion $A_{\mu} \hookrightarrow \text{Res}_{K_s/K}A_{\mu} \cong \text{Res}_{K_s/K}A_{et}$.

Since $K_s \otimes_K K' \cong \prod_{\sigma \in \text{Gal}(K_s/K')} K_r \sigma$ by sending $x \otimes y$ to $(x(\sigma)(y))_\sigma$ for any variety $X$ defined over $K_r$, we have $\text{Res}_{K_s/K}X \cong \prod_{\sigma} X^{\sigma}$, where $X^{\sigma} = X \otimes_{K_r, \sigma} K_r$. Thus $\tau \in \text{Gal}(K_r/K)$ acts on $\text{Res}_{K_s/K}X$ by a permutation: $x = (x_\sigma)_\sigma \mapsto \tau \cdot x := (x_{\tau(\sigma)})_\sigma$, and $\text{Gal}(K_r/K) \hookrightarrow \text{Aut}(\text{Res}_{K_s/K}X)$. 

ANALYTIC VARIATION OF TATE–SHAFAREVICH GROUPS 47
Thus $O_A[\text{Gal}(K/K)] \subset \text{End}(\text{Res}_{K/\mathbb{K}}A_p)$ by embedding $\text{Gal}(K/K)$ in this way. For $x = (x_r)_r \in \text{Res}_{K/\mathbb{K}}X(\mathbb{Q}_p)$, we have $x^r = \tau|_K \cdot (x^r_\sigma)$. Then the image of $A$ in $\text{Res}_{K/\mathbb{K}}A_p$ is given by $1_p(\text{Res}_{K/\mathbb{K}}A_p)$, where $1_p = [K : K]^{-1} \sum \psi^{-1} \sigma | A \in O_A[\text{Gal}(K/K)]$. Since $\tau \in \text{Gal}(\mathbb{Q}_p/K)$ acts on $x \in \text{Res}_{K/\mathbb{K}}A_p$ by $(x_\sigma)|_K = (x^r_\sigma)_\sigma$, writing the Galois action on $A_p$, as $x \mapsto x^n\sigma$, the action of $\sigma$ on $A_p$ is $A_p$. By the same argument, writing the Galois action on $A_\mathfrak{a}$ as $x \mapsto x^n\sigma$, the action of $\sigma$ on $A_{\mathfrak{a}}$ is $A_{\mathfrak{a}}$. Since $A_{\mathfrak{a}}^\mathfrak{a}$ is the formal Lie group whose Barsotti–Tate group is the potentially connected part of the Barsotti–Tate group of $A_{\mathfrak{a}}$. This formal Lie group descends to $W$ and is isomorphic $\hat{G}_m \otimes_{\mathbb{Z}} \mathfrak{A}$ over $W_{\mathfrak{a}}$ for the integer ring $W$ of $K$. Thus we have an identity $A \cong \hat{G}_m \otimes_{\mathbb{Z}} \mathfrak{A}(\psi)$ over $W_{\mathfrak{a}}$ and $A[p^\infty] = \mu_{p^\infty} \otimes_{\mathbb{Z}} \mathfrak{A}(\psi)$ over $W_{\mathfrak{a}}$, where $\mathfrak{A}(\psi) \cong A$ as $O_A$-modules on which $\text{Gal}(K/K)$ acts by $\psi$. Note that the second identity is valid over $W_{\mathfrak{a}}$ as this is the identity of Barsotti–Tate groups. From this, we get

**Lemma 16.1.** Assume $p > 2$. Let $a \in O_A$ be given by the action of Frob. Then we have, for $s \geq r$,

$$H^1(\text{Gal}(K_\infty/K), A[p^s](W_s)) \cong (\mathfrak{A}/(p^{s-1}, \nu_p^3)(\mathfrak{A}))|_s[a - 1]$$

which is finite and bounded independent of $s \geq r$.

*Proof.* The Frobenius element Frob acts on $A[p^s]$ via multiplication by $a$. Note that, for $s \geq r$

$$A[p^s](W_s) = (\mu_{p^s}(W_{s}^{\mathfrak{a}}) \otimes_{\mathbb{Z}} \mathfrak{A}(\psi))|_s[a - 1]$$

$$= \{ x \in \mu_{p^s}(W_{s}^{\mathfrak{a}}) \otimes_{\mathbb{Z}} \mathfrak{A}(\psi)|(a - 1)x = 0 \} \cong (\mathfrak{A}(\psi)/p^s(\mathfrak{A}(\psi))|_s[a - 1]$$

as $\text{Gal}(K_\infty/K)$-modules. Since $\text{Gal}(K_\infty/K)$ acts on $\mu_{p^s}$ by $\nu_p$, we conclude

$$H^1(\text{Gal}(K_\infty/K), A[p^s](W_s)) \cong (\mathfrak{A}/(p^{s-1}, \nu_p^3)(\mathfrak{A}))|_s[a - 1]$$

as desired. \hfill $\square$

### 17. Finiteness of the $p$-local error term

We assume (F) and $p > 2$. Here $K/\mathbb{Q}_p$ is a finite extension with $p$-adic integer ring $W$. Put $K_\infty = K[\mu_{p^\infty}]$ with integer ring $W_\infty$.

We studied the $\Lambda$-BT group $G_{\lambda,0,\omega,d}$ associated to the tower $\{X_1(Np^r)\}_{r}$ in [H14, §5], which is defined over $\mathbb{Z}[\mu_{p^\infty}]$. Here $\omega(d, a, d) = \omega(d)$. For the general tower $\{X_r\}_r$ determined by the fixed data $(\alpha, \beta, \xi)$, $J_r$ is a factor of $\text{Res}_{\mathbb{Q}/K}J_r(Np^\infty)$ again over $\mathbb{Q}[\mu_{p^\infty}]$ if $r \geq \epsilon$, since $F_\xi \subset \mathbb{Q}[\mu_{p^\infty}]$. Thus taking the tower of regular model $X_r/\mathbb{Z}[\mu_{p^\infty}]$ made out of the regular model $X_1(Np^r/\mathbb{Z}[\mu_{p^\infty}])$ (via the corresponding Weil restriction of scalars) and considering $J_{r}/\mathbb{Z}[\mu_{p^\infty}] := \text{Pic}_{X_r/\mathbb{Z}[\mu_{p^\infty}]}$, over $\mathbb{Z}[\mu_{p^\infty}]$, we are in the case of $[H14, §3]$ (replacing (CT) by (ct) in [H14, Remark 5.5] if $\xi = 1$). Though it is assumed that $p > 3$ in [H14, §3], the result there is valid for $p = 2, 3$. This is because the ordinary or nearly ordinary part is trivial if $Np \leq 3$ (and the assumption $p > 3$ is imposed to have $Np \geq 4$ for the representability of the elliptic moduli problem). We take its connected component $G_{\lambda,0,\omega,d}^{\mathfrak{a}}$ and put $G_{\lambda,0,\omega,d}^{\mathfrak{a}}/\mathbb{Z}[\mu_{p^\infty}] = G_{\lambda}^{\mathfrak{a}}(\mu_{p^\infty})$ which is a connected Barsotti–Tate group defined over $\mathbb{Z}[\mu_{p^\infty}]$. Write $G_{\lambda,0,\omega,d}^{\mathfrak{a}}/\mathbb{Z}[\mu_{p^\infty}]$ for the formal Lie group associated to the connected Barsotti–Tate group $G_{\lambda,0,\omega,d}^{\mathfrak{a}}/\mathbb{Z}[\mu_{p^\infty}]$ of [GME, 1.13.5].

We put $O_{\mathbb{Q}}/\mathbb{Z}[\mu_{p^\infty}] = \varprojlim G_s$, where the projection $G_{s+1} \to G_s$ is induced by the natural trace map $\pi^*_s : G_{s+1} \to G_s$ for $s > s$. We study $\text{Coker}(J_{r}^{\text{ord}}(K) \cong \varprojlim(J_{r}^{\text{ord}})(K))$. Identify $G_{r}^{\text{ord}}$ with $B_{r}^{\text{ord}}$ by $\pi^*_r$ and $\hat{A}_{r}^{\text{ord}}$ with $\hat{A}_{s,r}^{\text{ord}}$ by $\pi^*_s$. Let $A_{s} \cong A_{r}$ be the connected formal Lie group over $\mathbb{Z}[\mu_{p^\infty}]$ associated to the connected component of the Barsotti–Tate group of $G_{s}[\mu_{p^\infty}] \cong A_{r}[p^\infty]^{\text{ord}}$. 


We first study $\text{Coker}(G_s(W) \to \varpi(G_s)(W))$. We have an exact sequence of Barsotti–Tate groups over the integral base $\mathbb{Z}_p[\mu_{p^s}]$ [H14, §5]:

$$0 \to A_s[p^{\infty}] \to G_s^0 \to G_s^0/A_s[p^{\infty}] \to 0.$$  

(17.1)

This produces the following commutative diagram of formal Lie groups over $\mathbb{Z}_p[\mu_{p^r}]$ with exact rows:

$$\begin{array}{ccc}
A_s & \longrightarrow & G_s \\
\pi_{s,s'} \downarrow & & \downarrow \\
A_{s'} & \longrightarrow & G_{s'}/A_s \\
\end{array}$$

\vspace{1em}

Since $G_s^0/A_s[p^{\infty}]$ is a Barsotti–Tate group over $W_s$ by [H14, Theorem 5.4], $G_s/A_s$ is a smooth formal group over $W_s$ (e.g., [Sc87, Lemma 1]). Thus $G_s \cong (G_s/A_s) \times W_s A_s$ as formal schemes (but not necessarily as formal groups). Anyway, this shows that $G_s(W_{s'}) \to \varpi(G_s)(W_{s'})$ is surjective for all $s' \geq s$. Therefore, we get an exact sequence

$$0 \to A_s(W_{s'}) \to G_s(W_{s'}) \to \varpi(G_s)(W_{s'}) \to 0$$  

(17.2)

for all $s' \geq s \geq r$ including $s' = \infty$.

Since $\text{Gal}(\overline{K}/K^ur)$ acts on $A_s$ by the $p$-adic cyclotomic character, we find $A_s \cong \widehat{G}_m^d$ over $\widehat{W}_{ur}$ for $d = \dim A_s$. In Corollary in the introduction of [O00] (see also [H13a, Lemma 4.2]), Ohta shows that $\mathcal{T}G^0 := \lim_{\longrightarrow} G_s^0 \cong \text{h}$ (and hence $\mathcal{T}G^0 \cong \mathcal{T}$) canonically as $\text{h}$-modules. Assuming (F), we have $T_p G_s^0 \cong \text{h}_s$, so $G_s \cong \widehat{G}_m \otimes_{\mathbb{Z}_p} \text{h}_s$ over $\widehat{W}_{ur}$. Define $\mathfrak{A} \subset \text{h}_s$ by the annihilator of $G_s/A_s$ and $\mathfrak{B} := \ker(\text{h}_s \to \text{End}(A_{K_s}))$. Hence we have an exact sequence of formal groups:

$$0 \to \widehat{G}_m \otimes_{\mathbb{Z}_p} \mathfrak{A} \to G_s \xrightarrow{\varpi} G_s \to \widehat{G}_m \otimes_{\mathbb{Z}_p} \text{h}_s/\mathfrak{B} \to 0$$  

(17.3)

since $0 \to \mathfrak{A} \to \text{h}_s \xrightarrow{\varpi} \text{h}_s/\mathfrak{B} \to 0$ is an exact sequence of $\mathbb{Z}_p$-free modules. Thus we have $A_s \cong \widehat{G}_m \otimes_{\mathbb{Z}_p} \mathfrak{A}$ over $\widehat{W}_{ur}$, and $\mathfrak{A}$ is an $\text{h}_s$-ideal and is an $O_{A_s}$-module. This shows that $A_s$, $B_s$, $\overline{\varpi}(J^r_{\text{ord}}) := J^r_{\text{ord}} \otimes A^r_{\text{ord}}$ and $J_s$ all satisfy (A1–3) in Section 16.

The action of the Frobenius $[p] : \mathbb{Q}_p$ on $A_s[p^{\infty}]$, which is the multiplication by $a_p^{-1}$ in $O_{A_s}$ (where $a_p$ is the image of $U(p)$ in $O_{A_s}$). Thus $A_s(W_{ur}) = \mathfrak{A} \otimes \widehat{G}_m(W_{ur}) \cong \mathfrak{A} \otimes \mathbb{Z}_p(1+m_{\widehat{W}_{ur}})$ on which the natural Galois action on $\widehat{G}_m(W_{ur})$ is twisted by a character $\psi : \text{Gal}(\overline{K}_r/K) \cong \text{Gal}(\widehat{K}_{ur}/\widehat{K}_{ur}) \to A_s \otimes_{\mathbb{Z}_p} \mathfrak{A}$ induced by the nearly ordinary character $\psi$ sending $[z, \mathbb{Q}_p]$ for the image of $\text{h}_s$ of the Hecke operator in $\text{h}_s$ of the class of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in $\overline{G}_{r,\text{T}}$. Write simply $A$ for the abelian variety $A_s$. Let $O_{A_s} := \text{End}(A_{/\mathbb{Q}_p})$, which is an order of the Hecke algebra generated over $\mathbb{Q}$ by Hecke operators $T(n)$ in $\text{End}^0(A_{/\mathbb{Q}_p}) = \text{End}(A_{/\mathbb{Q}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}$.

Recall the Galois representation $\rho_A$ of $\text{Gal}(\overline{\mathbb{F}}_p/K)$ realized on $T_p A^r_{\text{ord}}$. Take the connected component $\text{Spec}(T)$ of $\text{Spec}(h)$ such that $h/\varpi h = T/\varpi T$. Write symbolically $\rho_A|_{\text{Gal}(\overline{\mathbb{F}}_p/K)} \cong \left(\begin{smallmatrix} \nu_p & \varphi \\ 0 & \varphi \end{smallmatrix}\right)$ and $\rho_T = \left(\begin{smallmatrix} \nu_p & \varphi \\ 0 & \varphi \end{smallmatrix}\right)$ for a deformation $\psi : \text{Gal}(K^ur/K) \to T/\varpi T$ of $\psi$. Here $\nu_p \varphi$ and $\nu_p \psi$ act on $T_p A_s[p^{\infty}]$ and on $TG^0$, respectively. This $\varphi$ (resp. $\varphi$) gives the action on $T_p A^r_{\text{ord}}/T_p A_s$ (resp. on $TG^0_{\text{r}}$). Note that $TG^0_{\text{r}} \cong \text{Hom}_A(T, A)$ as $T$-modules (so, if $T$ is Gorenstein, the above form $\rho_T$ of $2 \times 2$ matrix is literally true). We write $\text{Frob} \in \text{Gal}(K_{ur}/K)$ for the Frobenius element inducing the generator of $\text{Gal}(\overline{\mathbb{F}}_p/K)$ and an appropriate power of the identity id $= [p, \mathbb{Q}_p]$ on $K_{ur}/K$.

**Proposition 17.1.** Suppose (F). Let $G^0$ be the connected component of $G = G_{s', T}$, and take a generator $\sigma$ of $\text{Gal}(K^ur/K)$. Then we have $H^1(K_{ur}/K, G^0_{s', T}(W_{ur})) = (T_s/\langle \nu_p \psi(\sigma) - 1 \rangle T_s)_{\mathbb{Z}_p} \langle \nu_p \varphi(\text{Frob}) - 1 \rangle = 0$. If either $|\nu_p \psi(\sigma) - 1|_p = 1$ or $|\varphi(\text{Frob}) - 1|_p = 1$, then we have the vanishing $H^1(K_{ur}/K, G^0(W_{ur})) = 0$.

**Proof.** As we saw, under (F), we have $G^0_{s', T}(W_{ur}) \cong \nu_p(W_{ur}) \otimes_{\mathbb{Z}_p} T_s(\psi)$ as $\text{Gal}(K^ur/K)$-modules, where $\text{Gal}(K^ur/K)$ acts on $T_s(\psi) \cong T_s$ by $\psi$. We apply Lemma 16.1.2 to the formal Lie group $A$ with $A[p^{\infty}] = G^0_{s', T}$. Note that $a$ in the Lemma is the image of $\varphi(\text{Frob})$ in $O_A$ by [H14, (6-1)]. From this, the cohomology of $G^0$ vanishes if either $|\nu_p \psi(\sigma) - 1|_p = 1$ or $|\varphi(\text{Frob}) - 1|_p = 1$. We have then $H^1(K_{ur}/K, G^0(W_{ur})) = \lim_{\longrightarrow} H^1(K_{ur}/K, G^0(W_{ur})) = 0$. \qed
**Theorem 17.2.** Let the notation be as in Theorem 10.4. Let $K$ be a finite extension of $\mathbb{Q}_p$ for $p > 2$, and put $K_s = K[\mu_p^s]$ ($s = 1, 2, \ldots, \infty$). If $A_r$ does not have split multiplicative reduction over $W_r$, then the error term $E^\infty(K)_r$ is finite. If further $A_r$ has good reduction over $W_1 = W[\mu_p]$ with $|\varphi_{\text{Frob}}| - 1|_{\mu_p} = 1$, then $E^\infty(K)_r$ vanishes.

**Proof.** Let us first sketch the proof. As before, we write symbolically $\varpi(J_s)$ for the abelian variety quotient $J_s/A_s$, since $J_s/A_s^{\text{ord}} = J_s^{\text{ord}}/A_s^{\text{ord}} = \varpi(J_s^{\text{ord}})$ by definition. Thus $A_s(F) \hookrightarrow J_s(F) \xrightarrow{\varpi} \varpi(J_s)(F)$ is exact for any algebraic extension $F/K$, and hence $A_s^{\text{ord}}(F) \hookrightarrow J_s^{\text{ord}}(F) \xrightarrow{\varpi} \varpi(J_s)(F)$ is exact. We first assume that $J_s^{\text{ord}}$ is contained in an abelian subvariety of $J_s$ having good reduction over $W_\infty$ (so, we may assume that the subabelian variety has good reduction over $W_s$). Then the sequence

$$(17.4) \quad 0 \to A_s[p^\infty]^{\text{ord}} \to G_s \to \varpi(G_s) \to 0$$

is exact as Barsotti–Tate groups over $W_s$ (see [H14, §5] and a remark after Corollary 17.4). Since the complex of Néron models $A_s/W_s \to J_s/W_s \to \varpi(J_s/W_s)$ is exact up to $p$-finite errors [NMD, Proposition 7.5.3], the exactness of (17.4) shows the sequence $A_s^{\text{ord}}/W_s \hookrightarrow J_s^{\text{ord}}/W_s \to \varpi(J_s^{\text{ord}}/W_s)$ is exact as fppf sheaves over $W_s$. Since $0 \to A_r^{\text{ord}}/W_s(\mathbb{F}) \to J_s^{\text{ord}}/W_s(\mathbb{F}) \to \varpi(J_s^{\text{ord}}/W_s)(\mathbb{F}) \to H^1(\mathbb{F}, A_r^{\text{ord}}) = 0$ is exact, (17.2) shows that $J_s^{\text{ord}}(K_\infty) \to \varpi(J_s^{\text{ord}}(K_\infty))$ is onto.

By (15.1), we have

$$|H^1(K_{\infty}/K_r, A_r^{\text{ord}}(K_r))| = |A_r[p^\infty]^{\text{ord}}(\mathbb{F})|^2.$$ 

Since we have an exact sequence of the formal Lie groups, and applying the formal version [Sc87, Theorem 1] (particularly in the non-split multiplicative case), we get the finiteness for the connected part. The surjectivity (up to finite error) for the special fiber (of Néron models) will be shown below. Thus if $A_r$ has either good or non-split multiplicative reduction over $W_r$, we still have finiteness of $H^1(K_{\infty}/K_r, \hat{A}_r^{\text{ord}}(K_r))$ as above. Then by the inflation-restriction exact sequence:

$$H^1(K_r/K, \hat{A}_r^{\text{ord}}(K_r)) \to H^1(K_{\infty}/K_r, \hat{A}_r^{\text{ord}}(K_r))$$

$$\to H^0(K_r/K, H^1(K_{\infty}/K_r, \hat{A}_r^{\text{ord}}(K_r))) \to H^2(K_r/K, \hat{A}_r^{\text{ord}}(K_r)),$$

finiteness of $H^1(K_r/K, \hat{A}_r^{\text{ord}}(K_r))$ $(j = 1, 2)$ and $H^1(K_{\infty}/K_r, \hat{A}_r^{\text{ord}}(K_r))$ tells us finiteness of the cohomology $H^1(K_{\infty}/K_r, \hat{A}_r^{\text{ord}}(K_r))$, from which we conclude the finiteness of $E^\infty(K)_r$. If $r = 1$, $p \nmid [K_1 : K]$ and

$$H^q(K_1/K, \hat{A}_r^{\text{ord}}(K_1)) = 0 \quad \text{for} \quad q > 0.$$ 

Then, still assuming $r = 1$, we conclude

$$(17.5) \quad H^1(K_{\infty}/K, \hat{A}_r^{\text{ord}}(K_r)) \cong H^0(K_r/K, H^1(K_{\infty}/K_r, \hat{A}_r^{\text{ord}}(K_r))))$$

If in addition $|\varphi_{\text{Frob}}| - 1|_{\mu_p} = 1$ and $A_r$ has good reduction over $W_r$, from $|H^1(K_{\infty}/K_r, \hat{A}_r^{\text{ord}}(K_r))| = |A_r[p^\infty](\mathbb{F})|^2 = 0$, the groups in (17.5) vanish so, $E^\infty(K)_r = 0$. In any case, $H^1(K_{\infty}/K_r, \hat{A}_r^{\text{ord}}(K_r))$ is finite. Similarly, by the formal group version [Sc87, Theorem 1], we conclude the finiteness of $H^1(K_{\infty}/K, A_r(W_\infty))$.

We now give details of the proof in the general case. We first look into the identity connected components over $W_\infty$. By (17.2),

$$0 \to A_r(W_\infty) \to G_\infty(W_\infty) \to \varpi(G_\infty)(W_\infty) \to 0$$

is exact. Taking its Galois cohomology sequence, we get another exact sequence

$$0 \to A_r(W) \to G_\infty(W) \xrightarrow{\varpi} \varpi(G_\infty)(W) \to H^1(K_{\infty}/K, A_r(W_\infty)).$$
Since the cohomology group $H^1(K_{∞}/K, A_r(W_{∞}))$ is finite (cf. [Sc87, Theorem 1]), we find that  
\[ Coker(G_{∞}(W_{∞})^{Gal(K_{∞}/K)} \xrightarrow{\tilde{\phi}_{∞}} \varpi(G_{∞})(W_{∞})^{Gal(K_{∞}/K)}) \]
is finite.

As for the special fiber (of the Néron models), we have the exact sequence:
\[ 0 \to \hat{A}_r^{ord}(\mathbb{F}) \to \hat{J}_s^{ord}(\mathbb{F}) \to \varpi(\hat{J}_s^{ord})(\mathbb{F}) \to H^1(\mathbb{F}, \hat{A}_r^{ord}). \]
If $\varphi(\text{Frob}) \neq \pm 1$, $A_r$ has good reduction (not just semi-stable one) over $W_r$; so, by Lang's theorem [L56], $H^1(\mathbb{F}, \hat{A}_r^{ord}) \subset H^1(\hat{A}_r, A_r) = 0$. Even if $\varphi(\text{Frob}) = \pm 1$, from the exact sequence
\[ 0 \to A^0_r(\mathbb{F}_p) \to A_r(\mathbb{F}_p) \to \pi_0(A_r/\mathbb{F}_p) \to 0 \]
for the connected component $A^0_r$ of $A_r$, we find $H^1(\hat{A}_r, A_r) \cong H^1(\mathbb{F}, \pi_0(A_r))$ as $Gal(\mathbb{F}_p/\mathbb{F})$ has cohomological dimension 1. Thus $H^1(\hat{A}_r, A_r)$ is finite. After passing to the limit, we find
\[ |Coker(J_s^{ord}(\mathbb{F}) \xrightarrow{\tilde{\phi}_{∞}} \varpi(J_s^{ord})(\mathbb{F}))| \leq |\pi_0(A_r/\mathbb{F}_p)| < \infty. \]

We have the following exact sequence:
\[ 0 \to G^0(W_{∞})^G(K_{∞}/K) \to \varpi(G_{∞})(K_{∞}/K) \xrightarrow{\tilde{\phi}_{∞}} \varpi(G_{∞})(W_{∞}) \to 0. \]
Indeed, the maximal étale quotient $G^0$ of $G_{/W_{∞}}$ is a $\Lambda$-BT group by [H14, Proposition 6.3]; so, its closed points lifts to a $W_{∞}$-point as $W_{∞}$ is henselian. (Note that $G^0_{/W_{∞}}$ may not be an étale Barsotti–Tate group for finite $s$.) Taking the fixed point of $Gal(K_{∞}/K)$, we have
\[ 0 \to G^0(W_{∞})^G(K_{∞}/K) \to \varpi(G_{∞})(K_{∞}/K) \xrightarrow{\tilde{\phi}_{∞}} \varpi(G_{∞})(W_{∞}) \to 0. \]
Then by Proposition 17.1, $Coker(G(K) \xrightarrow{\tilde{\phi}_{∞}} \varpi(G(K)) \xrightarrow{\tilde{\phi}_{∞}} \varpi(G(W_{∞}))$ is $0$ (assuming either $|\varphi(\text{Frob}) - 1|_p = 1$ or $|\psi_{\mathbb{F}_p}(\sigma) - 1|_p = 1$), and in particular, $Coker(J_s^{ord}(K) \xrightarrow{\tilde{\phi}_{∞}} \varpi(J_s^{ord})(K) \xrightarrow{\tilde{\phi}_{∞}} \varpi(J_s^{ord})(W_{∞})$ is $0$.

We have the following commutative diagram with exact rows and columns:
\[
\begin{array}{ccc}
G_{∞}(W_{∞})^{Gal(K_{∞}/K)} & \longrightarrow & J_{s}^{ord}(K_{∞}/K) \\
\downarrow \varpi_{∞} & & \downarrow \varpi_{∞} \\
\varpi(G_{∞})(W_{∞})^{Gal(K_{∞}/K)} & \longrightarrow & \varpi(J_{s}^{ord}(K_{∞}/K)) \\
\downarrow & & \downarrow \\
Coker(\varpi_{∞}) & \longrightarrow & Coker(\varpi_{∞}) \\
\end{array}
\]
For the Frobenius endomorphism $\phi (= \varphi(\text{Frob}))$, we have
\[ J_{s}(\mathbb{F})[p^{\infty}]^{\ord} = J_{s}(\mathbb{F}_p)[p^{\infty}]^{\ord}[\phi - 1]. \]
Since $\phi \equiv \varphi(\text{Frob}) \mod m_{W}$, if $\varphi(\text{Frob}) \not\equiv 1 \mod m_{W}$ (\(\iff |\varphi(\text{Frob}) - 1|_p = 1\), $J_{s}(\mathbb{F})[p^{\infty}]^{\ord} = 0$, and $Coker(\varpi_{∞}) \to Coker(\varpi_{∞})$ is exact; so, $Coker(\varpi_{∞})$ is finite.

We need to argue more if $|\varphi(\text{Frob}) - 1|_p < 1$. We apply $X \mapsto X' := \text{Hom}(X, \mathbb{Q}_p/\mathbb{Z}_p)$ to the above diagram. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is $\mathbb{Z}_p$-injective, $X \mapsto X'$ is an exact contravariant functor, all arrows of (17.6) are reversed, but exactness is kept. Since $Coker(\varpi_{∞}) \to H^1(K, A_r)$, its Pontryagin dual module $Coker(\varpi_{∞})'$ is a $\mathbb{Z}_p$-module of finite type. Since this module killed by arithmetic prime ($\varpi$), we need to show the vanishing of the ($\varpi$)-localization $Coker(\varpi_{∞})'|_{\varpi}$ is $0$. Note that we have a surjective morphism of $A$-module: $Coker(\tilde{\phi}) \to Coker(\tilde{\phi})$ and that $Coker(\tilde{\phi})$ is killed by $(\nu, \varphi(\text{Frob}) - 1)(\gamma t - 1)$ by Proposition 17.1. Since ($\varpi$) is prime to $\gamma t - 1$, we have the vanishing of the localization $Coker(\tilde{\phi}) |_{\varpi}$ of (17.6) from the diagram obtained by applying $X \mapsto X'$, the localized sequence $0 = Coker(\varpi_{∞})'|_{\varpi} \to Coker(\varpi_{∞})'|_{\varpi} \to Coker(\varpi_{∞})'|_{\varpi}$ is exact. Since $Coker(\varpi_{∞})'$ is finite under $\psi(\sigma) \neq 1$ and $\varphi(\text{Frob}) \neq 1$, we conclude $Coker(\varpi_{∞})'|_{\varpi} = 0$; so, $Coker(\varpi_{∞})'|_{\varpi} = 0$ by the finiteness of $Coker(\varpi_{∞})$. Since $Coker(\varpi_{∞})'|_{\varpi} = 0$ is finite $\mathbb{Z}_p$-module, dualizing back, this shows finiteness of $Coker(\varpi_{∞})$ as desired. 
\[ \square \]
18. Twisted family

We describe, in down-to-earth terms, how to create $p$-adic analytic family of modular form associated to an irreducible component of $h_{0,q}$ from a $p$-ordinary family coming from an irreducible component of $\text{Spec}(h_{0,1,\phi_{ord}})$ for $\phi_{ord}(a,d)$ only dependent on $d$; i.e., $\phi_{ord}(a,d) = \phi(d)$ for some character $\phi$ of $(\mathbb{Z}/p\mathbb{Z})^\times$. We show that as a $\Lambda$-algebra $h_{0,p} := h_{0,1,\phi_{ord}}$ is isomorphic to $h_{0,q}$ for a specific choice $\xi$ depending on $\phi$ by $T(l) \mapsto \kappa(l)^{-1}T(l)$ regarding $l$ as an idele $l_l$ in $(\mathbb{A}^{(p\infty)})^\times$ supported on $Q$. Here $\kappa$ is a suitably chosen character of $(\mathbb{A}^{(p\infty)})^\times/Q^\times$ with values in $\Lambda$.

Let $S_k(\Gamma_0(Np^*), \varepsilon|_N)$ for a character $\varepsilon : \mathbb{Q}_p^\times/\mu_{p\infty}, \chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \overline{\mathbb{Q}}^\times$ and $\phi : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \overline{\mathbb{Q}}^\times$ be the space of cusp forms in $S_k(\Gamma_0(Np^*))$. Thus as above, we lift $\phi$ to a unique character $\chi$ of $G_{\kappa}$ for all $\phi$. Thus this operation $f \mapsto \kappa \circ f$ only dependent on $\kappa$. Let $S_k(\Gamma_0(Np^*), \psi_\kappa)$ defined in [H10, page 779]. Then $\psi_\kappa$ is a Hecke eigenform, the modular form $f \leftrightarrow \chi^\kappa$ and $\chi$ as above. Thus $f \in S_k(\Gamma_0(Np^*), \psi_\kappa)$ satisfies

$$f(\frac{az + b}{cz + d}) = \phi(\alpha \varepsilon_p(d)\chi(a \varepsilon_p(d)))$$

for all $(a \ b \ c \ d) \in \Gamma_0(Np^*)$. Write $\psi$ for the character of $\Gamma_0(Np^*)$ given by $\gamma = ((a \ b \ c \ d) \mapsto \varepsilon \phi(\alpha \varepsilon_p(d)\chi(a \varepsilon_p(d)))$ for $(a \ b \ c \ d) \in \Gamma_0(Np^*)$. We show that as a $\Lambda$-algebra

$$S_k(\Gamma_0(Np^*), \psi_\kappa) \cong S_k(\Gamma_0(Np^*), \psi_\kappa)$$

for $\psi_\kappa = \psi_\kappa \times \psi_\kappa$. Here $\psi_\kappa$ is a Hecke eigenform, the modular form $f \leftrightarrow \chi^\kappa$ and $\chi$ as above. Thus $f \in S_k(\Gamma_0(Np^*), \psi_\kappa)$ satisfies

$$f(\frac{az + b}{cz + d}) = \phi(\alpha \varepsilon_p(d)\chi(a \varepsilon_p(d)))$$

for all $(a \ b \ c \ d) \in \Gamma_0(Np^*)$. Lift $f$ to $GL_2(\mathbb{A})$ in the same manner as above, we get $f \in S_k(\Gamma_0(Np^*), \psi_\kappa)$ (for $\psi_\kappa = \psi_\kappa \times \psi_\kappa$) satisfying $f(\frac{zxu}{cz + d}) = \Phi(\alpha \varepsilon_p(d)\chi(a \varepsilon_p(d)))$ for all $\phi$. Thus this operation $f \mapsto \phi \circ \varphi$ preserves “ordinarity”.

**Lemma 18.1.** If $f$ as above satisfies $f|T(n) = \lambda(T(n))f$ (a Hecke eigenform) with $T(l) = U(l)$ if $l|Np$, we have $(f \circ \varphi)|T(l) = \varphi_\kappa \varepsilon_p(l)\lambda(T(l))(f \circ \varphi)$ for all primes $l$ prime to $p$, and for $U(p)$, we have $(f \circ \varphi)|U(p) = \lambda(U(p))(f \circ \varphi)$. Thus this operation $f \mapsto f \circ \varphi$ preserves “ordinarity”.

Here we have used the well known fact that $\varphi$ factors through the $p$-adic cyclotomic character whose value at $p$ is equal to $1$; thus, $\varphi_\kappa(p \mathbb{Z}) = 1$ and the formula $(f \circ \varphi)|U(p) = \lambda(U(p))(f \circ \varphi)$ is consistent with $(f \circ \varphi)|T(l) = \varphi_\kappa \varepsilon_p(l)\lambda(T(l))(f \circ \varphi)$.

By the lemma, if $f$ has weight $2$, from the abelian subvariety $A_f$ attached to $f$, we get an abelian variety $A_{f \circ \varphi}$ of $J_f$ (for a suitable choice of $H$) which is the $\varphi$-twist of $A_f$ (see (16.1)); i.e., we have an identity of $\Lambda$-adic Tate modules $T_1A_{f \circ \varphi} \cong (T_1A_f) \circ \varphi$. Analytic Variation of Tate–Shafarevich Groups

If $f$ (or starting $f$) is a Hecke eigenform, the modular form $f : GL_2(\mathbb{Q}) \rightarrow \mathbb{C}$ and its right translations $R(g)(f)(x) = f(xg) = f(xg)$ for $g \in GL_2(\mathbb{A})$ generate an irreducible automorphic
representation \( \pi = \pi_f \) of \( GL_2(\mathbb{A}) \). Similarly, the modular form \( f \otimes \varphi : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \to \mathbb{C} \) and its right translation \( R(g)(f \otimes \varphi)(x) = (f \otimes \varphi)(xg) = f(xg)\varphi_\mathbb{A}(\det(xg)) \) for \( g \in GL_2(\mathbb{A}) \) generate an irreducible automorphic representation \( \pi_f \otimes \varphi \). Plainly, we have \( \pi_f \otimes \varphi \cong \pi \otimes \varphi_\mathbb{A} \). Inside \( \pi_f \otimes \varphi \), we find a unique new vector which corresponds to a primitive Hecke eigenform \( f_\varphi \in S_k(\Gamma_0(C(\pi \otimes \varphi))) \), \( \phi \chi \varphi^2 \) for the conductor \( C(\pi \otimes \varphi) \). The form \( f_\varphi \) is usually not equal to \( f \otimes \varphi \) even if \( f \) is primitive (as their Neben types are plainly different). As explained in [H09, §3.1], \( f \otimes \varphi \) often has level smaller than the primitive form \( f_\varphi \). Unless the \( p \)-component \( \pi_p \) is super-cuspidal, \( \pi \otimes \varphi \) has non-zero \( U(p) \)-eigenvector with non-zero eigenvalue. Indeed, if \( \pi_p = \pi(\alpha, \beta) \), there are non-zero eigenspaces in \( \pi \otimes \varphi_\mathbb{A} \) on which \( U(p) \) acts by \( \alpha \varphi_\mathbb{A}(p) \) and \( \beta \varphi_\mathbb{A}(p) \) (if \( \alpha(p) \neq \beta(p) \)), the eigenspaces of each of the above value is one-dimensional). If \( \pi_p \) is special, we have one dimensional eigenspace with non-zero eigenvalue. Even if \( \varphi \) is highly ramified at \( p \), the eigenvalues of \( U(p) \) for \( f \otimes \varphi \) and \( f \) are equal.

Suppose that

\[
F_1 := \{ f_\varphi \in S_2(\Gamma_0(Np^r\mathbb{P})), \phi \chi \varphi \} \mid \varphi \in S_2(\Gamma_0(Np^r\mathbb{P})) \}
\]

is the ordinary \( p \)-adic analytic family for \( \chi \) modulo \( N, \phi \) modulo \( p^r \) and \( \varepsilon_p : \mathbb{Z}_p^\times / \mathbb{Q}_p \rightarrow \mathbb{F}_p \). Pick a positive integer \( b \) prime to \( p \) and a character \( \phi : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \mathbb{Z}_p^\times \). Then we consider the twisted family \( F_1 \). Since \( f_\varphi \in S_2(\Gamma_0(Np^r\mathbb{P})), \phi \chi \varphi \), we have \( f_\varphi \otimes \varepsilon_p^{1/b} \in S_2(\Gamma_0(Np^r\mathbb{P}), \psi \varphi) \), where \( \psi_p((b \choose c)_p) = \phi(a \mod p)\chi(\alpha \mod N)\varepsilon_p^{1/b}(a)\varepsilon_p^{1/b}(d)\varphi^{-1}(a)p^{-1}(d) \).

Thus in this case, \( \psi_p^{-1/p} \) factors through \( G/H \) for \( H \) defined for \( (\alpha, \delta) = (1, b - 1) \) and \( \xi(a, d) = \phi(a)\psi \). If one starts with \( f_\varphi \in S_2(\Gamma_0(Np)) \) whose L-function has root number \( \pm 1 \), the L-function \( f_\varphi \otimes \varepsilon_p^{1/2} \) has the same root number. Therefore, the most interesting case is when \( b = 2 \) (so, \( p > 2 \)), \( (\alpha, \delta) = (1, 1) \) and \( \phi \chi = 1 \). This process can be reversed by tensoring back \( \varepsilon_p^{1/b} \varphi \). Thus we have one-to-one correspondence of families of modular forms of \( \mathbf{h}_{0,1,\xi_{ord}} \) and \( \mathbf{h}_{1,1,\xi_{ord}} \), where \( \xi_\mathbb{A}(a, d) = \phi(d)\varphi(a)\varphi(d) \). This shows, writing \( \Lambda = \lim \mathbb{Z}_p[T] \) with \( \ell = 1 + T \).

**Proposition 18.2.** Let the notation be as above. Then the \( \mathbb{Z}_p \)-algebra \( \mathbf{h}_{1,1,\xi_{ord}} \) is isomorphic to \( \mathbf{h}_{0,1,\xi_{ord}}(\xi_{ord}(a, d) = \phi(d)) \) by \( T(l^n) \mapsto t^{-1/b\log_p(l^n)}(\gamma)^{-1}(l^n)^{1/2}T(l^n) \) for primes \( l \) as \( \mathbb{Z}_p \)-algebras, where \( \gamma = 1 + p^r \) and \( \log_p \) is the \( p \)-adic logarithm and we have written \( T(l^n) = U(l^n) \). The \( \Lambda \)-algebra structure of \( \mathbf{h}_{1,1,\xi_{ord}} \) is obtained twisting the \( \Lambda \)-algebra structure of \( \mathbf{h}_{0,1,\xi_{ord}} \) by the character \( \kappa : \mathbb{Z}_p^\times / \mathbb{Q}_p \to \mathbb{R}^\times \), which is given by \( \kappa(\gamma^s) = t \) for \( s \in \mathbb{Z}_p \). In particular, the algebra \( \mathbf{h}_{1,1,\xi_{ord}} \) is free of finite rank over \( \Lambda \) for all primes \( p \).

**References**

Articles


Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, U.S.A.

E-mail address: hida@math.ucla.edu