ANALYTIC VARIATION OF TATE–SHAFAREVICH GROUPS

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Abstract. Let $K$ be a number field. For a prime $p$, we study the inductive limit of the $p$-ordinary part of the Tate-Shafarevich groups and the Selmer groups (over $K$) of modular Jacobians of level $Np^r$ as $r \to \infty$ for a fixed integer $N$ prime to $p$. We prove control theorems of the $p$-ordinary $p$-primary part of $\Sha_K(A_p)$ over $p$-adic analytic family of abelian varieties $A_p$. In particular, under mild conditions, we show that if $\Sha_K(A_{p_0})^{\text{ord}}$ is finite for one member $A_{p_0}$ of the analytic family and the Mordell–Weil rank of $A_{p_0}$ is $\leq 1$ over its Hecke field, then $\Sha_K(A_p)^{\text{ord}}$ is finite for almost all members $A_p$.

1. Introduction

Fix a prime $p$ and a positive integer $N$ prime to $p$ throughout the paper. Let $\Spec(\mathcal{I})$ be an irreducible component of (the spectrum of) the $p$-ordinary big Hecke algebra $\mathfrak{h}$. Attached to $\mathcal{I}$ is the Mazur–Kitagawa $p$-adic $L$-function $L(k, s)$ for the weight variable $k$ and the cyclotomic variable $s$. The function $L$ is an element of the affine ring of the irreducible component of $\Spec(\mathfrak{h}^{\text{ord}})$ covering $\Spec(\mathcal{I})$ for the two variable nearly $p$-ordinary big Hecke algebra $\mathfrak{h}^{\text{ord}}$. We study in this paper the tower of modular curves $\{X_r\}$ whose coefficients $11F_{25}, 11F_{80}$. irreducible component of (the spectrum of) the algebra, Galois representation, Modular curve.

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and put \( \Pi^1(K^S/K, M)_p := \Pi^1(K^S/K, M) \otimes \mathbb{Z}_p \). Often we simply write \( \Pi^1 \) for \( \Pi^1 \). More generally, for a module \( M \), we define \( M_p \) by \( M \otimes \mathbb{Z}_p \) (so, \( M_p \) is the maximal \( p \)-power torsion submodule \( M[p^\infty] \) of \( M \) if \( M \) is torsion, and the maximal \( p \)-profinite quotient if \( M \) is profinite).

Throughout the paper, when \( M \) is related to an abelian variety, we always assume that \( S \) contains all finite places at which the abelian variety has bad reduction in addition to all \( p \)-adic and archimedean places of \( K \). Unless otherwise mentioned, we assume \( S \) to be chosen finite.

In addition to the divisible Mordell–Weil group \( J_r(K) \otimes \mathbb{Z}_p \mathbb{Q}_p / \mathbb{Z}_p \), we study the Tate–Shafarevich group \( \Pi^1_{K}(J_r) \), \( \Pi^1_{K}(K^S/K, J_r[p^\infty]) \) and the Selmer group

\[
\text{Sel}_K(J_r) = \text{Ker}(H^1(K^S/K, J_r[p^\infty]) \to \prod_{v \in S} H^1(K_v, J_r)).
\]

The Tate–Shafarevich group and the Selmer group of an abelian variety are independent of \( S \); so, we omitted "\( K^S / K \)" from the notation. The Hecke operator \( U(p) \) and its dual \( U^*(p) \) acts on \( \Pi^1_{K}(J_r) \) and their \( p \)-adic limit \( e = \lim_{n \to \infty} U(p)^n \) and \( e^* = \lim_{n \to \infty} U^*(p)^n \) are well defined on the above groups \( H \). We write \( H^{\text{ord}} := e(H) \). More generally, adding superscript or subscript "ord" (resp. "co-ord"), we indicate the image of \( e \) (resp. \( e^* \)) depending on the situation.

By Picard functoriality, we have injective limits \( \mathcal{G} := \lim_{r} \mathcal{G}_r \) with \( \mathcal{G}_r := J_r[p^\infty]^{\text{ord}} \) (a \( \Lambda \)-BT group in the sense of [H14], \( R \to J^\text{ord}(R) = \lim_j J^\text{ord}_j(R) \) for \( J^\text{ord}_j(R) = \lim_j J_j(R)/p^jJ_j(R) \) as an fppf sheaf over \( K \), \( \Pi^1_{K}(J^\text{ord}) = \lim_j \Pi^1_{K}(J^\text{ord}_j) \) \( \Pi^1_{K}(K^S/K, \mathcal{G}) = \lim_j \Pi^1_{K}(K^S/K, J_j[p^\infty]^{\text{ord}}) \), and \( \text{Sel}_K(J^\text{ord}_j) = \lim_{r} \text{Sel}_K(J^\text{ord}_j) \). We study control under Hecke operators acting on these arithmetic cohomology groups. These groups, we call \( \Lambda \)-BT groups, \( \Lambda \)-MW groups, \( \Lambda \)-TS groups and \( \Lambda \)-Selmer groups in order. For each Shimura’s abelian subvariety \( A_f \subset J(1)(Np) \) associated to a Hecke eigenform \( f \in S_2(\Gamma_1(Np)) \) [IAT, Theorem 7.14], we can think of the ordinary part of the Tate–Shafarevich group \( \Pi^1_{K}(A_f^{\text{ord}}) \) and the Selmer group \( \text{Sel}_K(A_f^{\text{ord}}) \) (see 1.2 and Section 8 of the text or [ADT, page 74] for the definition of these groups). Let \( h = h(N) \) be a big ordinary Hecke algebra of prime-to-\( p \) level \( N \), and pick a primitive connected component \( \text{Spec}(\mathbb{T}) \) of \( \text{Spec}(h(N)) \) in the sense of [H86a, §3]. Then points \( P \in \text{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p) \) correspond one-to-one to \( p \)-adic Hecke eigenforms \( f_P \) in a slope 0 analytic family. Assuming for example that \( T \) is a unique factorization domain, in a densely populated subset \( \Omega_T \subset \text{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p) \) of principal primes (indexed by \( \zeta^\alpha, \zeta^\beta = (\zeta, 1) \) for \( \zeta \in \mu_{p^\infty} \), \( f_P \) is classical, new at all prime factors of \( N \) and of weight 2 (a definition of \( \Omega_T \) will be given below Corollary 10.2). Write \( N_p^{(P)} \) for the minimal level of \( f_P \). Let \( A_{P/K} \) (resp. \( B_{P/Q} \)) be Shimura’s abelian subvariety (resp. abelian variety quotient) of \( J_1(Np^{(P)}) \) associated to \( f_P \). Write \( H_P \) for the subfield of \( \text{End}(A_{P/K}) \otimes \mathbb{Q}_p \) generated by the Hecke operators. In this introduction, for simplicity, we assume that \( A_P \) for every \( P \in \Omega_T \) has potentially good reduction at \( p \) and that \( A_P \) for some \( P \in \Omega_T \) has good reduction over \( \mathbb{Z}_p \). We prove control theorems for these arithmetic cohomology groups which imply

**Theorem A.** Suppose \( p > 2, |S| < \infty \) and that \( T \) is a unique factorization domain.

1. If \( |\Pi^1(K^S/K, A_{P/F}^{p\infty])^{\text{ord}}| < \infty \) for a single point \( P_0 \in \Omega_T \), then \( \Pi^1(K^S/K, A_{P/F}^{p\infty})^{\text{ord}} \) is finite for almost all \( P \in \Omega_T \) (Corollary 11.2).
2. If \( |\Pi^1_{K}(A_{P/F}^{p\infty})^{\text{ord}}| < \infty \) and \( \dim H_{P_0} A_{P_0}(K) \otimes \mathbb{Q} \leq 1 \) for a single point \( P_0 \in \Omega_T \), then \( \Pi^1_{K}(A_{P/F}^{p\infty})^{\text{ord}} \) is finite for almost all \( P \in \Omega_T \) (Corollary 13.4).
3. If \( |\Pi^1_{K}(A_{P/F}^{p\infty})^{\text{ord}}| < \infty \) and \( \dim H_{P_0} A_{P_0}(K) \otimes \mathbb{Q} \leq 1 \) for a single point \( P_0 \in \Omega_T \), then for almost all \( P \in \Omega_T \), \( \dim H_P A_{P}(K) \otimes \mathbb{Q} \) is independent of \( P \) with \( \dim H_P A_{P}(K) \otimes \mathbb{Q} \leq 1 \) (Corollary 12.5 and Corollary 12.2).
4. If \( \text{Sel}_K(A_{P/F}^{p\infty})^{\text{ord}} \) is finite for a single point \( P_0 \in \Omega_T \), then \( \text{Sel}_K(A_{P/F}^{p\infty})^{\text{ord}} \) is finite for almost all \( P \in \Omega_T \). Moreover if \( \text{Sel}_K(A_{P/F}^{p\infty})^{\text{ord}} = 0 \) for a single point \( P_0 \in \Omega_T \) such that \( A_{P_0}/K \) has good reduction at \( p \) and \( A_{P_0}/K \) has good reduction modulo each prime factor \( |N| \) in \( K \) with \( A_{P_0}((p)) = 0 \) for each prime factor \( p \) of \( K \), \( \text{Sel}_K(A_{P/F}^{p\infty})^{\text{ord}} = 0 \) for all \( P \in \Omega_T \) without exception (Corollary 10.5).
Here the words “almost all” means “except for finitely many”. The assertion (3) means that there exists a finite exceptional subset $E \subset \Omega$ such that the function: $\Omega \setminus E \ni P \mapsto \dim_{H_P} A_P(K) \otimes \mathbb{Q}$ is a constant function independent of $P$ satisfying the inequality (Corollary 12.5):

$$\dim_{H_P} A_P(K) \otimes \mathbb{Q} \leq 1.$$  

We could have $\dim_{H_P} A_P(K) \otimes \mathbb{Q} = 0$ all over $\Omega \setminus E$ but still $\dim_{H_{P_0}} A_{P_0}(K) \otimes \mathbb{Q} = 1$ (as $P_0$ could be in $E$). Taking the “self-dual” tower associated to $(\alpha, \delta) = (1, 1)$, as the root number $\epsilon$ is constant $\pm 1$ in the self-dual family, if we can determine the parity of $\dim_{H_P} A_P(K) \otimes \mathbb{Q}$ in terms of $\epsilon$ independently of the point $P$ over the entire $\Omega$, we would have

$$\dim_{H_P} A_P(K) \otimes \mathbb{Q} \equiv \dim_{H_{P_0}} A_{P_0}(K) \otimes \mathbb{Q} \mod 2,$$

and Theorem A (2) and (3) would imply the identity $\dim_{H_P} A_P(K) \otimes \mathbb{Q} = \dim_{H_{P_0}} A_{P_0}(K) \otimes \mathbb{Q}$ for all $P \in \Omega \setminus E$. See Conjecture 15.4 and a remark after the conjecture. Thus determining the parity is important, though we do not touch this topic in this paper.

The parity conjecture for $p$-Selmer group (for the self-dual tower) holds true under good circumstances by the results of Nekovár [N06, Theorem 12.2], [N07] and [N09] (particularly, the result in [N07] is valid over any number field $K$). Thus, by modifying $A_P$ by an isogeny so that the integer ring $O_P$ of $H_P$ is embedded into $\text{End}(A_{P/\mathbb{Q}})$, if $\text{corank}_{O_P} \text{Sel}_K(A_P)^{\text{ord}} \equiv \dim_{H_P} A_P(K) \otimes \mathbb{Q} \mod 2$ hold (i.e., $\text{corank}_{O_P} \text{III}_K(A_P)$ is even), there is some hope of getting the parity of $\dim_{H_P} A_P(K) \otimes \mathbb{Q}$.

General statements covering “exotic modular towers” (including “self-dual towers”) will be given as Theorems and Corollaries indicated in the theorem. The ring $\mathbb{T}$ is often a power series ring of one variable over a discrete valuation ring (and hence a unique factorization domain; see Theorem 5.3).

Let us now describe one technical idea and a most important tool for the proof. The technical idea is how to separate the $p$-primary part of the arithmetic cohomology groups by “(partially)” completing $p$-adically” the coefficients, and the important ingredient is the control by $\text{Gal}(X/Y)$ of rational points of Jacobians of a Galois covering $X \to Y$ of curves. Fix a base field $k = \mathbb{Q}$ or $\mathbb{Q}_l$. For an abelian variety $A$ over $k$, we consider the following Galois module

$$\hat{A}(\kappa) = \lim_{\longrightarrow} A(\kappa)/p^n A(\kappa) \quad \text{for a finite Galois extension } \kappa/k,$$

$$\hat{A}(\kappa) = \lim_{\longrightarrow} \hat{A}(F) \quad \text{for an infinite Galois extension } \kappa/k$$

with $F$ running over all finite Galois extensions $k$ inside $\kappa$. An explicit description of $\hat{A}(F)$ for a finite extension $F/k$ is given at the end of this introduction as Statement (S), and only when $\kappa/k/Q$ are finite extensions, we have the identity $\hat{A}(\kappa) = A(\kappa) \otimes \mathbb{Z}_p$. The following fact plays a key role to separate the $p$-primary part (and also the ordinary part of it):

(P) Though $\text{End}(A/Q) \otimes \mathbb{Z}_p = \text{End}(A[p^\infty]/Q)$ does not act on the abelian variety $A/Q$, it acts on the fpf/étale abelian sheaf $\hat{A}/Q$.

We consider the (continuous) Galois cohomology groups $H^q(K^S/K, \hat{A}(K^S))$ for a number field $K$ and $H^q(K, \hat{A}(\mathbb{R}))$ for $k = \mathbb{Q}_l$ putting discrete topology on $\hat{A}(\kappa)$ for $\kappa = K^S, \mathbb{R}$ and profinite topology on the Galois group. Here a number field means a finite extension of $\mathbb{Q}$. We write these cohomology groups as $H^q(A)$ for a statement valid globally and locally. Recall $M_p := M \otimes \mathbb{Z}_p$ for a $p$-torsion module $M$. Then we prove, as Lemma 7.2, $H^1(D) \cong H^1(A) \otimes \mathbb{Z}_p =: H^1(A)_p$, where $H^1(A)$ stands for $H^q(K^S/K, A(K^S))$ if $K$ is global and $H^q(K, \hat{A}(\mathbb{R}))$ if $K$ is local. Thus we conclude

$$\text{III}_K(A) := \text{Ker}(H^1(K^S/K, \hat{A}(K^S))) \to \prod_{v \in S} H^1(K_v, \hat{A}(\mathbb{R}_v)) \cong \text{III}_K(A)_p,$$

$$\text{Sel}_K(A) := \text{Ker}(H^1(K^S/K, A[p^\infty](K^S))) \to \prod_{v \in S} H^1(K_v, A[p^\infty](\mathbb{R}_v)) \cong \text{Sel}_K(A)_p.$$

To prove these identities, we have to be a bit careful as $\hat{A}(K_v)$ is not $A(K_v) \otimes \mathbb{Z}_p$ often (see Lemma 7.2). Anyway by (P), the $p$-part of the algebro-geometric $\text{III}$ and $\text{Sel}$ are translated into the
The closed subscheme \( P \) is finite if we have one point \( K/0 \) \((\text{Theorem 12.8})\): groups set, which is a mysterious set of bad exceptional primes against our control of the Mordell–Weil implies Theorem A (1)), and finiteness of \( W \) \( \mathbb{Z} \) so, \( U \) sheaf theoretic counterparts. Since \( p \) \( \text{write the minimalist condition as} \) up to finite error outside \( J \) \( \text{coefficients, it is essential to have the action of} \) \( p \) \( \text{in order to prove the facts corresponding to} \) and (ii) in these cases which result the modular \( p \)-adic deformation theory of ordinary modular forms and the Barsotti–Tate groups of the ordinary part of the Jacobian of \( X \) (this includes the \( p \)-adic deformation theory of modular Galois representations). Though it was clear at the time that if we had a well behaving contravariant functor \( X \to H(X) \) with a correspondence action, we would have deformation theory of the ordinary part of \( H(X) \). However there was not (at least to the author) a clear choice (other than \( H(X, \mathbb{Q}_p/\mathbb{Z}_p) \) and \( H(X, \mathbb{Q}^c) \) ) of the functor at the beginning. A few years after the publication of [H86a] and [H86b], the author realized that the functor \( r \) \( \text{would possibly work (though the application to} \) III is a more recent development). This paper in conjunction with [H15] and [H16] represents the endeavour for achieving the control properties (i) and (ii) for this functor (although the work should have been done earlier).

We now describe key steps towards the proof of Theorem A. The properties (i) and (ii) produce the following exact sequence of étale/fppf sheaves over \( \mathbb{Q} \) for each \( P = (\varpi) \in \Omega_T \) \((\text{Corollary 6.3})\):

\[
0 \to \tilde{A}_p^{\text{ord}} \to J_\infty^{\text{ord}} \varpi, J_\infty^{\text{ord}} \to \tilde{B}_p^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.
\]

Evaluating this sequence at a finite extension \( K/\mathbb{Q} \) and tensoring \( \mathbb{Q}_p/\mathbb{Z}_p \), we can easily prove that the Pontryagin dual \( J := (J_\infty^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \) is a \( T \)-module of finite type \((\text{Proposition 12.1})\). For the support \( Z \subset \text{Spec}(\mathbb{T}) \) of the maximal \( T \)-torsion submodule of \( J \), therefore \( Z(\mathbb{Q}_p) \) is a finite set, which is a mysterious set of bad exceptional primes against our control of the Mordell–Weil groups \( \tilde{A}_p^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \). Anyway for less mysterious finite sets of bad primes \( Z_j \subset \text{Spec}(\mathbb{T}) \) for \( j = g, p \), we prove exactness up to finite error \((\text{for} \ P \in \Omega_T - (Z \cup Z_g \cup Z_p) \)) of the following sequence \((\text{Theorem 12.8})\):

\[
(MW) \quad 0 \to \tilde{A}_p^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J_\infty^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J_r^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0.
\]

The closed subscheme \( Z_g \) \((\text{resp.} \ Z_p) \) of \( \text{Spec}(\mathbb{T}) \) represents infinite failure of \( \varpi(J_\infty^{\text{ord}}(K)) = \varpi(J_r^{\text{ord}})(K) \) \((\text{resp. infinite failure of the same error term replacing} \ K \ by \ its \ p \text{-adic completion})\). The scheme \( Z_p \) is made of points \( P \in \Omega_T - (Z \cup Z_g \cup Z_p) \) such that \( A_P \) has split multiplicative reduction at some \( p \)-adic place of \( K \); so, \( Z_p \) is finite. To show \( Z_g \) is finite, we need to assume the minimalist condition \((*_{P_0})\), where we write the minimalist condition as

\[
(*_{P_0}) : |\Pi_K(\tilde{A}_p^{\text{ord}})| < \infty \text{ and rank}_E A_P(K) \leq \dim A_P
\]

for \( P \in \Omega_T \). Actually \( Z_g \) is given by the support of the Pontryagin dual \( \Pi(\mathbb{K}^S/\mathbb{K}, \mathbb{G}) \), and \( Z_g \) is finite if we have one point \( P_0 \) outside \( Z_p \) with finite \( \Pi(\mathbb{K}^S/\mathbb{K}, \tilde{A}_p^{\text{ord}}[p^{\infty}]) \) \((\text{Theorem 11.1 which implies Theorem A (1)})\), and finiteness of \( \Pi(\mathbb{K}^S/\mathbb{K}, \tilde{A}_p^{\text{ord}}[p^{\infty}]) \) is implied by \((*_{P_0})\) \((\text{Proposition 13.1})\).

As is well known now \((\text{by Nekovár and some other people})\), we have a control exact sequence

\[
(Sel) \quad 0 \to \text{Sel}_K(\tilde{A}_p^{\text{ord}}) \to \text{Sel}_K(J_\infty^{\text{ord}}) \cong \text{Sel}_K(J_p^{\text{ord}}).
\]

up to finite error outside \( Z_p \). We reprove this as Theorem 10.4 in our way \((\text{and hence Thereom A (4)})\). Then applying the snake lemma to the top two exact rows of the following commutative
diagram:

\[
\begin{array}{c}
\xymatrix{
\tilde{A}_p^{\text{ord}}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p 
\ar[r]^i \ar[d] & J_{\infty,\varpi}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p 
\ar[r]^\varpi \ar[d] & J_{\infty,\varpi}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p 
\ar[d] & \ar[d] & \ar[d] & \\
\text{Sel}_K(\tilde{A}_p^{\text{ord}}) 
\ar[r] & \text{Sel}_K(J_{\infty,\varpi}^{\text{ord}}) 
\ar[r]^\varpi & \text{Sel}_K(J_{\infty,\varpi}^{\text{ord}}) 
\end{array}
\]

we conclude the control of $\text{III}_K(J_{\infty}^{\text{ord}})$ outside $Z \cup \mathbb{Z}_p \cup \mathbb{Z}_g$ (i.e., exactness up to finite error):

\[
\text{(III) } 0 \to \text{III}_K(\tilde{A}_p^{\text{ord}}) \to \text{III}_K(J_{\infty}^{\text{ord}}) \xrightarrow{\varpi} \text{III}_K(J_{\infty}^{\text{ord}}).
\]

In order to prove $T$-torsion property of $\text{III}_K(J_{\infty}^{\text{ord}})^{\vee}$, we use the following étale sheaf exact sequence (which is the limit of the isogeny $\varpi(J_{\infty}^{\text{ord}}) \xrightarrow{\varpi} \varpi(J_{\infty}^{\text{ord}})^{\vee}$ at finite level; see Corollary 6.4):

\[
0 \to \tilde{A}_p^{\text{ord}}[p^\infty] \to \varpi(J_{\infty}^{\text{ord}}) \xrightarrow{\varpi} \varpi(J_{\infty}^{\text{ord}}) \to 0,
\]

first to show $\text{III}_K(\varpi(J_{\infty}^{\text{ord}})^{\vee})$ is $T$-torsion, as $\text{III}(K^S/K, \tilde{A}_p^{\text{ord}}[p^\infty])$ is finite (taking $P = P_0$). Then by the exact sequence $0 \to \tilde{A}_p^{\text{ord}} \to J_{\infty}^{\text{ord}} \xrightarrow{\varpi} J_{\infty}^{\text{ord}} \to 0$, we sandwich $\text{III}_K(\varpi(J_{\infty}^{\text{ord}})^{\vee})$ between $T$-torsion $\text{III}_K(\tilde{A}_p^{\text{ord}})$ and $T$-torsion $\text{III}_K(\varpi(J_{\infty}^{\text{ord}}))$, which finishes the proof of Theorem A (2). Since this argument is valid for $P$ outside $Z \cup \mathbb{Z}_p \cup \mathbb{Z}_g$ and $P_0$ might be in $Z$, we cannot relate directly $\text{dim}_{H_p} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\text{dim}_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q}$, and hence some ambiguity in the assertion (3) of Theorem A.

Writing $\rho_T$ for the modular two-dimensional Galois representation associated to $T$ (see [GME, §4.3.1]), we can think of the Selmer group $\text{Sel}_Q(\rho_T \otimes \Phi)$ for a Galois character $\Phi$ with values in $\mathbb{T}^\times$ (cf. [Gr94] or [HML, §1.2.4]). The group $\text{Sel}_Q(\rho_T \otimes \Phi)$ has natural relation to the limit Mordell–Weil group studied in [H15] and [H16] and our ind $\lambda$-Selmer group $\text{Sel}_K(J_{\infty})^{\text{ord}}$. Nekovář studied control theorems for this type of Selmer group and Howard [Ho07] studied the Selmer groups via Heegner classes (for $\Phi = \sqrt{\nu \text{det}(p_T)^{-1}}$ with the $p$-adic cyclotomic character $\nu$). However variation of the Tate–Shafarevich groups over an analytic family has not been studied in depth for an arbitrary number field $K$, though when $K$ is an abelian extension of $\mathbb{Q}$, Kato’s Euler system argument is an obvious exception. Our points are:

(a) The Tate–Shafarevich groups (and the Mordell–Weil groups) have precise control over a given $p$-adic analytic family relative to the “inductive” limit over the tower. This is true even when the corresponding $p$-adic $L$-function is identically zero over the family (by unmatching parity of the root number).

(b) We studied in [H15] the $p$-adically completed injective limit of the Mordell–Weil groups in the family. We study here control of the $p$-divisible Mordell–Weil groups $J_{\infty}^{\text{ord}}(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p$, $\text{Sel}_K(J_{\infty}^{\text{ord}})$, $\text{III}(K^S/K, J_{\infty}[p^\infty]^{\text{ord}})$ and $\text{III}_K(J_{\infty}^{\text{ord}})$.

Our method is insensitive to the base field $K$. We choose $K$ to be a quintic field over $\mathbb{Q}$ with two complex places (whose Galois closure $K^{\text{rat}}$ has Galois group $A_5$ or $S_5$). If we choose a finite set $v$ of rational places well (including the infinite place), as $\text{Gal}(K^{\text{rat}}/\mathbb{Q})$ is almost $S_5$, we can make $((-1)^{v_1},(-1)^{v_1})$ to have a given pair of parity for the set $V$ of all places of $K$ over $v$. Start with $v$ having $(-1)^{v_1} = 1$ and $(-1)^{v_1} = -1$. Adjusting $v$ well, we find a semi-stable rational elliptic curve $E/\mathbb{Q}$ which has split multiplicative reduction at every finite place in $v$ and good reduction outside $v$. Note that the pair of root numbers of $E/\mathbb{Q}$ and $E/K$ is given by $((-1)^{v_1} = 1,(-1)^{v_1} = -1)$ (e.g., [DD11, §1.2]). Choose a prime $p$ ordinary for $E$ such that $E = A_{P_0}$ for $P_0 \in \Omega_T$ for a unique factorization domain $\mathbb{T}$. In addition to assuming good parity of $\dim_{H_p} A_P(K) \otimes \mathbb{Q}$, if we further assume $|E(\mathbb{Q})| < \infty$, $\dim_{E(K)} \otimes_{\mathbb{Z}} \mathbb{Q} = 1$ and $|\text{III}_K(E)^{\text{ord}}| < \infty$, then Theorem A would tell us that $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = [H_P : \mathbb{Q}]$ and $|\text{III}_K(A_{P}^{\text{ord}})| < \infty$ for most of $P \in \Omega_T$ without really studying $A_P(K)$ explicitly, though how we can achieve the above situation (i.e., $|E(\mathbb{Q})| < \infty$, $\dim_{E(K)} \otimes_{\mathbb{Z}} \mathbb{Q} = 1$ and $|\text{III}_K(E)| < \infty$) is another (statistical) question. One would expect that
the $K$-rank is 0 or 1 for most of $K$-rational elliptic curves $E$ which do not descend to a proper subfield of $K$ (in short, when $E$ is properly defined over $K$). On the other hand, the component of the “big” Hecke algebra for $\GL(2)/K$ carrying such an elliptic curve perhaps does not have much arithmetic points as $K$ has complex places (a conjecture of Calegari–Mazur [CM09]).

We may reformulate our result via congruence among abelian varieties. For such reformulation, we recall first the definition of the congruence. An $F$-simple abelian variety (with a polarization) defined over a number field $F$ is called, in this paper, “of $\GL(2)$-type” if we have a subfield $H_A \subset \End^0(A/F) = \End(A/F) \otimes_\bbZ \bbQ$ of degree $\dim A$ (stable under Rosati-involution). If $F = \bbQ$ (or more generally $F$ has a real place), for the two-dimensional compatible system $\rho_A$ of Galois representation of $A$ with coefficients in $H_A$, $H_A$ is generated by traces $\Tr(\rho_A(\text{Frob}_l))$ of Frobenius elements $\text{Frob}_l$ for $F$-primes $l$ of good reduction (i.e., the field $H_A$ is uniquely determined by $A$; see [GME, §5.3.1] and [Sh75, Theorem 0]). We always regard $F$ as a subfield of the algebraic closure $\overline{\bbQ}$. Thus $\O_A' := \End(A/F) \cap H_A$ is an order of $H_A$. Write $O_A$ for the integer ring of $H_A$. Replacing $A$ by the abelian variety representing the group functor $R \mapsto A(R) \otimes_{O_A'} O_A$, we may choose $A$ so that $O_A' = O_A$ in the $F$-isogeny class of $A$. Since finiteness of the Tate–Shafarevich group of $A$ (not necessarily its exact size) is determined by the $F$-isogeny class of $A$ for a field extension $K/F$, we hereafter assume that $\End(A/F) \cap H_A = O_A$ for any abelian variety of $\GL(2)$-type over $F$. For two abelian varieties $A$ and $B$ of $\GL(2)$-type over $F$, we say that $A$ is congruent to $B$ modulo a prime $p$ over $F$ if we have a prime factor $p_A$ (resp. $p_B$) of $p$ in $O_A$ (resp. $O_B$) and field embeddings $\sigma_A: O_A/p_A \hookrightarrow \overline{F}_p$ and $\sigma_B: O_B/p_B \hookrightarrow \overline{F}_p$ such that $(A[p_A] \otimes_{O_A/p_A,\sigma_A} \overline{F}_p)^{ss} \cong (B[p_B] \otimes_{O_B/p_B,\sigma_B} \overline{F}_p)^{ss}$ as $\text{Gal}(\overline{\bbQ}/F)$-modules, where the superscript “$ss$” indicates the semi-simplification. In this introduction, we assume that $A$ is $p_A$-ordinary meaning that the Barsotti–Tate group $A[p_A^\infty]$ has nontrivial $p$-divisible $p$-unramified quotient. Hereafter in this article, we always assume that the field of definition is equal to $\overline{\bbQ}$, but the coefficient field $K$ is any number field.

Let $E/Q$ be an elliptic curve. Writing the Hasse–Weil L-function $L(s, E)$ as a Dirichlet series $\sum_{n=1}^\infty a_n n^{-s}$ with $a_n \in \bbZ$ (i.e., $1 + p - a_p = |E(\bbF_p)|$) for each prime $p$ of good reduction for $E$, we call $p$ admissible for $E$ if $E$ has good reduction at $p$ and $(a_p \mod p)$ is not in $\Omega_E := \{\pm 1, 0\}$. Therefore, the maximal étale quotient of $E[p]$ over $\bbZ_p$ is not isomorphic to $\bbZ/p\bbZ$ up to unramified quadratic twists. By the Hasse bound $|a_p| \leq 2 \sqrt{p}$, $p \geq 7$ is not admissible if and only if $a_p \in \Omega_E$ (so, 2 and 3 are not admissible). Thus if $E$ does not have complex multiplication, the Dirichlet density of non-admissible primes is zero by a theorem of Serre as $L(s, E) = L(s, f)$ for a rational Hecke eigenform $f$. A proto-typical theorem we prove is as follows.

**Theorem B.** Let $E/Q$ be an elliptic curve with $|\Sha_K(E)| < \infty$ and $\dim_{\bbQ} E(K) \otimes_{\bbZ} \bbQ \leq 1$. Let $N$ be the conductor of $E$, and pick an admissible prime $p$ for $E$. Consider the set $\mathcal{A}_{E,p}$ made up of all $\bbQ$-isogeny classes of $\bbQ$-simple $p_A$-ordinary abelian varieties $A/Q$ of $\GL(2)$-type with prime-to-$p$ conductor $N$ congruent to $E$ modulo $p$ over $\overline{\bbQ}$. Then there exists an explicit (computable) finite set $S_E$ of primes depending on $N$ but independent of $K$ such that if $p \notin S_E$, almost all members $A \in \mathcal{A}_{E,p}$ have finite $\Sha_K(A)p_A$ and constant dimension $\dim_{\bbQ} A(K) \otimes \bbQ \leq 1$. If further $E(K) = \Sha_K(E) = 0$ (i.e., $\Sel(K)(E) = 0$ in short) and $E$ can be embedded into $J_r$ for some $r > 0$, then as long as every prime factor of $p$ in $K/Q$ has residue field $\bbF_p$, every $A \in \mathcal{A}_{E,p}$ has finite $\Sha_K(A)p_A$ and $\Sha_{E,K}(A)p_A$ as long as $p \notin S_E$.

Here for the prime $p_A|p$, we have $(A[p_A] \otimes_{O_A/p_A,\sigma_A} \overline{F}_p)^{ss} \cong (E[p] \otimes_{\bbF_p} \overline{F}_p)^{ss}$, and $\Sha_K(A)p_A$ (resp. $\Sel(K)(A)p_A$) is the $p_A$-primary part of $\Sha_K(A)p_A$ (resp. $\Sel(K)(A)p_A$). The definition of the set $S_E$ will be given in Definition 15.1, and a more general version of this theorem will be given as Theorem 15.2.

Taking $K = \bbQ$ and applying the above theorem to the modular elliptic curves $E = X_0(N)$ for small $N$, we get the following corollary:

**Corollary C.** Let $N$ be one of 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49 (all the cases when $X_0(N)$ is an elliptic curve with finite $X_0(N)(\bbQ)$). Pick an admissible prime $p$ for $X_0(N)$. Then we have $|\Sha_{\bbQ}(A)p_A| < \infty$ and $|\Sel_{\bbQ}(A)p_A| < \infty$ for almost all $A$ in $\mathcal{A}_{X_0(N),p}$. If further $X_0(N)(\bbQ)p = \Sha_{\bbQ}(X_0(N))p = 0$, $\Sel_{\bbQ}(A)p_A$ and $\Sha_{\bbQ}(A)p_A$ are both finite for all $A$ in $\mathcal{A}_{X_0(N),p}$ without exception.
By a celebrated theorem of Kolyvagin [K88] (with modularity of rational elliptic curves [BCDT01]), as long as the algebraic $Q$-rank of $E \leq 1$, we have $|\Pi_Q(E)| < \infty$. For the modular elliptic curves $X_0(N)$ listed above, the point of the above corollary is that we have $|X_0(N)(\mathbb{Q})| < \infty$ and that the exceptional set $S_E$ can be taken to be empty (or more precisely, $S_E$ is contained in non-admissible primes); so, the statement becomes more transparent. This corollary produces infinitely many examples of simple abelian varieties of (unbounded) dimension ($> 1$) with finite Tate–Shafarevich group $\Pi_Q(A)_{\mathbb{Z}_p}$, as we know $\dim_p A_p$ grows indefinitely in an analytic family [H11].

We have a version of this corollary for some elliptic curve factors $E/Q$ of $J_0(N)$ (e.g., $N = 37$) of root number $-1$ assuming rank$_Q E(\mathbb{Q}) = 1$ and $|\Pi_Q(E)| < \infty$ for the analytic family with constant root number $-1$ containing $E$ (such a family is associated to an exotic tower; see Theorem 15.2).

Instead of starting with the modular elliptic curve as listed above, we can start with a CM elliptic curve $E$ with finite Tate–Shafarevich group $\Pi_E$, found by Rubin [R87] and then we get a similar result for the CM family of abelian varieties containing the starting elliptic curve (although some CM cases are also covered by Corollary C).

For an extension $X$ of an abelian variety by a finite group scheme defined either over a number field $K$ or a local field $K$ of characteristic $0$, we define the fpfp abelian sheaf $\hat{X}$ explicitly as follows:

$$\hat{X}(R) = \begin{cases} X(R) \otimes_\mathbb{Z} \mathbb{Z}_p & \text{if } [K : \mathbb{Q}] < \infty, \\ X[p^n](R) & \text{if } [K : \mathbb{Q}] = 1 \text{ and } [\mathbb{Q}_l] < \infty (l \neq p) \text{ or } [K : \mathbb{R}] < \infty, \\ (X/X[p])(R) & \text{as a sheaf quotient if } [K : \mathbb{Q}_p] < \infty \end{cases}$$

for fpfp algebras $R/K$, where $X[p]$ is the maximal prime-to-$p$ torsion subgroup of $X$. If $R$ is a finite extension field of $K$ (except for the case of $K = \mathbb{R}, \mathbb{C}$), $\hat{X}(R) = \lim_n X(R)/p^n X(R)$ as already mentioned. Therefore, we could have defined $\hat{X}(R) := \lim_n X(R)/p^n X(R)$ except in the case where $K = \mathbb{R}, \mathbb{C}$ (and using this definition, the value $\hat{X}(R)$ is computed in [H15, (S) in page 228] as specified in (S) above). For $K = \mathbb{R}, \mathbb{C}$, this is just a convention as $H^q(K, ?)$ with coefficients in a $\mathbb{Z}_p$-module $M$ just vanishes if $p > 2$. Throughout the paper, we write $M^\vee$ for the Pontryagin dual module $\text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ for a $\mathbb{Z}_p$-module $M$.

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2. $U(p)$-isomorphisms

Replacing fppf cohomology we described in [H15, §3] by étale cohomology, we reproduce the results and proofs in [H15, §3] as it gives the foundation of our control result, though we need later to adjust technically the method described here to get precise control of the limit Tate–Shafarevich group. Let $S = \text{Spec}(K)$ for a field $K$. Let $X \to Y \to S$ be proper morphisms of noetherian schemes. We study

$$H^0_{\text{fppf}}(T, R^1 f_* \mathcal{G}_m) = H^0_{\text{ét}}(T, R^1 f_* \mathcal{G}_m) = R^1 f_* O^\times_X(T) = \text{Pic}_{X/S}(T)$$

for $S$-scheme $T$ and the structure morphism $f : X \to S$. Write the morphisms as $X \xrightarrow{\pi} Y \xrightarrow{g} S$ with $f = g \circ \pi$. We note the following general fact:

**Lemma 2.1.** Assume that $\pi$ is finite flat. Then the pull-back of line bundles: $\text{Pic}_{Y/S}(T) \ni \mathcal{L} \mapsto \pi^* \mathcal{L} \in \text{Pic}_{X/S}(T)$ induces the Picard functoriality which is a natural transformation $\pi^* : \text{Pic}_{Y/S} \to \text{Pic}_{X/S}$ contravariant with respect to $\pi$. Similarly, we have the Albanese functoriality sending $\mathcal{L} \in \text{Pic}_{X/S}(T)$ to $\bigwedge^{\text{deg}(X/Y)} \pi_* \mathcal{L} \in \text{Pic}_{Y/S}(T)$ as long as $X$ has constant degree over $Y$. This map $\pi_* : \text{Pic}_{X/S} \to \text{Pic}_{Y/S}$ is a natural transformation covariant with respect to $\pi$.

Hereafter we always assume that $\pi$ is finite flat with constant degree.

In [H15, §3], we assumed that $f$ and $g$ have compatible sections $S \xrightarrow{s_\pi} Y$ and $S \xrightarrow{s_g} X$ so that $\pi \circ s_f = s_g$. However in this paper, we do not assume the existence of compatible sections, but we limit ourselves to $T = \text{Spec}(\kappa)$ for an étale extension $\kappa$ of the base field $K$. Then we get (e.g., [NMD, Section 8.1] and [ECH, Chapter 3])

$$\text{Pic}_{X/S}(T) = H^0_{\text{fppf}}(T, R^1 f_* \mathcal{G}_m) \xrightarrow{(*)} H^1_{\text{fppf}}(X_T, O^\times_{X_T}) = H^1_{\text{ét}}(X_T, O^\times_{X_T})$$

$$\text{Pic}_{Y/S}(T) = H^0_{\text{fppf}}(T, R^1 g_* \mathcal{G}_m) \xrightarrow{(*)} H^1_{\text{fppf}}(Y_T, O^\times_{Y_T}) = H^1_{\text{ét}}(Y_T, O^\times_{Y_T})$$

for any $S$-scheme $T$. The identity at $\pi^*$ follows from the fact: $\text{Pic}_T = 0$, since $T$ is a union of points (i.e., $\kappa = k_1 \oplus \cdots \oplus k_m$ for finite separable field extensions $k_j/K$). Here $X_T = X \times_S T$ and $Y_T = Y \times_S T$. We suppose that the functors $\text{Pic}_{X/S}$ and $\text{Pic}_{Y/S}$ are representable by group schemes whose connected components are smooth (for example, if $X,Y$ are smooth proper and geometrically reduced (and $S = \text{Spec}(K)$ for a field $K$); see [NMD, 8.2.3, 8.4.2–3]). We then write $J_T = \text{Pic}^0_{X/S}$ ($= X,Y$) for the identity connected component of $\text{Pic}_{X/S}$. Anyway we suppose hereafter also that $X,Y,S$ are varieties (i.e., geometrically reduced separated schemes of finite type over a field).

For an fppf covering $U \to Y$ and a presheaf $P = P_Y$ on the fppf site over $Y$, we define via Čech cohomology theory an fppf presheaf $U \mapsto H^q(U, P)$ denoted by $\check{H}^q(P_U)$ (see [ECH, III.2.2 (b)]). The inclusion functor from the category of fppf sheaves over $Y$ into the category of fppf presheaves over $Y$ is left exact. The derived functor of this inclusion of an fppf sheaf $F = F_Y$ is denoted by $\check{H}^n(F_U)$ (see [ECH, III.15 (c)]). Thus $\check{H}^n(\mathbb{G}_m/Y)(U) = H^n_{\text{fppf}}(U, O^\times_U)$ for a $Y$-scheme $U$ as a presheaf (here $U$ varies in the small fppf site over $Y$).

To study control of the Picard groups under Galois action, assuming that $f$, $g$ and $\pi$ are all faithfully flat of finite presentation, we use the spectral sequence of Čech cohomology of the flat covering $\pi : X \to Y$ in the fppf site over $Y$ [ECH, III.2.7]:

$$\check{H}^p(X_T/Y_T, \check{H}^q(\mathbb{G}_m/Y)) \Rightarrow H^n_{\text{fppf}}(Y_T, O^\times_{Y_T}) \xrightarrow{\iota} H^n(Y_T, O^\times_{Y_T})$$

for each $S$-scheme $T$. Here $F \mapsto H^n_{\text{fppf}}(Y_T, F)$ (resp. $F \mapsto H^n(Y_T, F)$) is the right derived functor of the global section functor: $F \mapsto F(Y_T)$ from the category of fppf sheaves (resp. Zariski sheaves) over $Y_T$ to the category of abelian groups. The isomorphism $\iota$ is the one given in [ECH, III.4.9].
Write $H^*_T$ for $H^*(\mathbb{G}_m/Y_T)$ and $H^0(\mathbb{G}_m/Y_T)$ for $H^0(X_T/Y_T, H^0_{Y_T})$. From this spectral sequence, we have the following commutative diagram with exact rows:

$$
\begin{array}{c}
\begin{array}{ccc}
\hat{H}^1(H^*_T) & \longrightarrow & \text{Pic}_{Y/S}(T) \\
\downarrow & & \downarrow a \\
\hat{H}^0(\mathbb{G}_m/Y_T) & \longrightarrow & \hat{H}^2(H^0_{Y_T})
\end{array}
\quad \text{(2.2)}
\end{array}
\begin{array}{ccc}
\hat{H}^1(H^0_{Y_T}) & \longrightarrow & \text{Pic}_{Y/S}(T) \\
\downarrow & & \downarrow b \\
\hat{H}^0(X_T/Y_T, \text{Pic}_X(T)) & \longrightarrow & \hat{H}^2(H^0_{Y_T})
\end{array}
\text{ (2.3)}
\end{array}

Here the horizontal exactness at the top two rows follows from the spectral sequence (2.1).

Take a correspondence $U \subset Y \times_S Y$ given by two finite flat projections $\pi_1, \pi_2 : U \to Y$ of constant degree (i.e., $\pi_j_*\mathcal{O}_U$ is locally free of finite rank $\deg(\pi_j)$ over $\mathcal{O}_Y$). Consider the pullback $U_X \subset X \times_S X$ given by the Cartesian diagram:

$$
\begin{array}{ccc}
U_X = U \times_{Y \times_S Y} (X \times_S X) & \longrightarrow & X \times_S X \\
\downarrow & & \downarrow \\
U & \leftarrow & Y \times_S Y
\end{array}
$$

Let $\pi_{j,X} = \pi_j \times_S \pi : U_X \to X$ ($j = 1, 2$) be the projections.

Consider a new correspondence $U_X^{(q)} = \underbrace{U_X \times_Y U_X \times_Y \cdots \times_Y U_X}_{q}$, whose projections are the iterated product

$$
\pi_{j,X}^{(q)} = \pi_{j,X} \times_Y \cdots \times_Y \pi_{j,X} : U_X^{(q)} \to X^{(q)} \quad (j = 1, 2).
$$

Here is a first step to get a control result of $\Lambda$-TS groups:

**Lemma 2.2.** Let the notation and the assumption be as above. In particular, $\pi : X \to Y$ is a finite flat morphism of geometrically reduced proper schemes over $S = \text{Spec}(K)$ for a field $K$. Suppose that $X$ and $U_X$ are proper schemes over a field $K$ satisfying one of the following conditions:

1. $U_X$ is geometrically reduced, and for each geometrically connected component $X^o$ of $X$, its pull back to $U_X$ by $\pi_{2,X}$ is also connected; i.e., $\pi^0(X) \to \pi^0(U_X)$.
2. $(f \circ \pi_{2,X})_*\mathcal{O}_{U_X} = f_*\mathcal{O}_X$.

If $\pi_2 : U \to Y$ has constant degree $\deg(\pi_2)$, then, for each $q > 0$, the action of $U^{(q)}$ on $H^0(X, \mathcal{O}_X^{(q)})$ factors through the multiplication by $\deg(\pi_2) = \deg(\pi_{2,X})$.

This result is given as [H15, Lemma 3.1, Corollary 3.2]. Though in [H15, §3], an extra assumption of requiring the existence of compatible sections to $X \to Y \to S$, this assumption is nothing to do with the proof of the above lemma, and hence the proof there is valid without any modification.

To describe the correspondence action of $U$ on $H^0(X, \mathcal{O}_X^{(q)})$ in down-to-earth terms, let us first recall the Čech cohomology: for a general $S$-scheme $T$,

$$
\begin{array}{c}
\hat{H}^0(X_T/Y_T, \mathbb{G}_m/Y_T) = \\
\{(c_{i_0}, \ldots, i_q) \in H^0(X_T^{(q+1)}, \mathcal{O}_X^{(q+1)}) \text{ and } \prod_j (c_{i_0, j, \ldots, i_q} \circ p_{i_0, j, \ldots, i_q})^{(-1)^j} = 1\} \\
\{db_{i_0, \ldots, i_q} = \prod_j (h_{i_0, j, \ldots, i_q} \circ p_{i_0, j, \ldots, i_q})^{(-1)^j} | b_{i_0, j, \ldots, i_q} \in H^0(X_T^{(q)}, \mathcal{O}_X^{(q)})\}
\end{array}
\text{ (2.3)}
$$

where we agree to put $H^0(X_T^{(0)}, \mathcal{O}_X^{(0)}) = 0$ as a convention,

$$
X_T^{(q)} = \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{q} \times_S T, \mathcal{O}_X^{(q)} = \underbrace{\mathcal{O}_X \times_{\mathcal{O}_Y} \mathcal{O}_X \times_{\mathcal{O}_Y} \cdots \times_{\mathcal{O}_Y} \mathcal{O}_X}_{q} \times_{\mathcal{O}_S} \mathcal{O}_T.
$$
the identity $\prod (c \circ p_{i_0\ldots i_j})\{-i\}^j = 1$ takes place in $O_{X^{(q+2)}}$ and $p_{i_0\ldots i_j} : X^{'(q+2)} \rightarrow X^{'(q+1)}$ is the projection to the product of $X$ the $j$-th factor removed. Since $T \times_T T \cong T$ canonically, we have $X^{(q)} \cong X_T \times_T \cdots \times_T X_T$ by transitivity of fiber product.

Consider $\alpha \in H^0(X, \mathcal{O}_X)$. Then we lift $\pi^* \alpha \circ \pi_X \in H^0(U_X, \mathcal{O}_{U_X})$. Put $\alpha_U := \pi^* \alpha \circ \pi_X$. Note that $\pi_X^* \mathcal{O}_{U_X}$ is locally free of rank $d = \deg(\pi_X)$ over $\mathcal{O}_X$, the multiplication by $\alpha_U$ has its characteristic polynomial $P(T)$ of degree $d$ with coefficients in $\mathcal{O}_X$. We define the norm $N_U(\alpha_U)$ to be the constant term $P(0)$. Since $\alpha$ is a global section, $N_U(\alpha_U)$ is a global section, as it is defined everywhere locally. If $\alpha \in H^0(X, \mathcal{O}_X)$, $N_U(\alpha_U) \in H^0(X, \mathcal{O}_X)$. Then define $U(\alpha) = N_U(\alpha_U)$, and in this way, $U$ acts on $H^0(X, \mathcal{O}_X)$.

For a degree $q$ Čech cohomology class $[c] \in \check{H}^q(X/Y, \mathcal{L}^0(\mathbb{G}_m/Y))$ with a Čech $q$-cocycle $c = (c_{i_0\ldots i_q})$, $U([c])$ is given by the Čech cohomology class of the Čech cocycle $U(c) = (U(c_{i_0\ldots i_q}))$, where $U(c_{i_0\ldots i_q})$ is the image of the global section $c_{i_0\ldots i_q}$ under $U$. Indeed, $(\pi^*_U c_{i_0\ldots i_q})$ plainly satisfies the cocycle condition, and $(N_U(\pi^*_U c_{i_0\ldots i_q}))$ is again a Čech cocycle as $N_U$ is a multiplicative homomorphism. By the same token, this operation sends coboundaries to coboundaries, and define the action of $U$ on the cohomology group. Thus we get the following vanishing result:

**Proposition 2.3.** Suppose that $S = \text{Spec}(K)$ for a field $K$. Let $\pi : X \rightarrow Y$ be a finite flat covering of (constant) degree $d$ of geometrically reduced proper varieties over $K$, and let $Y \xrightarrow{\pi_1} U \xrightarrow{\pi_2} Y$ be two finite flat coverings (of constant degree) identifying the correspondence $U$ with a closed subscheme $U$ of $\pi^*_1 \pi^*_2 Y \times_S Y$. Write $\pi_{1, X} : U_X = U \times_Y X \rightarrow X$ for the base-change to $X$. Suppose one of the conditions (1) and (2) of Lemma 2.2 for $(X, U)$. Then

1. The correspondence $U \subset Y \times_S Y$ sends $\check{H}^q(H_1^0)$ into $\deg(\pi_2)(\check{H}^q(H_1^0))$ for all $q > 0$.
2. If $d$ is a $p$-power and $\deg(\pi_2)$ is divisible by $p$, $\check{H}^q(H_1^0)$ for $q > 0$ is killed by $U^M$ if $p^M \geq d$.

This follows from Lemma 2.2, because on each Čech $q$-cocycle (whose value is a global section of iterated product $X^{'(q+1)}$), the action of $U$ is given by $U^{(q+1)}$ by (2.3). See [H15, Proposition 3.3] for a detailed proof.

Assume that a finite group $G$ acts on $X/Y$ faithfully. Then we have a natural morphism $\phi : X \times G \rightarrow X \times_Y X$ given by $\phi(x, x') = (x, \sigma(x))$. Suppose that $\phi$ is surjective; for example, if $Y$ is a geometric quotient of $X$ by $G$; see [GME, §1.8.3]). Under this map, for any fppf abelian sheaf $F$, we have a natural map $\hat{H}^0(X/Y, F) \rightarrow H^0(G, F(X))$ sending a Čech 0-cocycle $c \in H^0(X, F) = F(X)$ (with $p^*c = p^*c$) to $c \in H^0(G, F(X))$. Obviously, by the surjectivity of $\phi$, the map $\hat{H}^0(X/Y, F) \rightarrow H^0(G, F(X))$ is an isomorphism (e.g., [ECH, Example III.2.6, page 100]). Thus we get

**Lemma 2.4.** Let the notation be as above, and suppose that $\phi$ is surjective. For any scheme $T$ fppf over $S$, we have a canonical isomorphism: $\hat{H}^0(X_T/Y_T, F) \cong H^0(G, F(X_T))$.

We now assume $S = \text{Spec}(K)$ for a field $K$ and that $X$ and $Y$ are proper reduced connected curves. Then we have from the diagram (2.2) with the exact middle two columns and exact horizontal rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\uparrow & & \downarrow \text{deg} \text{onto} & & \downarrow \text{deg} \text{onto} & & \uparrow \\
\hat{H}^1(H^0_Y) & \longrightarrow & \text{Pic}_{Y/S}(T) & \longrightarrow & \hat{H}^0(\frac{\mathcal{X}_Y}{T}, \text{Pic}_{X/S}(T)) & \longrightarrow & \hat{H}^2(H^0_Y) \\
\uparrow & & \cup & & \cup & & \uparrow \\
?_1 & \longrightarrow & J_Y(T) & \longrightarrow & \hat{H}^0(\frac{\mathcal{X}_Y}{Y_T}, J_X(T)) & \longrightarrow & ?_2, \\
\end{array}
$$

Thus we have $?_j = \hat{H}^j(H^0_Y)$ ($j = 1, 2$).

By Proposition 2.3, if $q > 0$ and $X/Y$ is of degree $p$-power and $p|\deg(\pi_2)$, $\hat{H}^q(H^0_Y)$ is a $p$-group, killed by $U^M$ for $M \gg 0$. 

3. Exotic modular curves

We study a more general tower \( \{ X_r \} \), different from the standard one \( \{ X_1(Np^r) \} \), considered in the introduction (thus hereafter, \( X_r \) could be no longer \( X_1(Np^r) \)). We introduce open compact subgroups of \( \text{GL}_2(\mathbb{A}^{(\infty)}) \) giving rise to the general tower \( \{ X_r \} \).

Let \( \Gamma := 1 + p^r \mathbb{Z}_p \subset \mathbb{Z}_p^\times \), where \( \epsilon = 2 \) if \( p = 2 \) and \( \epsilon = 1 \) otherwise. Let \( \gamma = 1 + p^r \), which is a topological generator of \( \Gamma = \gamma \mathbb{Z}_p^\times \). We define the Iwasawa algebra \( \Lambda := \mathbb{Z}_p[[\Gamma]] = \lim_{n \to \infty} \mathbb{Z}_p[\Gamma/G^{p^n}] \) and identify it with the power series ring \( \mathbb{Z}_p[[T]] \) sending \( \gamma \) to \( t = 1 + T \). The group \( \hat{\Gamma} \) is a maximal torsion-free subgroup of \( \mathbb{Z}_p^\times \). Fix an exact sequence of profinite groups \( 1 \to H_p \to \Gamma \to \mathbb{Z}_p^\times \to 1 \), and regard \( H_p \) as a subgroup of \( \Gamma \times \Gamma \). This implies

\[
\pi_\Gamma(a, d) = a^\alpha d^{-\delta}
\]

for a pair \((\alpha, \delta) \in \mathbb{Z}_p^2\) with \( a\mathbb{Z}_p + \delta \mathbb{Z}_p = \mathbb{Z}_p \) and hence \( H_p = \{(a, d) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times | a^\alpha d^{-\delta} = 1\} \).

Writing \( \mu \) for the maximal torsion subgroup of \( \mathbb{Z}_p^\times \), we pick a character \( \xi : \mu \times \mu \to \mathbb{Z}_p \) and define

\[
H = H_\xi = H_{\alpha, \delta, \xi} := H_p \times \text{Ker}(\xi) \text{ in } \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times = \Gamma = \Gamma \times \Gamma \times \mu \times \mu.
\]

We can take \( \xi(\zeta, \zeta') = \zeta^{\alpha'} \zeta'^{-\delta'} \) for \((\alpha', \delta') \in \mathbb{Z}_p^2\). Write \( \pi := \pi_\Gamma \times \xi : \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \) and the image of \( H \) in \((\mathbb{Z}_p^\times)^2 / (\mathbb{Z}_p^\times)^2 \) as \( H_r \).

Then define, for \( \hat{\mathbb{Z}} = \prod_{\text{primes}} \mathbb{Z}_l \),

\[
\hat{\Gamma}_0(M) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(\hat{\mathbb{Z}}) \big| c \in M \hat{\mathbb{Z}} \right\} , \quad \hat{\Gamma}_1(M) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(M) \big| d - 1 \in M \hat{\mathbb{Z}} \right\} , \quad \hat{\Gamma}_s := \hat{\Gamma}_{H,s} := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \hat{\Gamma}_0(p^r) \cap \hat{\Gamma}_1(N) \big| (a, d_p) \in H_s \right\} , \quad \hat{\Gamma}_r := \hat{\Gamma}_{H,s} := \hat{\Gamma}_0(p^r) \cap \hat{\Gamma}_1(N) (s \geq r).
\]

By definition, \( \hat{\Gamma}_r \cap \text{SL}_2(\mathbb{Q}) = \Gamma_1(Np^r) \) as in the introduction if \( H_p = \Gamma \times \{1\} \) (i.e., \((\alpha, \delta) = (0, 1)\)) and \( \xi(a, d) = \omega(d) \) for \( \omega(a) = \lim_{n \to \infty} a^{p^n} \) if \( p \) is odd and otherwise \( \omega(a) = \left( \frac{2a(\sqrt{-1})}{a} \right) \) (the quadratic residue symbol). We write this \( \xi \) as \( \omega_d \).

Consider the moduli problem over \( \mathbb{Q} \) of classifying the following triples

\[
(\mathcal{E}, \mu_N \overset{\phi_N}{\rightarrow} \mathcal{E}, \mu_p \overset{\phi_p}{\rightarrow} \mathcal{E}[p^r] \overset{\varphi_{p^r}}{\rightarrow} \mathbb{Z}/p^r \mathbb{Z})/\mathcal{R},
\]

where \( \mathcal{E} \) is an elliptic curve defined over a \( \mathbb{Q} \)-algebra \( \mathcal{R} \) and the sequence \( \mu_{p^n} \hookrightarrow E[p^r] \to \mathbb{Z}/p^r \mathbb{Z} \) is meant to be exact in the category of finite flat group schemes. As is well known (e.g., [AME]), the triples are classified by a modular curve \( U_r/\mathbb{Q} \), and we write \( Z_r \) for the compactification of \( U_r \) smooth at cusps. In Shimura's terminology, writing \( Z_r' \) for the canonical model attached to \( U_r := \hat{\Gamma}_1(p^r) \cap \hat{\Gamma}_1(N) \), the curve \( Z_r' \) is defined over \( \mathbb{Q}(\mu_{p^r}) \) and is geometrically irreducible, while we have \( Z_r = \text{Res}_{\mathbb{Q}(\mu_{p^r})/\mathbb{Q}} Z_r' \) (when \( N \geq 4 \)) which is not geometrically irreducible. We have the identity of the complex points \( Z_r(\mathbb{C}) - \{\text{cusps}\} = \text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\hat{\mathbb{A}})/U_r \mathbb{R}^+ \text{SO}_2(\mathbb{R}) \).

Each element \((u, a, d)\) of the group \( G := (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \) acts on \( Z_r \) by sending \( (\mathcal{E}, \mu_N \overset{\phi_N}{\rightarrow} \mathcal{E}, \mu_p \overset{\phi_p}{\rightarrow} \mathcal{E}[p^r] \overset{\varphi_{p^r}}{\rightarrow} \mathbb{Z}/p^r \mathbb{Z}) \) to

\[
(\mathcal{E}, \phi_N \circ u : \mu_N \overset{\phi_N \circ u}{\rightarrow} \mathcal{E}, \mu_p \overset{\phi_p \circ d}{\rightarrow} \mathcal{E}[p^r] \overset{\varphi_{p^r} \circ d_p}{\rightarrow} \mathbb{Z}/p^r \mathbb{Z}),
\]

where \( a \circ \varphi_{p^r}(x) = a \varphi_{p^r}(x) \) and the action on \( \phi_{p^r} \) and \( \phi_N \) is the one we have described in the introduction. For \( z = (z_N, z_p) \in (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \), we write the action of \((u, a, d) = (z_N, z_p, z_p)\) as \( (z) \). Via the inclusion \( \Gamma \times \Gamma \subset G \), the two variable Iwasawa algebra \( \Lambda := \mathbb{Z}_p[[\Gamma \times \Gamma]] \) is embedded into \( \mathbb{Z}_p[[G]] = \Lambda[(\mathbb{Z}/N\mathbb{Z})^\times \times \mu \times \mu] \) for the maximal torsion subgroup \( \mu \) of \( \mathbb{Z}_p^\times \).

We consider the quotient curves \( X_r := Z_r/H \). The complex points of \( X_r \) removed cusps is given by \( Y_r(\mathbb{C}) = \text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\hat{\mathbb{A}})/\hat{\Gamma}_r \mathbb{R}^+ \text{SO}_2(\mathbb{R}) \). Indeed, the action of \((a, d_p) \in H \) regarded as an element \( \left( \begin{array}{cc} a & 0 \\ 0 & d_p \end{array} \right) \in \text{GL}_2(\mathbb{Z}_p) \subset \text{GL}_2(\hat{\mathbb{Z}}) \) is given by \( (\phi_{p^r}, \varphi_{p^r}) \mapsto (\phi_{p^r} \circ d_p, \varphi_{p^r} \circ a_p) \). If \( \text{det}(\hat{\Gamma}_r) = \hat{\mathbb{Z}}^\times \),
by [IAT, Chapter 6], $X_r$ is a geometrically connected curve canonically defined over $\mathbb{Q}$. We have an adelic expression of their complex points.

$$X_r^i(\mathbb{C}) \setminus \{\text{cusps}\} = GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A})/\hat{\Gamma}_r \times SO_2(\mathbb{R}) = \Gamma_r \setminus \mathbb{H}$$ and $X_r(\mathbb{C}) = \Gamma_r \setminus \mathbb{H}$,

where $\hat{\Gamma}_r = \hat{\Gamma}_r^s \times SL_2(\mathbb{Q})$ and $\Gamma_r = \hat{\Gamma}_r \cap SL_2(\mathbb{Q})$. If $\det(\hat{\Gamma}_r) \not\subseteq \hat{\mathbb{Z}}^\times$, our curve $X_r^i = \text{Res}_{F_r/\mathbb{Q}} V_{F_r}$, and $X_r = \text{Res}_{F_r/\mathbb{Q}} V_{F_r}$ for Shimura’s geometrically irreducible canonical model $V_S$ defined over $\hat{F}_S$ for $S = \hat{\Gamma}_r^s$ and $\hat{\Gamma}_r$ (see [IAT, Chapter 6]). In any case, these curves are geometrically reduced curves defined over $\mathbb{Q}$ with equal number of geometrically connected components (i.e., it is $[F_r : \mathbb{Q}]$ for Shimura’s field of definition $F_r \subset \mathbb{Q}_{ab}$ fixed by $\det(\hat{\Gamma}_r) \subset \hat{\mathbb{Z}}^\times \cong \text{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})$).

We fix a $\mathbb{Z}_p$ basis $(\zeta_{np^r} = \exp(\frac{2\pi i}{np^r}))_{r \in \mathbb{Z}/(p-1)} \times (\mathbb{Z}/(nz))(1)$. Then we identify $\mu_{np^r}$ with $(\mathbb{Z}/np^r\mathbb{Z})$ by $\zeta_{np^r} \mapsto (m \mod np^r)$. For a triple $(E, \mu_q, \phi_{np^r}, E, \mu_{p^r}, \phi_{p^r}) \cong E[p^r]$, we have a unique generator $v \in E[\mu_{np^r}]$, by the canonical duality \langle \cdot, \cdot \rangle on $E[\mu_{np^r}]$, we have a unique generator $v \in E[\mu_{np^r}]$, such that $\langle v, \phi_{np^r}(\zeta_{np^r}) \rangle = \zeta_{np^r}$. Then the quotient $E' := E/(\text{Im}(\phi_{np^r}))$ has an inclusion $\mu_{np^r} \to E'$ given by sending $\zeta_{np^r}$ to $(\phi_{np^r} \cdot v)$. This gives a new triple $(E', \phi_{np^r}, \phi_{p^r})$, where $\phi_{p^r}$ is determined by $\langle \phi_{p^r}, (\phi_{np^r}(z)) \rangle = (\phi_{np^r}(z))^n$ for $z \in E'[p^r]$. We define an operator $w_r = w_{\zeta_{np^r}}$ acting on $Z_r$ by sending $(E, \mu_q, \phi_q) \to E[p^r]$, and $\phi_{p^r}$ to the above $(E', \phi_{np^r}, \phi_{p^r})$. We have the following fact from the definition:

**Lemma 3.1.** The tower $\{X_{r/q}\}$, with respect to $(\alpha, \delta, \xi)$, is isomorphically sent by $w_r$ defined over $\mathbb{Q}$ to the tower over $\mathbb{Q}$ with respect to $(-\delta, -\alpha, \xi')$ for $\xi'((a, d), \delta) = (\xi(a, d), \alpha)$. In other words, $H$ defining the tower $\{X_{r/q}\}$, is send to $H'$ defining the other by the involution $(a, d) \mapsto (d, a)$. Regarding $w_r$ as an involution of $X_r$ defined over $\mathbb{Q}(\mu_{np^r})$, if $\sigma_z \in \text{Gal}(\mathbb{Q}(\mu_{np^r})/\mathbb{Q})$ for $z \in (\mathbb{Z}/nz)^\times \leftrightarrow \mathbb{Z}_{np^r}$ is given by $\sigma_z(\mu_{np^r}) = \zeta_{np^r}$, we have $w_{\zeta_{np^r}} = (z \circ w_r = w_{\zeta_{np^r}} = (z)^{-1}$.

The last assertion of the lemma follows from $w_{\zeta_{np^r}} \circ w_{\zeta_{np^r}} = w_{\zeta_{np^r}} \circ w_{\zeta_{np^r}} = (z \circ w_r = w_{\zeta_{np^r}}$ and $w_{\zeta_{np^r}} = id$.

The group $\hat{\Gamma}_r^s (s > r)$ normalizes $\hat{\Gamma}_s$, and we have $\hat{\Gamma}_r^s/\hat{\Gamma}_r^s = \hat{\Gamma}_s/\hat{\Gamma}_s$ is canonically isomorphic to $(\mathbb{Z}_{np^r}^s \times \mathbb{Z}_{np^r}^s)/H \mod p^s$ by sending coset $((a, b), (c, d)) \mapsto (a, c, d, p^s)$ to $(\mathbb{Z}/p^s) \mod p^s$, and the moduli theoretic action of $H$ coincides with the action of $\text{Gal}(\mathbb{Q}(\mu_{np^r})/\mathbb{Q}) \cong (\mathbb{Z}_p^s \times \mathbb{Z}_p^s)/H \mod p^s$, where $\hat{\Gamma}_s \mapsto (\hat{\Gamma}_r^s \times \hat{\Gamma}_s^s)/H \cong \hat{\mathbb{Z}}^\times$ is the one variable Iwasawa algebra $\Lambda$ for $\hat{\Gamma}_r^s/\hat{\Gamma}_s^s$. The problem is reduced to the study of the determinant map at $p$. By $\alpha Z_{np^r} + \delta Z_{np^r} = \mathbb{Z}_p^s$, it is easy to see by definition, embedding diagonally $H$ into $GL_2(\mathbb{Z}_p^s)$, that

$$\det(H) \to \Gamma$$ is an isomorphism if and only if $p \nmid (\alpha + \delta)$ or $\alpha \cdot \delta = 0$.

If $(\alpha', \delta') \in \mathbb{Z}^2$ with $\alpha' Z + \delta' Z = \mathbb{Z}$ and $\xi(a, d) = \omega(d)^{\alpha'} \omega(d) - \delta'$, then $H \cap \mu_{np^r} \to H$ is an isomorphism if $\alpha' + \delta'$ is prime to $2 \cdot (p - 1)$ or $\alpha' \cdot \delta' = 0$.

The second condition becomes also a necessary condition if we replace $\alpha' \cdot \delta' = 0$ by $\alpha' \cdot \delta' \equiv 0$ mod $p - 1$ if $p$ is odd and by $\alpha' \cdot \delta' \equiv 0$ mod $2$ if $p = 2$. If $\alpha' = \delta' = i$, then $\text{Ker}(\xi) \cong (\xi, \xi)$, and hence $\det(H) \equiv \mu^2$. To have a non-trivial element in $\text{Ker}((\omega'))$ in $\mu \setminus \mu^2$, $\omega'$ has to have odd order.

$$\det(H) \cap \mu_{np^r} \equiv \mu \text{ if } \alpha' = \delta' = i \text{ and } \omega' \text{ has odd order.}$$

The image $\text{det}(H)$ can be a proper subgroup in $\mathbb{Z}_p^s$, and the curves $X_r$ and $X_r^i$ become reducible over the subfield $F = F_r/\mathbb{Q}$ fixed by $\text{det}(H)$ identifying $\text{Gal}(\mathbb{Q}(\mu_{np^r})/\mathbb{Q})$ with $\mathbb{Z}_p^s$.

The most interesting case is when $\xi(a, d) = \omega(d)^{\alpha} \omega^{-1}(d) (i = 0, 1, \ldots, p - 2)$ and $\alpha = \delta = 1$. Suppose $\alpha' = \delta' = i$ for $0 \leq i < p$ (so, $\alpha' Z + \delta' Z = i\mathbb{Z}$). In this case, the L-function $L(s, f_p)$ can have root number $\pm 1$ (so, Birch–Swinnerton Dyer conjecture would force the non-triviality of the
Mordell–Weil group of $A_P$ if the root number is $-1$). By (3.6), \( \det : \ker(\xi) \to \mu \) is onto if and only if \( \omega^\epsilon \) has odd order (including the case where \( i = 0 \)), and hence \( \det(\hat{\Gamma}_{H,i}) = \mathbb{Z}^\times \) if \( p > 2 \) and \( \omega^\epsilon \) has odd order. Otherwise, if \( p > 2 \), \( F_\xi \) is a unique quadratic extension of \( \mathbb{Q} \) inside \( \mathbb{Q}(\mu_p) \). If \( p = 2 \), if \( \alpha = \delta = 1 \) and \( \alpha' = \delta' = 0 \), \( F_\xi = \mathbb{Q}[\sqrt{2}] \), and if \( \alpha = \delta = 1 \) and \( \alpha' = \delta' = 0 \), then \( F_\xi = \mathbb{Q}(\sqrt{-1}, \sqrt{2}) \).

Taking \( (X,Y,U)/S \) to be \( (X_*/Q, X_{*/Q}, U(p))/Q \) for \( s > r \geq 1 \), to the projection \( \pi : X_s \to X_r^\epsilon \), the result of the previous section is plainly applicable if \( X_r^\epsilon \) is geometrically irreducible, since \( U(p) \) is also geometrically irreducible as it is the image of \( X_r^* \to X_r^\epsilon \) by the diagonal product of two degeneration maps from \( X_r^* \to X_r^\epsilon \). If not, writing \( X_{s,i}/\mathbb{Q} \) for geometrically irreducible components \( X_r^*, \) then \( U(p) \) restricted in each \( X_{r,s,i}^* \) is geometrically irreducible by the same argument above and its degree is a \( p \)-power independent of the components; so, we can apply the argument in Section 2 in these geometrically reducible cases.

**Corollary 3.2.** Let \( F \) be a number field or a finite extension of \( \mathbb{Q}_l \) for a prime \( l \). Then we have, for integers \( r, s \) with \( s > r \geq \epsilon \),

\[
\text{(u)} \quad \pi^*: J^*_{s/Q}(F) \to \bar{H}^0(X_*/X^\epsilon_*, J_{s/Q}(F)) \quad \text{is a } U(p)\text{-isomorphism,}
\]

where \( J_{s/Q}(F)[\gamma^{p^{s-r}}-1] = \ker(\gamma^{p^{s-r}}-1 : J_s(F) \to J_s(F)) \) and \( \epsilon = 1 \) if \( p > 2 \) and \( \epsilon = 2 \) if \( p = 2 \).

Here the identity at (s) follows from Lemma 2.4. The kernel \( A \mapsto \ker(\gamma^{p^{s-r}}-1 : J_s(A) \to J_s(A)) \) is an abelian fpf sheaf (as the category of abelian fpf sheaves is abelian and regarding a sheaf as a presheaf is a left exact functor), and it is represented by the scheme theoretic kernel \( J_{s/Q}[\gamma^{p^{s-r}}-1] \) of the endomorphism \( \gamma^{p^{s-r}}-1 \) of \( J_s/Q \). From the exact sequence \( 0 \to J_s[\gamma^{p^{s-r}}-1] \to J_s \to J_{s/Q}[\gamma^{p^{s-r}}-1] \to J_s \), we get another exact sequence: \( 0 \to J_s[\gamma^{p^{s-r}}-1](F) \to J_s(F) \to J_{s/Q}[\gamma^{p^{s-r}}-1](F) \). Thus

\[
J_{s/Q}(F)[\gamma^{p^{s-r}}-1] = J_{s/Q}[\gamma^{p^{s-r}}-1](F).
\]

By a simple Hecke operator identity (e.g., [H15, (3.1)]), we have the following commutative diagram (i.e., the contraction property of the \( U(p)\)-operator):

\[
\begin{array}{ccc}
J_{r/R} & \xrightarrow{\pi^*} & J^*_{s/R} \\
\downarrow u & & \downarrow u' \\
J_{r/R} & \xrightarrow{\pi^*} & J^*_{s/R} \\
\end{array}
\]

where the middle \( u' \) is given by \( U_s^*(p^{s-r}) = [\Gamma_s \left( \frac{1}{p^{s-r}} \right) \Gamma_r] \) and \( u \) and \( u'' \) are \( U_s^*(p^{s-r}) \). Thus

\[
\text{(u1)} \quad \pi^*: J^*_{r/R} \to J^*_{s/R} \text{ is a } U(p)\text{-isomorphism (for the projection } \pi : X^\epsilon_* \to X_r^\epsilon).\]

The above (u) combined with (u1) and (3.7) implies the sheaf identity (u2) below for integers \( r, s \) with \( s \geq r \geq \epsilon \):

\[
\text{(u2)} \quad \pi^*: J^*_{s/Q} \to J_{s/Q}[\gamma^{p^{s-r}}-1] = \ker(\gamma^{p^{s-r}}-1 : J_{s/Q} \to J_{s/Q}) \text{ is a } U(p)\text{-isomorphism.}
\]

We reformulate the above statement (u2) as follows:

**Lemma 3.3.** For integers \( r, s \) with \( s \geq r \geq \epsilon \), we have morphisms

\[
\iota^*_s : J_{s/Q}[\gamma^{p^{s-r}}-1] \to J^*_{s/Q} \quad \text{and} \quad \iota^*_{s,s} : J_{s/Q} \to J_{s/Q}/(\gamma^{p^{s-r}}-1)(J_{s/Q})
\]

satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
J^*_{s/Q} & \xrightarrow{\pi^*} & J_{s/Q}[\gamma^{p^{s-r}}-1] \\
\downarrow u & \nearrow \iota^*_s & \downarrow u'' \\
J^*_{s/Q} & \xrightarrow{\pi^*} & J_{s/Q}[\gamma^{p^{s-r}}-1], \\
\end{array}
\]

and

\[
\begin{array}{ccc}
J^*_{s/Q} & \xleftarrow{\pi^*} & J_{s/Q}/(\gamma^{p^{s-r}}-1)(J_{s/Q}) \\
\uparrow u^* & \nearrow \iota^*_s & \uparrow u'^* \\
J^*_{s/Q} & \xleftarrow{\pi^*} & J_{s/Q}/(\gamma^{p^{s-r}}-1)(J_{s/Q}), \\
\end{array}
\]
where \( u \) and \( u' \) are \( U(p^{s-r}) = U(p)^{s-r} \) and \( u^* \) and \( u'^* \) are \( U^*(p^{s-r}) = U^*(p)^{s-r} \). In particular, for an fppf extension \( T/\mathbb{Q} \), the evaluated map at \( T \): \( (J_{s/\mathbb{Q}}/(\gamma p^{s-r} - 1))(J_{s/\mathbb{Q}})(T) \xrightarrow{\pi} J'_s(T) \) (resp. \( J'_s(T) \xrightarrow{\pi^{-1}} J_s[\gamma p^{s-r} - 1](T) \)) is a \( U^*(p) \)-isomorphism (resp. a \( U(p) \)-isomorphism).

Proof. We first prove the assertion for \( \pi^* \). We note that the category of groups schemes fppf over a base \( S \) is a full subcategory of the category of abelian fppf sheaves. We may regard \( J'_s/\mathbb{Q} \) and \( J_s[\gamma p^{s-r} - 1]/\mathbb{Q} \) as abelian fppf sheaves over \( \mathbb{Q} \) in this proof. Since these sheaves are represented by (reduced) algebraic groups over \( \mathbb{Q} \), we can check being \( U(p) \)-isomorphism by evaluating the sheaf at a field \( K \) of characteristic 0 (e.g., [EAI, Lemma 4.18]). Since the degree of \( \pi \) is a full subcategory of the category of abelian fppf sheaves. We may regard \( J_s \) as \( X_s/\mathbb{Q}r \), every morphism appearing in the identity \( \pi^* \circ \pi^*_s = U(p)^{s-r} \) as a morphism of abelian fppf sheaves. Since the category of groups schemes fppf over a base \( S \) is a full subcategory of the category of abelian fppf sheaves, all morphisms appearing in the identity \( \pi^* \circ \pi^*_s = U(p)^{s-r} \) are morphism of group schemes. This proves the assertion for \( \pi^* \).

Take a number field so that \( X_s(K) \neq \emptyset \) (for example, the infinity cusp of \( X_s \) is rational over \( \mathbb{Q}(\mu_{p^n}) \)). Then \( \text{Pic}^0_{J'_s/K} \cong J'_s \) for any \( s \geq r \geq 0 \) by the self-duality of the Jacobian variety. Note that the second assertion is the dual of the first under this self-duality; so, over \( K \), it can be proven reversing all the arrows and replacing \( J_s[\gamma p^{s-r} - 1]/\mathbb{Q} \) (resp. \( U(p) \)) by the quotient \( J_s/(\gamma p^{s-r} - 1)J_s \) as fppf abelian sheaves (resp. \( \pi^*, U^*(p) \)). By Lemma 2.1, every morphism and every abelian variety of the diagram in question are all well defined over \( \mathbb{Q} \). In particular \( J_s/(\gamma p^{s-r} - 1)(J_s) \) is an abelian variety quotient over \( \mathbb{Q} \) (cf., [NMD, Theorem 8.2.12] combined with [ARG, §V.7]). Then by Galois descent for projective varieties (e.g., [GME, §1.11]), the diagram descends to \( \mathbb{Q} \). Since being \( U^*(p) \)-isomorphism or \( U(p) \)-isomorphism is insensitive to the descent process, we get the final assertion. □

Remark 3.4. For a finite extension \( k \) of \( \mathbb{Q} \) and an abelian variety \( A/k \), recall \( \hat{A}(k) := \lim_{\rightarrow} A(k)/p^n A(k) \) and for an infinite Galois extension \( \kappa/k \), \( \hat{A}(\kappa) := \lim_{\leftarrow} \hat{A}(F) \) with \( F \) running over all finite Galois extensions \( k \) inside \( \kappa \) (here note that \( \hat{A}(k) \) is not equal to \( A(k) \otimes_\mathbb{Z} \mathbb{Z}_p \) if \( k \) is a finite extension of \( \mathbb{Q} \); see (S) in the introduction). Thus this process of taking projective limit and then possibly an inductive limit with respect to \( F \) preserves the commutative diagrams (3.8) and (3.9), and the statements Corollary 3.2, (u), (u1), (u2) and Lemma 3.3 are also valid replacing the abelian varieties \( A \) in each statement by \( \hat{A} \).

4. Hecke algebras for exotic towers

Hereafter, we fix the data \( (\alpha, \delta, \xi) \) which defines the exotic tower \{\( X_r \_r \}. We introduce the Hecke algebra \( h_{\alpha, \delta, \xi} \) for the exotic tower \{\( X_r \_r \} \text{ defined for } (\alpha, \delta, \xi). We assume in the rest of the paper the following condition:

(F) The Hecke algebra \( h_{\alpha, \delta, \xi} \) is \( \Lambda \)-free.

In practice, if the local ring \( T \) of \( h_{\alpha, \delta, \xi} \) we are dealing with is \( \Lambda \)-free, our argument works. However there is not a good way to confirm directly \( \Lambda \)-freeness of \( T \); so, we assume (F). If \( (\alpha, \delta) = (0, 1) \) and \( \xi(a, d) \) only depends on \( d \), this is always true, and as we see in this section, the \( \Lambda \)-freeness of \( h_{\alpha, \delta, \xi} \) holds for \( p \geq 5 \) without any other assumptions, and even for \( p = 3 \), for most of \( (\alpha, \delta) \) including the self-dual case of \( \alpha = \delta = 1 \), the \( \Lambda \)-freeness of \( h_{\alpha, \delta, \xi} \) holds still true (see Proposition 18.2).
Let \( \{ X_r/\mathbb{Q} \} \) be the exotic tower as in Section 3. As described in (3.3), \( z \in \mathbb{Z}_p^\times \) acts on \( X_r \). Recall that \( J_r/\mathbb{Q} \) (resp. \( J_r/\mathbb{Q} \)) is the Jacobian of \( X_r \) (resp. \( X_r^* \)). We regard \( J_r \) as the degree 0 component of the Picard scheme of \( X_r \). For an extension \( K/\mathbb{Q} \), we consider the group of \( K \)-rational points \( J_r(K) \).

For each prime \( l \), we consider \( \pi_l := (\delta, \gamma) \in \text{GL}_2(\mathbb{Z}_l) \), and regard \( \pi_l \in \text{GL}_2(\mathbb{A}) \) so that its component at each place \( v \mid l \) is trivial. Then \( \Delta := \pi_l^{-1} \Gamma_r \pi_l \cap \Gamma_r \) gives rise to a modular curve \( X(\Delta) \) whose \( \mathbb{C} \)-points (outside cusps) is given by \( \text{GL}_2(\mathbb{Q}) \setminus \text{GL}_2(\mathbb{A}(\infty)) / (\delta \cup \delta') / \Delta \). We have a projection \( \pi_l^* : X(\Delta) \rightarrow X_r^* \) given by \( \delta \ni z \mapsto z/l \in \delta \) in addition to the natural one \( \pi_l : X(\Delta) \rightarrow X_r^* \) coming from the inclusion \( \Delta \subset \Gamma_r^* \). Then embedding \( X(\Delta) \) into \( X_r^* \times X_r^* \) by these two projections, we get the modular correspondence written by \( T(l) \) if \( l \nmid Np \) and \( U(l) \) if \( l \mid Np \). We can extend this definition to \( T(n) \) for all \( n > 0 \) prime to \( Np \) via Picard/Albanese functoriality (see Lemma 2.1). We use the same symbol \( T(n) \) and \( U(l) \) to indicates the endomorphism (called the Hecke operator) given by the corresponding correspondence \( T(n) \) and \( U(l) \). The Hecke operator \( U(p) \) acts on \( J_r(K) \) and the \( p \)-adic limit \( e = \lim_{n \to \infty} U(p)^n \) is well defined on the Barsotti–Tate group \( J_r[p^{\infty}] \) and the completed Mordell–Weil group \( J_r(K) \) as defined in (S) above.

Let \( \Gamma \) be the maximal torsion-free subgroup of \( \mathbb{Z}_p^\times \) given by \( 1 + p^r \mathbb{Z}_p \) for \( \epsilon = 1 \) if \( p > 2 \) and \( \epsilon = 2 \) if \( p = 2 \). Writing \( \gamma = 1 + p \gamma' \in \Gamma, \gamma' \) is a topological generator of the multiplicative group \( \Gamma = \mathbb{Z}_p^\times. \) As described in (3.3), \( (u, a, d) \in \mathbb{G} = (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \) acts on \( J_r \) through the quotient \( G/H \cong (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times. \) This action of \( (u, a, d) \in \mathbb{G} \), we write as \( (u, a, d) \); so, for a prime \( l \nmid Np, \) \( (l) = (u, a, d) \) for \( u = (l \bmod N) \) and \( a = d = l \in \mathbb{Z}_l^\times. \)

Define \( h_r(Z) \) by the subalgebra of \( \text{End}(J_r) \) generated by \( T(l) \) with \( l \) prime to \( Np \) and \( U(l) \) with \( l \neq Np \). Define \( h_r \) for \( s > r \) inducing a projective system \( \{ h_r \} \), whose limit gives rise to the big ordinary Hecke algebra \( h = \lim_r h_r \).

Writing \( (l) \) (the diamond operator) for the action of \( l \in (\mathbb{Z}/Np\mathbb{Z})^\times \) identified with \( \text{Gal}(X_r/X_0(Np^r)) \), we have an identity \( l(l)^2 = T(l)^2 - T(l^2) \in h_r(Z_p) \) for all primes \( l \nmid Np \).

Since \( \Gamma \subset \mathbb{Z}_p^\times \subset G/H \), we have a canonical \( \Lambda \)-algebra structure \( \Lambda = \mathbb{Z}_p[[\Gamma]] \hookrightarrow h \) sending \( \gamma \) to \( (1, a, d) \) for \( a, d \in \Gamma \) such that \( \pi_l(a, d) = \gamma \) as in (3.1). If \( (a, \delta) = (0, 1) = (\alpha', \delta') \), it is now well known that \( h \) is a free of finite rank over \( \Lambda \) and \( h_\infty = h \otimes \Lambda \otimes (\gamma_p^{r-1} - 1) \) (cf. [H68a], [K13] or [GME, §3.2.6]). More generally, by [PAF, Corollary 4.31], assuming \( p \geq 5 \), the same facts hold (and we expect this to be true without any assumption on primes). Anyway, if \( p = 2, 3 \), the specialization map \( h \otimes \Lambda \otimes (\gamma_p^{r-1} - 1) \to h_r \) is onto with finite kernel, and \( h \) is a torsion-free \( \Lambda \)-module of finite type. We will prove the \( \Lambda \)-freeness of \( h_{a', \delta} \otimes \Lambda \)-isomorphisms \( h \otimes \Lambda \otimes (\gamma_p^{r-1} - 1) \cong h_r \) for most cases of \( p = 3 \) in Section 18 for the sake of completeness.

A prime \( P \in \omega_\infty := \bigcup_{r > 0} \text{Spec}(h_r)(\overline{\mathbb{Q}}_p) \subset \text{Spec}(h)(\overline{\mathbb{Q}}_p) \) is called an arithmetic point of weight 2 in \( \text{Spec}(h) \). For a connected (resp. irreducible) component \( \text{Spec}(T) \) (resp. \( \text{Spec}(I) \)) of \( \text{Spec}(h) \), we put

\[
\omega_T := (h_\infty \cap \text{Spec}(T)) \text{ (resp. } \omega_I := (h_\infty \cap \text{Spec}(I))\text{).}
\]

In this paper, we only deal with arithmetic point of weight 2; so, we often omit the word “weight 2” and just call them arithmetic points/primes. Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism \( \lambda : h \rightarrow \overline{\mathbb{Q}}_p \) killing \( \gamma_p^{r-1} - 1 \) for \( r \geq 0 \) to a classical Hecke eigenform, we need to fix (once and for all) an embedding \( \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p \) of the algebraic closure \( \overline{\mathbb{Q}} \) in \( \mathbb{C} \) into a fixed algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \). We write \( i_\infty \) for the inclusion \( \overline{\mathbb{Q}} \subset \mathbb{C} \).

More generally, for the jacobian variety \( J(Z_r) \) of the curve \( Z_r \) defined above (3.3), we define \( h^{n, \text{ord}}_r \) to be the maximal \( \Lambda \)-algebra direct summand of \( \text{End}(J(Z_r)) \otimes \mathbb{Z}_p \) in which \( U(p) \) is invertible. Then as before we put \( h^{n, \text{ord}, \varphi}_r = h^{n, \text{ord}}_r \otimes \mathbb{A}_r W/\mathbb{A}_r h^{n, \text{ord}}_r \otimes \mathbb{Z}_p W \).
where $a_\varphi$ is the kernel of the algebra homomorphism $W[[Z_p^\times \times Z_p^\times]] \to W[[Z_p^\times]]^\times$ induced by the character $(a, d) \mapsto \varphi(a, d)\xi(a, d) : Z_p^\times \times Z_p^\times \to Z_p^\times$. If we take $\varphi(a, d) = a^\alpha d^{-\delta}$ for $(a, d) \in \Gamma \times \Gamma$ and $W = Z_p$ with $\delta : \mu \times \mu \to Z_p^\times$, we have $h^{n.ord.\varphi} = h_{s, \delta, \xi}(N)$ under present notation. Then by [PAF, Corollary 4.31], $h^{n.ord.\varphi}$ is $\Lambda$-free of finite rank for $\Lambda = Z_p[[I^2/H_p]]$.

Proposition 4.1. Assume $p \geq 5$ or $(\alpha, \delta, \xi) = (0, 1, \omega_d)$, where $\omega_d(a, d) = \omega(d)$. Then $h_{s, \delta, \xi}(N)$ is $\Lambda$-free of finite rank for $\Lambda = Z_p[[I^2/H_p]]$.

Remark 4.2. For $p \leq 3$, we will prove in Proposition 18.2 $\Lambda$-freeness of $h_{s, \delta, \xi}(N)$ if it is obtained by systematic twists of $h_{0, 1, \omega_d}(N)$. This covers the interesting cases of analytic families of abelian varieties, including some corresponding to the $p$-adic $L$-function $k \mapsto L(2k, k)$ as in the introduction.

We have injective limits $J_\infty(K) = \varinjlim J_r(K)$ and $J_\infty[p^\infty](K) = \varinjlim J_r[p^\infty](K)$ via Picard functoriality, on which $e$ acts. Write $G = G_{s, \delta, \xi} := e(J_\infty[p^\infty])$, which is called the $\Lambda$-adic Barsotti–Tate group in [H14] and whose arithmetic properties are scrutinized there. Adding superscript or subscript “ord”, we indicate the image of $e$.

The compact cyclic group $\Gamma$ acts on these modules by the diamond operators. Thus $J_\infty(K)^{ord}$ is a module over $\Lambda := Z_p[[\Gamma]] \cong Z_p[[T]]$ by $\gamma \mapsto t = 1 + T$ for a fixed topological generator $\gamma$ of $\Gamma = \gamma^2 \times \gamma$. The big ordinary Hecke algebra $h$ acts on $J_\infty^{ord}$ as endomorphisms of functors.

Let $\text{Spec}(\mathbb{T})$ be a connected component of $\text{Spec}(h)$ and $\text{Spec}(\mathbb{I})$ be a primitive irreducible component of $\text{Spec}(\mathbb{T})$. For each $\mathbb{I}$-module $M$, we put $M_T := \mathbb{M} \otimes \mathbb{I} \mathbb{T}$; in particular, $J_\infty^{ord} := J_\infty^{ord} \otimes \mathbb{I} \mathbb{T}$ as an fpf sheaf. We write $w_1 = \bigcup_{n=0}^{\infty} \text{Spec}(\mathbb{I} / (\gamma^{p^n} - 1))(\mathbb{Q}_p)$ (which is the set of all arithmetic points of weight 2). For $P \in w_1$ with $P \in \text{Spec}(\mathbb{I} / (\gamma^{p^n} - 1))(\mathbb{Q}_p)$, we write $r(P)$ for the minimal $r$ with this property. Then the corresponding Hecke eigenform $f_P$ belongs to $S_2(\Gamma_0(Np^r), \varepsilon_p, \chi)$ for a character $\varepsilon_p : Z_p^\times \to \mu_{p^r}$ and a character $\chi : \mu \times (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Here $f$ in this space satisfies

$$f((a \ b \ c \ d)) = \varepsilon_p(a^{-\alpha} d^\delta) \chi_p(\xi(a, d)) \chi(d)f$$

for $(a \ b \ c \ d) \in \Gamma_0(Np^r)$. Here $\chi_p = \chi|_\mu$ and $\chi_N = \chi|_{(\mathbb{Z}/N\mathbb{Z})^\times}$. The corresponding adelic form $f$ satisfies

$$f(\alpha xu) = \varepsilon_p(a^\alpha d^{-\delta}) \chi_p(\xi^{-1}(a, d)) \chi(dN)f$$

for all $u = (a \ b \ c \ d) \in \Gamma_0(Np^r)$. Here $\chi(d) = \chi_N(d^{-1}) = \chi(d^{(N\infty)})$ regarding $d \in \mathbb{A}^\times$.

5. Abelian factors of $J_r$.

We give a description of abelian factors $A_s$ and $B_s$ of the modular jacobian varieties $\{J_s\}_s$ of the exotic modular tower which behave coherently in the limit process under the Hecke operator action. Let $\pi_{s, r} : J_s \to J_r$ for $s > r$ be the morphism induced by the covering map $\pi_{s, r} : X_s \to X_r$ through Albanese functoriality. Then we define $\pi_r^s = w_r \circ \pi_{s, r} \circ w_s$. Note that $\pi_r^s$ is well defined over $\mathbb{Q}$ (cf. Lemma 3.1), and satisfies $T(n) \circ \pi_r^s = \pi_r^s \circ T(n)$ for all $n$ prime to $Np$ and $U(q) \circ \pi_r^s = \pi_r^s \circ U(q)$ for all $q|Np$ (as $w_r \circ h \circ w_g = h^*$ for $h \in h_0(\mathbb{Z})$ (7 = s, r) by [MF, Theorem 4.5.5].

Let $\text{Spec}(\mathbb{T})$ be a connected component of $\text{Spec}(h(N))$. Write $m_T$ for the maximal ideal of $\mathbb{T}$ and $1_T$ for the idempotent of $\mathbb{T}$ in $h(N)$. We assume the following condition

(A) We have $\varpi \in m_T$ such that $(\varpi) \cap \Lambda$ is a factor of $(\gamma^{p^s} - 1)$ in $\Lambda$ and that $\mathbb{T}/(\varpi)$ is free of finite rank over $\mathbb{Z}_p$.

We call a prime ideal $P$ satisfying the above condition (A) a principal arithmetic point of $\text{Spec}(\mathbb{T})$. Write $\varpi_s$ for the image of $\varpi \oplus (1 - 1_T)$ in $h_s(s \geq r)$ and define an $h_s(\mathbb{Z})$-ideal by

$$a_s = (\varpi_s h_s + (1 - e) h_s(\mathbb{Z}_p)) \cap h_s(\mathbb{Z}).$$

Write $A_s$ for the identity connected component of $J_s|_{a_s} = \bigcap_{a \in a_s} J_s(a)$, and put $B_s = J_s/a_s J_s$, where $a_s J_s$ is a rational abelian subvariety of $J_s$ given by $a_s J_s(\mathbb{Q}) = \sum_{a \in a_s} a(J_s(\mathbb{Q})) \subset J_s(\mathbb{Q})$.

Taking a finite set $G$ of generators of $a_s$, $a_s J_s$ is the image of $a : \bigoplus_{g \in G} J_s(\mathbb{Q}) \to J_s(\mathbb{Q})$ of $a(g(x)) \mapsto a_s J_s$. The kernel $J_s|_{a_s} = \text{Ker}(a)$ is a well defined fpf sheaves, which is represented by an extension of the abelian variety $A_s$ by a finite étale group scheme both over $\mathbb{Q}$. Then by [NMD, Theorem 8.2.12],
the quotient $aJ_s = (\bigoplus_{g \in G} J_s)/\text{Ker}(a)$ is well defined as an abelian scheme and is the sheaf fppf quotient. Then again $\hat{B}_s := J_s/a_sJ_s$ is the fppf sheaf quotient and also abelian variety quotient again by [NMD, Theorem 8.2.12]. By definition, $A_s$ is stable under $h_s(Z)$ and $h_s(Z)/a_s \to \text{End}(A_s)$.

**Lemma 5.1.** Assume (F) and (A). Then we have $\hat{A}^\text{ord}_s = \hat{J}^\text{ord}_s[\varpi_s]$ and $\hat{J}_s[a_s] = \hat{A}_s$. The abelian variety $A_s$ ($s > r$) is the image of $A_r$ in $J_s$ under the morphism $\pi^* = \pi_{s,r}^* : J_r \to J_s$ induced by Picard functoriality from the projection $\pi = \pi_{s,r} : X_s \to X_r$ and is $\mathbb{Q}$-isogenous to $B_s$. The morphism $J_s \to B_s$ factors through $J_s \xrightarrow{\pi^*} J_r \to B_r$. In addition, the sequence

$$0 \to \hat{A}_s^\text{ord} \to \hat{J}_s^\text{ord} \xrightarrow{\varpi_s} \hat{J}_s^\text{ord} \to \hat{B}_s^\text{ord} \to 0$$

is an exact sequence of fppf sheaves.

Passing to the limit, we get the following exact sequence of fppf sheaves:

$$0 \to \hat{A}_s^\text{ord} \to \hat{J}_s^\text{ord} \xrightarrow{\varpi_s} \hat{J}_s^\text{ord} \to \hat{B}_s^\text{ord} \to 0,$$

where $\hat{J}_s^\text{ord} = \text{lim}_{s \to s'} \hat{J}_s^\text{ord}$ and $\hat{X}_s^\text{ord} = \text{lim}_{s \to s'} \hat{X}_s^\text{ord}$ for $X = A, B$.

**Proof.** Taking a finite set $G$ of generators of $a_s$ containing $\varpi_s$, we get an exact sequence $0 \to J_s[a_s] \to J_s \xrightarrow{x \mapsto (g(x))_{g \in G}} \bigoplus_{g \in G} J_s$. Since $X \mapsto \hat{X}$ as in (S) is left exact, we have $\hat{A}_s \subset \bigcap_{a \in a_s} \hat{J}_s[a]$ with finite quotient. Applying further the idempotent, since $a_s = (\varpi_s) \oplus (1 - e)h_s(Z_p)) \cap h_s(Z)$, we find

$$\hat{J}_s[a_s]^\text{ord} = \bigcap_{a \in a_s} \hat{J}_s[a] = \hat{J}_s^\text{ord}[\varpi_s].$$

We have an exact sequence

$$0 \to J_s[a_s][p^\infty]^\text{ord} \to J_s[p^\infty]^\text{ord} \xrightarrow{\varpi_s} J_s[p^\infty]^\text{ord} \to \text{Coker}(\varpi_s) \to 0,$$

and $\text{Coker}(\varpi_s)$ is $p$-divisible and is dual to $J_s[a_s][p^\infty]^\text{ord}$ under the $w_s$-twisted self Cartier duality of $J_s[p^\infty]^\text{ord}$ (over $\mathbb{Q}$; see [H14, §4]). This shows $\hat{J}_s[a_s][p^\infty]^\text{ord}$ is $p$-divisible (so, $(J_s[a_s]/A_s)^\text{ord}$ has order prime to $p$), and hence $\hat{A}_s^\text{ord} = \hat{J}_s[a_s]^\text{ord}$.

Plainly by definition, $\pi^*(J_s[a_s]) \subset J_s[a_s]$. Since we have the following commutative diagram:

$$
\begin{array}{c}
\require{AMScd}
\begin{CD}
\text{h}_s(Z) @>>> \text{h}_r(Z) \\
\downarrow @. \downarrow \\
\text{h}_s(Z_p)/(\varpi, h_p + (1 - e)h_r(Z_p)) @>>> \text{h}_r(Z_p)/(\varpi, h_p + (1 - e)h_r(Z_p))
\end{CD}
\end{array}
$$

we have $\text{dim} A_s = \text{rank}_k h_s(Z)/a_s = \text{rank}_k h_r(Z)/a_s = \text{dim} A_r$; so, $A_s = \pi^*(A_r)$.

The above commutative diagram also tells us that $a_s \supset b_s := \text{Ker}(h_s(Z) \to h_r(Z))$ in $h_s(Z_p)$. Thus the projection $J_s \to J_s/a_sJ_s = B_s$ factors through $J_r = J_s/b_sJ_s$. Indeed, the natural projection: $J_s/b_sJ_s \to J_r/Q$ has to be a finite morphism (as the tangent space at the origin of the two are isomorphic), and we conclude $J_s/b_sJ_s = J_r$ by the universality of the categorical quotient $J_s/a_sJ_s$ (cf. [NMD, page 219]).

Assuming $J_s(K) \neq \emptyset$ for a finite extension $K/Q$, the dual sequence (over $K$) of the exact sequence of fppf sheaves: $0 \to J_s[a_s] \to J_s \xrightarrow{x \mapsto (g(x))_{g \in G}} \bigoplus_{g \in G} J_s$ is $\bigoplus_{g \in G} J_s \xrightarrow{x \mapsto \sum g(x)} J_s \to B_s \to 0$. Thus $A_s$ is isogenous to $B_s$ over $K$, and by Galois descent, $A_s$ is $\mathbb{Q}$-isogenous to $B_s$. Indeed, for the complementary abelian subvariety $A_s^\perp$ in $J_s$ of $A_s$, we have $J_s/A_s^\perp = B_s$, and the $\mathbb{Q}$-isogeny follows without taking duality. Here note that the quotient $J_s/A_s^\perp$ exists as an abelian variety and also as an fppf sheaves by [NMD, Theorem 8.2.12] (and [ARG, V.7]).

As explained just below (A), we have $\text{Im}(\bigoplus_{g \in G} J_s \xrightarrow{x \mapsto \sum g(x)} J_s) = a_sJ_s$ as fppf sheaves. Then applying the argument of [H15, Section 1] to the exact sequence

$$0 \to J_s[a_s] \to \bigoplus_{g \in G} J_s \to aJ_s \to 0$$

of fppf sheaves, we confirm the exactness of
\[ 0 \to J_s[a_s] \to \bigoplus_{g \in G} J_s \to aJ_s \to 0 \]
as fppf sheaves. Thus applying the idempotent \( e \), we confirm
\[ \text{Im}(\bigoplus_{g \in G} J_{s, g}^{\text{ord}} \to J_{s, g}^{\text{ord}}) = a_s J_{s, g}^{\text{ord}}. \]

Since the morphism \( \bigoplus_{g \in G} J_{s, g}^{\text{ord}} \xrightarrow{x - \sum_s g(x)} J_{s, g}^{\text{ord}} \) factors through \( \varpi_s(J_{s, g}^{\text{ord}}) \) as all \( g = \varpi_s x \) with \( x \in \mathfrak{h}_s \), noting \( \varpi \in G \), \( \varpi_s(J_{s, g}^{\text{ord}}) \to a_s(J_{s, g}^{\text{ord}}) \). Thus \( a_s J_{s, g}^{\text{ord}} = \varpi_s(J_{s, g}^{\text{ord}}) \) as fppf sheaves. This shows the exactness:
\[ 0 \to \hat{A}_s^{\text{ord}} \to J_s^{\text{ord}} \xrightarrow{\varpi_s} a_s J_s = \varpi_s(J_s^{\text{ord}}) \to 0. \]

Since \( B_s = J_s/a_s J_s \) as fppf sheaves, we see the exactness of
\[ 0 \to \varpi_s(J_s^{\text{ord}}) \to J_s^{\text{ord}} \to B_s^{\text{ord}} \to 0 \]
as fppf sheaves. Combining the two exact sequences, we obtain the exactness of the last sequence in the lemma.

Assuming \( X_s(K) \neq \emptyset \), \( J_s \cong \text{Pic}^0_{J_s/K} \) via the polarization of the canonical divisor (e.g., [ARG, VII.6]). The Rosati involution \( h \mapsto h^* \) and \( T(n) \mapsto T^*(n) \) brings \( h_s(Z) \to h_s^*(Z) \subset \text{End}(J_s/K) \). At the level of double coset operator \( [\Gamma \alpha \Gamma'] \), the involution has the following effect \( [\Gamma \alpha \Gamma']^* = [\Gamma' \alpha \Gamma] \).

Because of this fact, the involution \( h \mapsto h^* \) itself is well defined giving \( h_s(Z) \to h_s^*(Z) \in \text{End}(J_s/K) \) (even if \( X_s(Q) = \emptyset \)). Note that \( X_1(Np^r)(Q) \) contains the infinity cusp; so, for the standard tower, we have \( X_s(Q) \neq \emptyset \).

The Weil involution \( w_s = [\Gamma_s \left( \begin{smallmatrix} 0 & -1 \\ Np^{-1} & 0 \end{smallmatrix} \right) \Gamma_s] \) has the effect that \( w_s[\Gamma_s^* \Gamma_s] = [\Gamma_s \alpha \Gamma_s]w_s \) as easily verified. Thus \( w_s \circ T^*(n) = T(n) \circ w_s \) for all \( n \) including \( T(l) = U(l) \) for \( l \mid Np \). We write \( \{X^r_{s/N} \}_{s > r} \) for the dual tower corresponding to \( \{[\hat{\Gamma}_s^r] = w_s \hat{\Gamma}_s^r w_s^{-1} \}_{s > r} \) with the main involution \( \iota \) given by \( x' = \det(x) \). Thus \( \{X^r_{s/N} \}_{s > r} \) corresponds to the triple \( (-\alpha, -\delta, \xi') \) for \( \xi'(a,d) = \xi(d,a) \), and the \( l \)-component of \( \hat{\Gamma}_s^r \) for \( l \mid N \) is given by
\[ \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{GL}_2(Z_l) \mid c \in NZ_l, a - 1 \in NZ_l \right\}. \]

Then \( w_s \) gives an isomorphism \( w_r : X^r_s \to X^{r'}_s \) defined over \( Q \). Note that the fixed isomorphism \( \mu_{p^r} \cong \mathbb{Z}/p^r \mathbb{Z} \) \( \iota \) induces an isomorphism \( X^r_s \cong X^{r'}_s \) over \( Q(\mu_{p^r}) \). As an automorphism of \( X^r_{s/Q(\mu_{p^r})} \), \( w_s \) satisfies \( w^r_{s/Q} = \langle z \rangle \circ w_s = w_s \circ \langle z \rangle^{-1} \) for \( z \in \hat{\mathbb{Z}}^\times \) (see Lemma 3.1).

Take a connected component \( \text{Spec}(T) \) of \( \text{Spec}(h) \) and an irreducible component \( \text{Spec}(I) \) of \( \text{Spec}(T) \). Assume that \( I \) is primitive in the sense of [H86a, Section 3]. For each arithmetic \( P \in \text{Spec}(I)(\overline{Q}_p) \), the corresponding cusp form \( f_P \) is a \( p \)-stabilized Hecke eigenform of weight 2 new at each prime \( l \mid N \) if and only if \( I \) is primitive.

We get directly from Lemma 5.1 the following proposition giving sufficient conditions for the validity of (A) for \( A_{f,s} \) when \( f \mid f_P \) is in a \( p \)-adic analytic family indexed by \( P \in \text{Spec}(I) \).

**Proposition 5.2.** Let \( \text{Spec}(T) \) be a connected component of \( \text{Spec}(h) \) and \( \text{Spec}(I) \) be a primitive irreducible component of \( \text{Spec}(T) \). Assume \( \Lambda \)-freeness of \( T \) (i.e., (F)). Then the condition (A) holds for the choices of \( (\varpi, A_s, B_s) \):

1. Suppose that an eigen cusp form \( f = f_P \) new at each prime \( l \mid N \) belongs to \( \text{Spec}(T) \) and that \( T = I \) is regular (or more generally a unique factorization domain). Then writing the level of \( f_P \) as \( Np^r \), the algebra homomorphism \( \lambda : T \to \overline{Q}_p \) given by \( f(T(l)) = \lambda(T(l))f \) gives rise to the prime ideal \( P = \text{Ker}(\lambda) \). Since \( P \) is of height 1, it is principal generated by \( \varpi \in T \). This \( \varpi \) has its image \( a_s \in T_s = T \otimes \Lambda \Lambda_s \) for \( \Lambda_s = \Lambda/(\gamma p^{r-1} - 1) \). Write \( h_s = h \otimes \Lambda \Lambda_s = T_s \oplus 1_h, h_s \) as an algebra direct sum for an idempotent \( 1_s \). Then, the element \( \varpi_s = a_s \oplus 1_s \in h_s \) for the identity \( 1_s \) of \( h_s \) satisfies (A).

2. Fix \( r > 0 \). Then \( \varpi \in m_T \) for a factor \( \varpi \) in \( (\gamma p^{r-1} - 1) \) in \( \Lambda \), satisfies (A).
Let \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \) be a Dirichlet character, and consider the space \( S_2(\Gamma_0(N), \chi) \) of cusp forms of weight 2 with Neben character \( \chi \). Write \( \mathbb{Z}[\chi] \) for the subalgebra of \( \mathbb{C} \) generated by the values of \( \chi \). Then we can consider the Hecke algebra \( h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) \) inside \( \mathrm{End}_{\mathbb{C}}(S_2(\Gamma_0(N), \chi)) \) generated over \( \mathbb{Z}[\chi] \) by all Hecke operators \( T(n) \) and \( U(l) \). Then this Hecke algebra is free of finite rank over \( \mathbb{Z} \), and hence its reduced part (modulo the nilradical) has a well-defined discriminant \( D_\chi \) over \( \mathbb{Z} \). Here is a criterion from [F02, Theorem 3.1] for regularity of \( T \):

**Theorem 5.3.** Assume \( \Lambda \)-freeness of \( h_{\alpha,\delta,\xi} \). Let \( f \) be a Hecke eigenform of conductor \( N \), of weight 2 and with Neben character \( \chi \), and define \( a_p \in \mathbb{Q} \) by \( f(T(p)) = a_p f \). Let \( p \) be a prime outside \( 6D_\chi N\mathcal{P}(N) \) (for \( \mathcal{P}(N) = \mathbb{Z}/(\mathbb{Z}/N\mathbb{Z})^\times \)). Suppose that for the prime ideal \( p \) of \( \mathbb{Z}[a_p] \), \( a_p \mod p \) is different from 0 and \( \pm\sqrt{\chi(p)} \). Then for the connected component \( \Sigma(T) \) of \( \mathrm{Spec}(\mathbb{C}) \) acting non-trivially on the \( p \)-stabilized Hecke eigenform corresponding to \( f \) in \( S_2(\Gamma_0(Np), \chi) \), \( T \) is a regular integral domain isomorphic to \( W \otimes_{\mathbb{Z}_p} \Lambda = W[[T]] \) for a complete discrete valuation ring \( W \) unramified at \( p \).

The result is valid always for \( p \geq 5 \) and for \( p = 3 \) under (F) (see Propositions 4.1 and 18.2). Here is a proof of this fact since [F02, Theorem 3.1] is slightly different from the above theorem.

**Proof.** Let \( e^o := \lim_{\to} \langle a \rangle T(p)^{n!} \in h_2(\Gamma_0(N), \chi; A) \) for \( \mathbb{Z}[\chi] \)-algebra \( A \). Put \( h_2^\text{ord}(\Gamma_0(N), \chi; A) := e^o h_2(\Gamma_0(N), \chi; A) \). Since \( U(p) = T(p) \mod p \) on \( A[[q]] \), the natural algebra homomorphism:

\[
h_2^\text{ord}(\Gamma_0(Np), \chi; A) \to h_2^\text{ord}(\Gamma_0(N), \chi; A)
\]

sending \( U(p) \) to the unit root of \( X^2 - T(p)X + \chi(p)p \in h_2^\text{ord}(\Gamma_0(N), \chi; A)[X] \) and \( T(l) \) to \( T(l) \) for all primes \( l \neq p \) is a well defined surjective \( A \)-algebra homomorphism.

Since \( p \nmid 6D_\chi N\mathcal{P}(N) \), we have \( p > 3 \) and \( p \nmid \mathcal{P}(N) \). Write \( h \) for \( h_{\alpha,\delta,\xi}(N) \). Then \( h \) is \( \Lambda \)-free by (F) and an exact control is valid (see Propositions 4.1 and 18.2). By the diamond operators \( z \) for \( z \in (\mathbb{Z}/Np\mathbb{Z})^\times \), \( h \) is an algebra over \( \mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \). We can decompose \( h = \otimes_p h(\psi) \) so that the diamond operator \( z \) for \( z \in (\mathbb{Z}/N\mathbb{Z})^\times \) acts by \( \psi(z) \) on \( h(\psi) \), where \( \psi \) runs over all even characters of \( (\mathbb{Z}/Np\mathbb{Z})^\times \). From the exact control \( h/Th \cong h_1(T = 1) \), we thus get

\[
h(\chi)/Th(\chi) \cong h_2^\text{ord}(\Gamma_0(Np), \chi; \mathbb{Z}_p[\chi]) =: h
\]

for the character \( \chi \) of \( (\mathbb{Z}/Np\mathbb{Z})^\times \), where

\[
h_2(\Gamma_0(Np), \chi; \mathbb{Z}_p[\chi]) = h_2(\Gamma_1(Np), \chi; \mathbb{Z}) \otimes_{\mathbb{Z}[(\mathbb{Z}/Np\mathbb{Z})^\times]} \mathbb{Z}_p[\chi]
\]

and \( \mathbb{Z}_p[\chi] \) is the \( \mathbb{Z}_p \)-subalgebra of \( \mathbb{Q}_p \) generated by the values of \( \chi \). Here the tensor product is with respect to the algebra homomorphism \( \mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \to \mathbb{Z}_p[\chi] \) induced by \( \chi \). Writing \( \Sigma = \mathrm{Hom}_\mathbb{alg}(h(\chi), \mathbb{F}_p) \), for each \( \lambda \in \Sigma \), \( \Sigma := \{ m_\lambda = \ker h(\chi) / \lambda \in \Sigma \} \) is the set of all maximal ideals of \( h(\chi) \). Thus we have compatible decompositions \( h(\chi) = \bigoplus_{\lambda \in \Sigma} h(\chi)_m \) and \( h = \bigoplus_{\lambda \in \Sigma} h(\chi)_m \) (see [BCM, III.4.6]). Here the subscript \( "m" \) indicates the localizations at the maximal ideal \( m \).

Identify \( \Sigma \) with \( \mathrm{Hom}_\mathbb{alg}(h, \mathbb{F}_p) \). Write \( \Sigma^o \) for the subset of \( \Sigma = \mathrm{Hom}_\mathbb{alg}(h, \mathbb{F}_p) \) made of \( \lambda \)'s such that there exists

\[
\lambda_0 \in \mathrm{Hom}_\mathbb{alg}(h_2^\text{ord}(\Gamma_0(N), \chi; \mathbb{F}_p[\chi]), \mathbb{F}_p[\lambda]) \quad \text{with} \quad \lambda(T(l)) = \lambda_0(T(l)) \quad \text{for all primes} \quad l \nmid pN.
\]

Here we put

\[
h_2^\text{ord}(\Gamma_0(N), \chi; \mathbb{F}_p[\chi]) := h_2^\text{ord}(\Gamma_0(N), \chi; \mathbb{Z}_p[\chi]) \otimes_{\mathbb{Z}} \mathbb{F}_p.
\]

Accordingly let \( \Sigma^o \) denote the set of maximal ideals corresponding to \( \lambda \in \Sigma^o \). Since \( p \)-new forms in \( S_2(\Gamma_0(Np), \chi) \) have \( U(p) \)-eigenvalues \( \pm\sqrt{\chi(p)} \) (see [MFM, Theorem 4.6.17]), by \( a_p \neq \pm\sqrt{\chi(p)} \mod p \), we have further decomposition \( h = h_N \oplus h' \) so that \( h_N \) is the direct sum of \( h_{m_\lambda} \) for \( \lambda \) running over \( \Sigma^o \). Since \( h(\chi)/Th(\chi) \cong h \), by Hensel’s lemma (e.g., [BCM, III.4.6]), we have a unique algebra decomposition \( h(\chi) = h_N \oplus h' \) so that \( h_N/Th_N = h' \oplus h'/Th' = h'/Th' \).

Since \( T(p) \equiv U(p) \mod p \) in \( h_N \), we get \( h_N \cong h_2^\text{ord}(\Gamma_0(N), \chi; \mathbb{Z}_p[\chi]) \). Since \( p \nmid D_\chi \), the reduction map modulo \( p \): \( \mathrm{Hom}_\mathbb{alg}(h, \mathbb{Q}_p) \to \Sigma \) is a bijection. In particular, we have \( h = h_\text{new} \oplus h_\text{old} \) where \( h_\text{new} \) is the direct sum of \( h_{\lambda_0} \) for \( \lambda \) coming from the eigenvalues of \( N \)-primitive forms. Again by Hensel’s lemma, we have the algebra decomposition \( h_N = h_\text{new} \oplus h_\text{old} \) with \( h'/Th' = h'/Th' \) for
of Nakayama’s lemma, we have $\iota: \text{Im}(f) \to \mathbb{T}$ that $\text{Im}(f) \to \mathbb{T}$ for $T = \mathbb{T} \cap \mathbb{Z}[f] = W$. Since $W$ is unramified over $\mathbb{Z}_p$, by the theory of Witt vectors [BCM, IX.1], we have a unique section $W \to \mathbb{T}$ of $W[|T]] \subset \mathbb{T}$ which induces a surjection after reducing modulo $T$. Then by Nakayama’s lemma, we have $W = W[|T]] = W \otimes_{\mathbb{Z}_p} \Lambda$ as desired.

6. Limit abelian factors

We recall some elementary but useful facts (e.g. [H15, §6, after (6.6)]) with their proof. Let $\iota: C_{r/\mathbb{Q}} \subset J_{r/\mathbb{Q}}$ (resp. $\pi: J_{r/\mathbb{Q}} \to D_{r/\mathbb{Q}}$) be an abelian subvariety (resp. an abelian variety quotient) stable under Hecke operators (including $U(l)$ for $l|Np$) and $w_r$. Here the stability means that $\text{Im}(\iota)$ and $\text{Ker}(\pi)$ are stable under Hecke operators. Then $\iota$ and $\pi$ are Hecke equivariant. Let $\iota_s: C_s := \pi_{s,r}(r(C) \subset J_s$ for $s > r$ and $D_s$ is the quotient abelian variety of $\pi_s: J_s \overset{p^\infty}{\to} J_r \to D_r$, where $\pi_s = w_r \circ \pi_{s,r} \circ w_s$. The twisted projection $\pi_s$ is rational over $\mathbb{Q}$ as $w_r \circ \pi_{s,r} \circ w_s = w_s \circ (\iota)^{-1}$ for $z \in \hat{\mathbb{Z}}^\times$.

Since the two morphisms $J_r \to J_s$ and $J_s \to J_r$ are $\mathbb{Z}[p^{r-s}] - 1$ (Picard functoriality) are $U(p)$-isomorphism of fppf abelian sheaves by (u1) and Corollary 3.2, we get the following two isomorphisms of fppf abelian sheaves for $s > r > 0$:

$$C_r[p^\infty]_s \to C_s[p^\infty]_s \overset{p^r}{\to} \hat{C}_r[p^\infty]_s,$$

since $\hat{C}_r$ is the isomorphic image of $\hat{C}_r \subset \hat{J}_r$ in $\hat{J}_s[\mathbb{Z}[p^{r-s}]] - 1$]. By the $w$-twisted Cartier duality [H14, §4], we have

$$D_s[p^\infty]_s \to D_r[p^\infty]_s.$$

This isomorphism (6.2) is over $\mathbb{Q}$ not over the discrete valuation ring $\mathbb{Z}_p = \mathbb{Z}_p \cap \mathbb{Q}$ as explained in [H16, §11] (after the proof of Proposition 11.3 in [H16]), but the isomorphism (6.1) is often valid over $\mathbb{Z}_p$ (see the argument in Section 17). Thus by Kummer sequence, we have the following commutative diagram

$$\begin{array}{ccc}
\hat{D}_s[p^\infty]_s \otimes \mathbb{Z}/p^m\mathbb{Z} & \overset{\sim}{\to} & D_s[p^\infty]_s \otimes \mathbb{Z}/p^m\mathbb{Z} \\
\pi_s & \downarrow & \downarrow \iota \\
\hat{D}_r[p^\infty]_s \otimes \mathbb{Z}/p^m\mathbb{Z} & \overset{\sim}{\to} & D_r[p^\infty]_s \otimes \mathbb{Z}/p^m\mathbb{Z}.
\end{array}$$

This shows the injectivity of the following map

$$\hat{D}_s[p^\infty]_s \otimes \mathbb{Z}/p^m\mathbb{Z} \to \hat{D}_r[p^\infty]_s \otimes \mathbb{Z}/p^m\mathbb{Z}.$$

Taking the $w$-twisted dual $C_s$ of $D_s$ (which may interchange $(\alpha, \delta)$ to $(\delta, \alpha)$), from $\hat{C}_s \cong \hat{C}_r \cong \hat{C}_s \cong \hat{C}_r \cong \hat{C}_s \cong \hat{C}_r \cong \hat{C}_s \cong \hat{C}_r$, the source and the target of the above map has the same order; so, it is an isomorphism. Passing to the projective/injective limit, we get

$$\hat{D}_s[p^\infty]_s \overset{\sim}{\to} \hat{D}_r[p^\infty]_s \text{ and } (D_s \otimes_{\mathbb{Z}} \mathbb{T}_p)[p^\infty]_s \overset{\sim}{\to} (D_r \otimes_{\mathbb{Z}} \mathbb{T}_p)[p^\infty]_s$$

as fppf abelian sheaves. In short, we get

**Lemma 6.1.** Suppose that $\kappa$ is a field extension of finite type of either a number field or a finite extension of $\mathbb{Q}_p$. Then we have the following isomorphism

$$\hat{C}_r(p[p^\infty]_s \overset{\sim}{\to} \hat{C}_s(p[p^\infty]_s \text{ and } \hat{D}_s(p[p^\infty]_s \overset{\sim}{\to} \hat{D}_r(p[p^\infty]_s$$

for all $s > r$ including $s = \infty$. 

? = new, old. Since $h^{new}$ is reduced by the theory of new forms ([H86a, §3] and [MFM, §4.6]) and unramified over $\mathbb{Z}_p$ by $p \nmid D_s[\phi(N)]$, we conclude $h^{new} \cong \bigoplus W$ for discrete valuation rings $W$ finite unramified over $\mathbb{Z}_p$, and the direct summand $W$ acts on $f$ non-trivially; i.e., $W$ given by $\mathbb{Z}_p[f] = \mathbb{Z}_p[\alpha_{n}]$ for $n = 1, 2, \ldots \subset \mathbb{Z}_p$ for $T(n)$-eigenvalues $\alpha_n$ of $f$. Thus again by Hensel’s lemma, we have a unique algebraic direct factor $\mathbb{T}$ of $h^{new}$ such that $\mathbb{T}/\mathbb{T} = \mathbb{Z}_p[f] = W$. Since $W$ is unramified over $\mathbb{Z}_p$, by the theory of Witt vectors [BCM, IX.1], we have a unique section $W \to \mathbb{T}$ of $W[\mathbb{T}] \subset \mathbb{T}$ which induces a surjection after reducing modulo $T$. Then by Nakayama’s lemma, we have $\mathbb{T} = W[\mathbb{T}] = W \otimes_{\mathbb{Z}_p} \Lambda$ as desired. 

\[ \square \]
Taking $C_s$ to be $A_s$ (and hence $D_s = B_s$ by Lemma 5.1) and applying this lemma to the exact sequence (5.1), we get a new exact sequence (for $\varpi$ in (A)):

\[
0 \to \hat{A}_r^\ord \to J_r^\ord \xrightarrow{\varpi} J_r^\ord \to \hat{B}_r^\ord \to 0,
\]

since $\hat{A}_r^\ord = \lim \hat{A}_s^\ord \cong \hat{A}_r^\ord$ by the lemma.

We make $\hat{B}_r^\ord$ explicit. By computation, $\pi_s^* \circ \pi_r^* = p^s r U(p^r)$. To see this, as Hecke operators from $\Gamma_s$-coset operations, we have $\pi_s^* = [\Gamma_s]$ (restriction with respect to $\Gamma_r/\Gamma_s$) and $\pi_r, s = [\Gamma_r]$ (trace map with respect to $\Gamma_r/\Gamma_s$). Thus we have

\[
(6.4) \quad \pi_s^* \circ \pi_r^* = [\Gamma_s] \cdot w_s \cdot [\Gamma_r] \cdot w_r = [\Gamma_s] \cdot [w_s w_r] \cdot [\Gamma_r] = [\Gamma_r^r][\Gamma_r^r \left( \begin{smallmatrix} 1 & 0 \\ 0 & p^r \end{smallmatrix} \right) \Gamma_r] = p^s r U(p^r).
\]

**Lemma 6.2.** We have the following two commutative diagrams for $s' > s$

\[
\begin{array}{ccc}
\hat{C}_s & \xrightarrow{\sim} & \hat{C}_s^\ord \\
\pi_s^* \downarrow & & \downarrow p^{s'-r} U(p^{s'-s}) \\
\hat{C}_s^\ord & \xrightarrow{\pi_s^*} & \hat{C}_s^\ord
\end{array}
\]

and

\[
\begin{array}{ccc}
\hat{D}_s & \xrightarrow{\sim} & \hat{D}_s^\ord \\
\pi_s^* \downarrow & & \downarrow p^{s'-s} U(p^{s'-s}) \\
\hat{D}_s^\ord & \xrightarrow{\pi_s^*} & \hat{D}_s^\ord
\end{array}
\]

In particular, we get $\hat{D}_r^\ord := \lim \hat{D}_r^\ord = \hat{D}_r^\ord \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

**Proof.** By $\pi_r^*$ (resp. $\pi_s^*$), we identify $\hat{C}_s^\ord$ with $\hat{C}_r^\ord$ (resp. $\hat{D}_s^\ord$ with $\hat{D}_r^\ord$) as in Lemma 6.1. Then the above two diagrams follow from (6.5).

For a free $\mathbb{Z}_p$-module $F$ of finite rank, we suppose to have a commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{p^n} & F \\
\downarrow & & \downarrow p^{-n} \\
F & \xrightarrow{p^{-n}} & p^{-n} F
\end{array}
\]

Thus we have $\lim_{n \to \infty} F = F$.

Identifying $\hat{D}_s^\ord$ with $\hat{D}_r^\ord$ by $\pi_r^*$ for all $s \geq r$, the transition map of the inductive limit $\lim_s \hat{D}_s^\ord$ is given by the following commutative diagram

\[
\begin{array}{ccc}
\hat{D}_r^\ord & \to & \hat{D}_r^\ord \\
\downarrow & & \downarrow \pi_r^* \\
\hat{D}_r^\ord & \to & \hat{D}_r^\ord
\end{array}
\]

where the top arrow is induced by $\pi_r^*$. Thus applying the above result for $M = \hat{D}_s^\ord(K)$, we find

\[
\lim_s \hat{D}_s^\ord(K) = \hat{D}_r^\ord(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\]

Applying this lemma to $D_s = B_s$, we get from (6.4), the following exact sequence:
Corollary 6.3. Assume $\Lambda$-freeness of $\mathfrak{h}_{\delta, \xi}$. Let $K$ be either a number field or a finite extension of $\mathbb{Q}_l$ for a prime $l$. For $(\varpi, A_r, B_r)$ satisfying (A), we get the following natural 4-term exact sequence of étale sheaves over $\text{Spec}(K)$:

$$0 \to \hat{A}_r^{\text{ord}} \to J_\infty^{\text{ord}} \xrightarrow{\varpi} J_\infty^{\text{ord}} \xrightarrow{p^\infty} \hat{B}_r \otimes \mathbb{Z}_p \mathbb{Q}_p \to 0.$$ 

In particular, for $K' = K^S$ if $K$ is a number field and $K' = \bar{K}$ if $K$ is local, we have the following exact sequence of Galois modules: $0 \to \hat{A}_r^{\text{ord}}(K') \to J_\infty^{\text{ord}}(K') \xrightarrow{\varpi} J_\infty^{\text{ord}}(K') \to B_r^{\text{ord}}(K') \otimes \mathbb{Z}_p \mathbb{Q}_p \to 0$.

Proof. Since a finite étale extension $R$ of $K$ is a product of finite field extensions of $K$, we may assume that $R$ is a field extension of $K$. Then by (S), $\hat{B}_s(R)^{\text{ord}} \cong \hat{B}_s(R)^{\text{ord}}$ is a $\mathbb{Z}_p$-module of finite type. Then by the above lemma Lemma 6.2, taking $D_r$ to be $B_s$, we find that $\lim_s \hat{B}_s(R)^{\text{ord}} = B_r^{\text{ord}} \otimes \mathbb{Z}_p \mathbb{Q}_p$.

Since passing to injective limit is an exact functor, this proves the first exact sequence:

$$0 \to \hat{A}_r^{\text{ord}} \to J_\infty^{\text{ord}} \xrightarrow{\varpi} J_\infty^{\text{ord}} \to \hat{B}_r \otimes \mathbb{Z}_p \mathbb{Q}_p \to 0.$$ 

Since $\hat{A}_r^{\text{ord}}(K') = \lim_{K'/F} \hat{A}_r^{\text{ord}}(F)$ for $F$ running a finite extension of $K$, we get the exactness of

$$0 \to \hat{A}_r^{\text{ord}}(K') \to J_\infty^{\text{ord}}(K') \xrightarrow{\varpi} J_\infty^{\text{ord}}(K').$$

Since $0 \to \hat{A}_r^{\text{ord}}(K') \to J_\infty^{\text{ord}}(K') \xrightarrow{\varpi} J_\infty^{\text{ord}}(K') \to B_r^{\text{ord}}(K') \to 0$ is an exact sequence of Galois modules, passing to the limit, we still have the exactness of

$$0 \to \hat{A}_r^{\text{ord}}(K') \to J_\infty^{\text{ord}}(K') \xrightarrow{\varpi} J_\infty^{\text{ord}}(K') \to \\ \hat{B}_s^{\text{ord}}(K') \otimes \mathbb{Z}_p \mathbb{Q}_p \to 0.$$ 

Note here that $\hat{B}_s^{\text{ord}}(K')$ is $p$-divisible, and hence

$$\hat{B}_s^{\text{ord}}(K') \otimes \mathbb{Z}_p \mathbb{Q}_p = \hat{B}_s^{\text{ord}}(K')/\hat{B}_s^{\text{ord}}[p^{\infty}](K') \cong \hat{A}_s^{\text{ord}}(K')/\hat{A}_s^{\text{ord}}[p^{\infty}](K'),$$

and we have the sheaf identity: $\hat{B}_s^{\text{ord}} \otimes \mathbb{Z}_p \mathbb{Q}_p = \hat{B}_s^{\text{ord}}(K')/\hat{B}_s^{\text{ord}}[p^{\infty}] \cong \hat{A}_s^{\text{ord}}(K')/\hat{A}_s^{\text{ord}}[p^{\infty}].$ 

Corollary 6.4. We have a sheaf isomorphism $i : (\hat{A}_r^{\text{ord}} \oplus \varpi(J_\infty^{\text{ord}}))/\hat{A}_r^{\text{ord}}[p^{\infty}] \cong J_\infty^{\text{ord}}$ with $\hat{A}_r^{\text{ord}} \cong \hat{A}_r^{\text{ord}}$, where $i(a \oplus x) = a + x$ for $a \in \hat{A}_r^{\text{ord}}$ and $x \in \varpi(J_\infty^{\text{ord}})$ and $a \in \hat{A}_r^{\text{ord}}[p^{\infty}]$ is sent to $(a \oplus -a) \in \hat{A}_r^{\text{ord}} \oplus \varpi(J_\infty^{\text{ord}})$. In particular, identifying $\hat{A}_r^{\text{ord}} \cong \hat{A}_r^{\text{ord}}$, the sequence $0 \to \hat{A}_r^{\text{ord}}[p^{\infty}] \to J_\infty^{\text{ord}} \xrightarrow{\varpi} J_\infty^{\text{ord}} \to 0$ is exact as fppf abelian sheaves.

Proof. Consider the composite $f_s$ of the inclusion $\hat{A}_s^{\text{ord}} \to \hat{A}_r^{\text{ord}}$ and the projection: $\hat{J}_s^{\text{ord}} \to \hat{B}_s^{\text{ord}}$. Since $\hat{A}_s^{\text{ord}} \cap \varpi(J_s^{\text{ord}}) \cong \text{Ker}(\hat{A}_s^{\text{ord}} \to \hat{B}_s^{\text{ord}})$, we may think $\text{Ker}(f_s)$ also as a group subscheme of $\varpi(J_s^{\text{ord}})$. Identifying $\hat{B}_s^{\text{ord}}$ with $\hat{B}_s^{\text{ord}}$ by $\pi_s^*$ for all $s \geq r$, the transition map of the inductive limit $\lim_s \hat{B}_s^{\text{ord}}$ satisfies the following commutative diagram:

$$\begin{array}{ccc}
\hat{B}_r^{\text{ord}} & \xrightarrow{\pi_{r,s}} & \hat{B}_s^{\text{ord}} \\
\downarrow & & \downarrow \\
\hat{A}_s^{\text{ord}} & \xrightarrow{\pi_{r,s}} & \hat{A}_r^{\text{ord}} \\
\end{array}$$

Thus composing with the inclusion $\hat{A}_r^{\text{ord}} \xrightarrow{\pi_{r,s}} \hat{A}_s^{\text{ord}} \subset \hat{J}_s^{\text{ord}}$, we conclude $f_s = p^{s-r}U(p)^{s-r} \circ f_r$ and hence $\text{Ker}(f_s) \supset f_r^{-1}(\hat{B}_s^{\text{ord}}[p^{s-r}]) \supset \hat{A}_r^{\text{ord}}[p^{s-r}]$. Since $\text{Ker}(f_s)/\hat{A}_r^{\text{ord}}[p^{s-r}] \cong \text{Ker}(f_r)$ which is bounded independent of $s$, passing to the limit, we find

$$\hat{A}_r^{\text{ord}}[p^{\infty}] \cong \hat{A}_r^{\text{ord}}[p^{\infty}] = \lim_s \text{Ker}(f_s) = \hat{A}_s^{\text{ord}} \cap \varpi(J_s^{\text{ord}})$$

inside $J_\infty^{\text{ord}}$. Since we have the sheaf identity $\hat{A}_r^{\text{ord}}/\hat{A}_r^{\text{ord}}[p^{\infty}] = \hat{A}_s^{\text{ord}} \otimes \mathbb{Z}_p \mathbb{Q}_p \cong \hat{B}_s^{\text{ord}} \otimes \mathbb{Z}_p \mathbb{Q}_p$ and $J_\infty^{\text{ord}}/\varpi(J_\infty^{\text{ord}}) = \hat{B}_s^{\text{ord}} \otimes \mathbb{Z}_p \mathbb{Q}_p$, the morphism of sheaves $\hat{A}_s^{\text{ord}} \oplus \varpi(J_\infty^{\text{ord}}) \supset (a \oplus x) \mapsto a + x \in J_\infty^{\text{ord}}$ is an epimorphism of sheaves.

By the expression in the corollary, we have $\text{Ker}(\varpi(J_\infty^{\text{ord}})) \cong \hat{A}_s^{\text{ord}}[p^{\infty}] \cong \hat{A}_r^{\text{ord}}[p^{\infty}]$. Thus $0 \to \hat{A}_r^{\text{ord}}[p^{\infty}] \to \varpi(J_\infty^{\text{ord}}) \xrightarrow{\varpi} \varpi(J_\infty^{\text{ord}})$ is sheaf exact. For finite $s$, $\varpi(J_s^{\text{ord}}) \xrightarrow{\varpi} \varpi(J_s^{\text{ord}})$ is an
isogeny and hence a sheaf epimorphism. Passing to the limit, we conclude \( \varpi(J_{\infty}^{ord}) \xrightarrow{\varpi} \varpi(J_{\infty}^{ord}) \) is a sheaf epimorphism, concluding the proof.

Let \( G_{\ast} = J_{\ast}[p_{\infty}]^{ord} \). Then we have an exact sequence \( A_{\ast}[p_{\infty}]^{ord} \xrightarrow{\varpi} G_{\ast} \xrightarrow{\varpi} B_{\ast}[p_{\infty}]^{ord} \) of fppf sheaves. In this case, \( \lim_{\varphi \rightarrow \varphi'} B_{\ast}[p_{\infty}]^{ord} = 0 \) as \( B_{\ast}[p_{\infty}]^{ord} \) is \( p \)-torsion. Passing to the limit, we recover the following fact proven in [H14, §3, (DV) and §5]:

**Corollary 6.5.** We have an exact sequence of fppf sheaves over \( \mathbb{Q} \):

\[
0 \rightarrow A_{\ast}[p_{\infty}]^{ord} \rightarrow G_{\infty} \xrightarrow{\varpi} G_{\infty} \rightarrow 0
\]

which extends canonically to an exact sequence of fppf sheaves over \( \mathbb{Z}[p_{\infty}] \).

### 7. Generality of Galois cohomology

We prove some general result on Galois cohomology for our later use. Let \( S \) be a set of places of a number field \( K \). Suppose that \( S \) contains all archimedean places and \( p \)-adic places of \( K \) (and primes for bad reduction of the abelian varieties when we deal with abelian varieties). Let \( K^{S} \) be the maximal extension unramified outside \( S \).

**Lemma 7.1.** Let \( \{M_n\}_n \) be a projective system of finite \( \mathbb{Z}[	ext{Gal}(K^{S}/K)] \)-modules \( M_n \). Write \( M_{\infty} := \lim_n M_n \) and \( M^{\vee}_{\infty} := \lim_n M^{\vee}_n \) for the Pontryagin dual \( M^{\vee}_n \) of \( M_n \). Write \( G \) (resp. \( G_{\ast} \) for a place \( v \) of \( K \)) for the (point by point) stabilizer of \( M^{\vee}_n \) in \( \text{Gal}(K^{S}/K) \) (resp. \( \text{Gal}(K_{\ast}/K_v) \) and \( G = \text{Gal}(K^{S}/K)/G \) (resp. \( G_{\ast} := \text{Gal}(K_{\ast}/K_v)/G_v) \)). Then, we have

1. \( \Pi^1(K^{S}/K, M_{\infty}) = \lim_n \Pi^1(K^{S}/K, M_n) \), and if \( S \) is a finite set, we have \( \Pi^1(K^{S}/K, M^{\vee}_{\infty}) = \lim_n \Pi^1(K^{S}/K, M^{\vee}_n) \).

2. \( \Pi^2(K^{S}/K, M^{\vee}_{\infty}) = \lim_n \Pi^2(K^{S}/K, M^{\vee}_n) \), and if \( S \) is a finite set, we have \( H^2(K^{S}/K, M_{\infty}) = \lim_n H^2(K^{S}/K, M_n) \) and \( \Pi^2(K^{S}/K, M^{\vee}_{\infty}) = \lim_n \Pi^2(K^{S}/K, M^{\vee}_n) \).

**Proof.** We first prove the assertion in (1) for projective limit. Since \( H^0(?, M_n) \) (resp. \( K^{S}/K \) and \( K_v \)) is finite for all \( n \), we have \( \lim_n H^1(?, M_n) = H^1(?, M_{\infty}) \) for \( ? = K^{S}/K \) and \( K_v \) by [CNF, Corollary 2.7.6]. By definition, we have an exact sequence:

\[
0 \rightarrow \Pi^1(K^{S}/K, M_n) \rightarrow H^1(K^{S}/K, M_n) \rightarrow \prod_{v \in S} H^1(K_v, M_n).
\]

Since any continuous cochain with values in \( \lim_n M_n \) is a projective limit of continuous cochains with values in \( M_n \), we have a natural map \( H^1(?, \lim_n M_n) \rightarrow \lim_n H^1(?, M_n) \) for \( ? = K^{S}/K \) and \( K_v \). Passing to the limit, we get the following commutative diagram with exact rows

\[
\begin{array}{ccc}
\Pi^1(K^{S}/K, \lim_n M_n) & \xrightarrow{\iota} & H^1(K^{S}/K, \lim_n M_n) \\
\downarrow & & \downarrow \\
\lim_n \Pi^1(K^{S}/K, M_n) & \xrightarrow{\iota} & \lim_n H^1(K^{S}/K, M_n)
\end{array}
\]

This shows

\[
\Pi^1(K^{S}/K, \lim_n M_n) = \lim_n \Pi^1(K^{S}/K, M_n)
\]

as desired. As for the injective limit, we first note that the cohomology functor commutes with the limit. However it may not commute with infinite product; so, we need to assume that \( S \) is finite (this fact is pointed out to the author by D. Harari).

As for (2), since cohomology functor commutes with injective limit, the assertion (2) for injective limits follows from the same argument as in the case of (1), noting that by local Tate duality, the direct product \( \prod_{v \in S} H^2(K_v, M_n) \) in the definition of \( \Pi^2 \) can be replaced by the direct sum; so, we have the assertion for the injective limit. If \( S \) is finite, \( H^2(K^{S}/K, M_{\infty}) \) is finite (e.g., [ADT, I.5.1]). Thus by [CNF, Corollary 2.7.6], we have \( \lim_n H^2(?, M_n) = H^2(?, \lim_n M_n) \) for \( ? = K^{S}/K \) and \( K_v \), and hence once again the same argument works (replacing \( H^1 \) by \( H^2 \)).
Let $A$ be an abelian variety over a field $K$. Since the Galois group $\text{Gal}(\overline{K}/K)$ and $\text{Gal}(K^S/K)$ is profinite and $A(\overline{K})$ and $A(K^S)$ are discrete modules, for $q > 0$, the continuous cohomology group $H^q(K^S/K, A)$ for a number field $K$ and $H^q(K, A)$ for a local field $K$ are torsion discrete modules (see [MFG, Corollary 4.26]).

**Lemma 7.2.** If $K$ is either a number field or a local field of characteristic 0, we have a canonical isomorphism for $0 < q \in \mathbb{Z}$:

\[(7.1) \quad H^q(\hat{A}) \cong H^q(A)[p^\infty],\]

where $H^q(\cdot) = H^q(K^S/K, \cdot)$ if $K$ is a number field, and $H^q(\cdot) = H^q(K, \cdot)$ if $K$ is local.

**Proof.** By (S), if $K$ is a number field, and we have

\[H^q(K^S/K, \hat{A}) \cong H^q(K^S/K, A) \otimes_{\mathbb{Z}} \mathbb{Z}_p = H^q(K^S/K, A)_p,\]

as $H^q(K^S/K, A)$ is a torsion module. Here the identity (*) follows from the universal coefficient theorem (e.g., [CNF, 2.3.4] or [CGP, (0.8)]).

Now suppose that $K$ is an $l$-adic with $l \neq p$ or archimedean local field. Then $\hat{A} = A[p^\infty]$, and we have a natural inclusion $0 \to \hat{A}(\overline{K}) \to A(\overline{K}) \to Q \to 0$ for the quotient Galois module $Q$. Thus $Q$ is $p$-torsion-free and $p$-divisible; i.e., the multiplication by $p$ is invertible on $Q$. Therefore $H^q(K, Q)_p := H^q(K, Q) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a $\mathbb{Q}_p$-vector space for $q \geq 0$ (so, $H^q(K, Q)_p = 0$ though we do not need this vanishing). By the exact sequence $H^{q-1}(K, Q)_p \to H^q(K, A)_p \to H^q(K, A)_p \to H^q(K, Q)_p$, we conclude $H^q(K, \hat{A}) = H^q(K, A)_p$ as the two modules are $p$-torsion.

If $l = p$, we have $A(\overline{K}) = \hat{A}(\overline{K}) = A[p^\infty]$ under the notation of (S), and hence $H^q(K, A)_p = H^q(\hat{A})_p \cong H^q(K, A(p))_p$. Since $\hat{A}(\overline{K})$ is a union of $p$-profinite group, we have $H^q(\hat{A})_p = H^q(\hat{A})$. Since $A[p^\infty]$ is prime-to-$p$ torsion, we have $H^q(K, A(p))_p = 0$. Thus we get

\[H^q(K, A)_p = H^q(K, A) \otimes_{\mathbb{Z}} \mathbb{Z}_p = H^q(K, \hat{A}) \cong H^q(K, A)_p \otimes_{\mathbb{Z}} \mathbb{Z}_p = H^q(K, \hat{A})\]

as desired. \hfill \Box

**8. Diagrams of Selmer groups and Tate–Shafarevich groups**

We describe a commutative diagram involving different cohomology groups and Tate–Shafarevich groups, which lays a base of the proof of the control result in later sections. We assume $p > 2$ for simplicity.

Recall the definition of the $p$-part of the Selmer group and the Tate–Shafarevich group for an abelian variety $A$ defined over a number field $K$:

\[\text{Sel}_K(A)_p = \text{Ker}(H^1(K^S/K, A)_p) \xrightarrow{\text{Res}} \prod_{v \in S} H^1(K_v, A)_p,\]

\[\text{Sel}_K(A)_p = \text{Ker}(H^1(K^S/K, A[p^\infty])) \xrightarrow{\text{Res}} \prod_{v \in S} H^1(K_v, A)_p.\]

As long as $S$ is sufficiently large containing all bad places for $A$ in addition to all archimedean and $p$-adic places, these groups are independent of $S$ (see [ADT, I.6.6]) and are $p$-torsion modules.

**Lemma 8.1.** We can replace $A$ in the above definition by $\hat{A}$, and we get

\[(8.1) \quad \text{Sel}_K(A)_p = \text{Sel}_K(\hat{A}) = \text{Ker}(H^1(K^S/K, \hat{A}) \xrightarrow{\text{Res}} \bigoplus_{v \in S} H^1(K_v, \hat{A})),\]

\[\text{Sel}_K(\hat{A}) = \text{Ker}(H^1(K^S/K, A[p^\infty]) \xrightarrow{\text{Res}} \bigoplus_{v \in S} H^1(K_v, \hat{A})).\]

**Proof.** It is known that image of global cohomology classes lands in the direct sum $\bigoplus_{v \in S} H^1(K_v, \hat{A})$ in the product $\prod_{v \in S} H^1(K_v, \hat{A})$ (see [ADT, I.6.3]).

By Lemma 7.2, we have $\text{III}_K(\hat{A}) = \text{III}_K(A)_p = \text{III}_K(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Thus we may replace the $p$-primary part of the traditional $\text{III}$-functor $A \mapsto \text{III}_K(A)_p$ by the completed one $A \mapsto \text{III}_K(\hat{A})$. \hfill \Box
Since $A \to \mathbb{III}_K(\hat{A})$ is a covariant functor from abelian varieties defined over a number field $K$ to an abelian groups, from Lemma 3.3 (and Remark 3.4), we get the commutative diagram for $X = \mathbb{III}$ and Sel:

\[
\begin{array}{ccc}
X_K(\tilde{J}_r) & \xrightarrow{\pi'} & X_K(\tilde{J}_s) \\
\downarrow u & \searrow u' & \downarrow u'' \\
X_K(\hat{J}_r) & \xrightarrow{\pi''} & X_K(\hat{J}_s),
\end{array}
\]

(8.2)

Similarly to the diagram as above, from Corollary 3.2, we get the following commutative diagram:

\[
\begin{array}{ccc}
X_K(\tilde{J}_r) & \xrightarrow{\pi'} & X_K(\tilde{J}_s[\gamma^{p^{r-s}} - 1]) \\
\downarrow u & \searrow u' & \downarrow u'' \\
X_K(\hat{J}_r) & \xrightarrow{\pi''} & X_K(\hat{J}_s[\gamma^{p^{r-s}} - 1]).
\end{array}
\]

(8.3)

These diagrams provide us the following canonical isomorphisms

\[
X_K(\tilde{J}_r)_{\text{ord}} \cong X_K(\tilde{J}_s[\gamma^{p^{r-s}} - 1])_{\text{ord}} \quad \text{for} \quad X = \mathbb{III} \text{ and Sel.}
\]

(8.4)

For any group subvariety $A/\mathbb{Q}$ of $J_s$ proper over $\mathbb{Q}$ stable under $U(p)$ or any abelian variety quotient $A/\mathbb{Q}$ of $J_s$ stable under $U(p)$, we have $\hat{A} = \hat{A}_{\text{ord}} \oplus (1 - e)\hat{A}$, and hence $H^q(?, \hat{A}) = H^q(?, \hat{A}_{\text{ord}}) \oplus H^q(?, (1 - e)\hat{A})$ for $? = \mathbb{K}$ and $K^S$. This shows $H^q(?, \hat{A}_{\text{ord}}) = H^q(?, \hat{A}_{\text{ord}})$, and hence $X_K(\hat{A}_{\text{ord}})_{\text{ord}} = X_K(\hat{A})_{\text{ord}}$ for $X = \mathbb{III}$ and Sel. Thus hereafter, we attach the superscript “ord” inside the cohomology/Tate–Shafarevich group if the coefficient is $p$-adically completed in the sense of (S).

We define the ind $\Lambda$-TS group and the ind $\Lambda$-Selmer group by

\[
\mathbb{III}_K(J_{\infty})_{\text{ord}} := \mathbb{III}_K(J_{\infty}^\text{ord}) = \lim_{\rightarrow \mathbb{III}_K(J_{\infty}^\text{ord})} = \lim_{\rightarrow \mathbb{III}_K(J_{\infty})_{\text{ord}}},
\]

(8.5)

\[
\text{Sel}_K(J_{\infty})_{\text{ord}} := \text{Sel}_K(J_{\infty}^\text{ord}) = \lim_{\rightarrow \text{Sel}_K(J_{\infty}^\text{ord})} = \lim_{\rightarrow \text{Sel}_K(J_{\infty})_{\text{ord}}}
\]

which are naturally $h$-modules.

Write $H^1_0(M) = \bigoplus_{v \in S} H^1(K_v, M)$ and $H^q(M) = H^q(K^S/K, M)$ for a Gal($K^S/K$)-module $M$. By [ADT, I.6.6], $\mathbb{III}(K^S/K, A)_p = \mathbb{III}_K(A)_p$ for an abelian variety $A/K$ as long as $S$ contains all bad places of $A$ and all archimedean and $p$-adic places. Consider a triple $(\varpi, A_s = J_s[a_s], B_s = J_s/a_sJ_s)$ satisfying the condition (A) of Section 5 and (F) in Section 4. Note that $J^\text{ord}_s[\varpi_s] = J^\text{ord}_s(A^\text{ord}_s)$ (see Lemma 5.1), we have $H^q(?, J^\text{ord}_s[\varpi_s]) = H^q(?, J^\text{ord}_s)$. This implies $\mathbb{III}_S(J^\text{ord}_s[\varpi_s]) = \mathbb{III}_S(J^\text{ord}_s) \cong \mathbb{III}_K(\hat{A}_{\text{ord}})$, where the last identity follows from [ADT, I.6.6]. Recall the following exact sequence from Corollary 6.3:

\[
0 \to \hat{A}_{\text{ord}}(K') \to J^\text{ord}_s(K') \xrightarrow{\varpi} J^\text{ord}_s(K') \to \hat{B}_{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0,
\]

(8.6)

where $J^\text{ord}_s = \lim_{\leftarrow J^\text{ord}_s}$ and $K' = K^S$ and $\mathbb{K}_v$. We separate it into two short exact sequences:

\[
0 \to \hat{A}_{\text{ord}}(K') \to J^\text{ord}_s(K') \xrightarrow{\varpi} J^\text{ord}_s(K') \to \hat{B}_{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.
\]

(8.7)

Define

\[
\text{Sel}_K(\varpi(J^\text{ord}_s)) := \text{Ker}(i : H^1(K^S/K, \varpi(J^\text{ord}_s)[p^\infty]) \to H^1_0(\varpi(J^\text{ord}_s))),
\]

(8.8)

\[
\mathbb{III}_K(\varpi(J^\text{ord}_s)) := \text{Ker}(i : H^1(K^S/K, \varpi(J^\text{ord}_s)) \to H^1_0(\varpi(J^\text{ord}_s))),
\]

\[
E_{MW}(K_v) := \text{Coker}(J^\text{ord}_s(K_v) \to \hat{B}_{\text{ord}}(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).
\]

Here we have written $H^1_0(X) := \prod_{v \in S} H^1(K_v, X)$. 

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Look into the following commutative diagram of sheaves with exact rows:

\[
\begin{array}{ccccccccc}
A_r[p^\infty]_{\text{ord}} & \longrightarrow & J_\infty^{\text{ord}}[p^\infty] & \xrightarrow{\varpi[p^\infty]} & J_\infty^{\text{ord}}[p^\infty] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A_r^{\text{ord}} & \longrightarrow & J_\infty^{\text{ord}} & \xrightarrow{i} & J_\infty^{\text{ord}} & \longrightarrow & \hat{B}_r^{\text{ord}} \otimes \mathbb{Q}_p.
\end{array}
\]

(8.9)

Since \(\hat{B}_r^{\text{ord}} \otimes \mathbb{Q}_p\) is a sheaf of \(\mathbb{Q}_p\)-vector spaces and \(J_\infty^{\text{ord}}[p^\infty]\) is \(p\)-torsion, the inclusion map \(i\) factors through the image \(\text{Im}(\varpi) = \varpi(J_\infty^{\text{ord}})\); so, for a finite extension \(K\) of \(\mathbb{Q}\) or \(\mathbb{Q}_l\),

\[
\varpi(J_\infty^{\text{ord}})[p^\infty] = J_\infty^{\text{ord}}[p^\infty].
\]

(8.10)

From the exact sequence, \(\varpi(J_\infty^{\text{ord}}) \hookrightarrow J_\infty^{\text{ord}} \rightarrow \hat{B}_r^{\text{ord}} \otimes \mathbb{Q}_p\), taking its cohomology sequence, we get the bottom sequence of the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\varpi(J_\infty^{\text{ord}})[p^\infty]) & \longrightarrow & H^1(J_\infty^{\text{ord}}[p^\infty]) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{v \in S} E_{MW}(K_v) & \longrightarrow & H^1_S(\varpi(J_\infty^{\text{ord}})) & \longrightarrow & H^1_S(J_\infty^{\text{ord}}).
\end{array}
\]

By the snake lemma, we get an exact sequence

\[
0 \rightarrow \text{Sel}_K(\varpi(J_\infty^{\text{ord}})) \rightarrow \text{Sel}_K(J_\infty^{\text{ord}}) \rightarrow \prod_{v \mid p} E_{MW}(K_v),
\]

(8.11)

since \(\hat{B}_r^{\text{ord}}(K_v) \otimes \mathbb{Q}_p = E_{MW}(K_v) = 0\) if \(v \nmid p\) by (S) in the introduction.

Define error terms

\[
E^*_S(\varpi)(F) := \frac{\varpi(J_\infty^{\text{ord}})[p^\infty](F)}{\varpi(J_\infty^{\text{ord}})[p^\infty](F)} , \quad E_S(\varpi)(F) := \frac{J_\infty^{\text{ord}}[p^\infty](F)}{J_\infty^{\text{ord}}[p^\infty](F)} = \lim_s \frac{\varpi(J_\infty^{\text{ord}})[p^\infty](F)}{\varpi(J_\infty^{\text{ord}})[p^\infty](F)}.
\]

(8.12)

\[
E^*_S(F) := \frac{\varpi(J_\infty^{\text{ord}})(F)}{\varpi(J_\infty^{\text{ord}})(F)} , \quad E_S(F) := \frac{J_\infty^{\text{ord}}(F)}{J_\infty^{\text{ord}}(F)} = \lim_s \frac{\varpi(J_\infty^{\text{ord}})(F)}{\varpi(J_\infty^{\text{ord}})(F)}
\]

for \(F = K, K_v\), and put \(E^*_S(K) = \prod_{v \in S} E^*_S(K_v)\). If \(F\) is a finite extension of \(\mathbb{Q}_l\) for \(l \neq p\), by (S) in the introduction, we have \(E_S(\varpi)(F) = E^*_S(\varpi)(F)\). If \(A_r = A_p\) for an arithmetic point \(P\), we often write \(E^*_S(F)\) for \(E^*_S(F)\) as it depends on \(P\). Noting \(\mathcal{G} \xrightarrow{\varpi} \mathcal{G}\) is an isomorphism of sheaves for \(\mathcal{G} = J_\infty^{\text{ord}}[p^\infty]\) by Corollary 6.5, we then get the following commutative diagram with two bottom exact rows and columns:

\[
\begin{array}{ccccccccc}
\text{Ker}(t_{\text{Sel},*}) & \longrightarrow & \text{Sel}_K(\hat{A}_r^{\text{ord}}) & \xrightarrow{\varpi_{\text{Sel},*}} & \text{Sel}_K(J_\infty^{\text{ord}}) \\
\uparrow & & \uparrow & & \uparrow \\
E_{\text{Sel}}(K) & \longrightarrow & H^1(\hat{A}_r^{\text{ord}}[p^\infty]) & \xrightarrow{i_*} & H^1(J_\infty^{\text{ord}}[p^\infty]) \\
\downarrow & & \downarrow & & \downarrow \\
E^*_S(K) & \longrightarrow & H^1_S(\hat{A}_r^{\text{ord}}) & \xrightarrow{\varpi_{*,*}} & H^*_S(J_\infty^{\text{ord}}).
\end{array}
\]

(8.13)

Here the last map \(\varpi_{*,*}\) could have 2-torsion finite cokernel if \(p = 2\).

We look into \(\Lambda\)-TS groups. Let \(\varpi \in \mathfrak{h}\) coming from \(\varpi_r \in \text{End}(J_{r/\mathbb{Q}})\) and suppose that \((\varpi) = \varpi \mathfrak{h} \supset (\gamma^{p-\varepsilon} - 1)\). The long exact sequence obtained from (8.7) produces the following commutative
diagram with exact columns and bottom two exact rows:

\[
\begin{array}{cccc}
\text{Ker} & \rightarrow & \mathbb{III}_K(\hat{A}_r^\text{ord}) & \rightarrow \\
\cap & \downarrow & \cap & \downarrow \\
E^\infty(K) & \rightarrow & H^1(\hat{A}_r^\text{ord}) & \rightarrow \\
\cap & \downarrow & \cap & \downarrow \\
E^\infty_S(K) & \rightarrow & H^1_S(\hat{A}_r^\text{ord}) & \rightarrow \\
\cap & \downarrow & \cap & \downarrow \\
(8.14) & & & \\
\end{array}
\]

(8.14)

By the vanishing of \(H^2_S(\hat{A}_r)\) ([ADT, Theorem I.3.2] and Lemma 7.2), \(\varpi_{S,*}\) are surjective. In each term of the diagram (8.14), we can bring the superscript “ord” inside the functor \(\mathbb{III}\) and \(H^1\) to outside the functor as the ordinary projector acts on \(\hat{J}_s\), \(J_\infty\) and \(\hat{A}_r\) and gives direct factor of the sheaf. The diagram “ord” inside is the one obtained directly from the short exact sequence of Corollary 6.3. Thus we get from [BCM, Proposition I.1.4.2] the following fact:

**Lemma 8.2.** Suppose (A) and (F). If \(E^\infty_S(K)\) is finite, the sequence

\[ \mathbb{III}_K(\hat{A}_r^\text{ord}) \rightarrow \mathbb{III}_K(J_\infty^\text{ord}) \rightarrow \mathbb{III}_K(\varpi(J_\infty^\text{ord})) \]

is exact up to finite error.

### 9. VANISHING OF THE ERROR TERM FOR \(l\)-ADIC FIELDS WITH \(l \neq p\).

In this section, we prove vanishing of the error term \(E^\infty(K)\) for local fields of residual characteristic \(l \neq p\), which combined with a similar (but more difficult) result for \(p\)-adic fields given in Section 17 will be used in the following sections to prove the control result up to finite error of the limit Selmer group, the finite Mordell–Weil group and the limit Tate–Shafarevich group.

More generally, for the moment, we denote by \(K\) either a number field or an \(l\)-adic field (the prime \(l\) can be \(p\) unless we mention that \(l \neq p\)).

**Lemma 9.1.** Let \(K\) either a number field or an \(l\)-adic field. Then the Pontryagin dual \(E^\infty(K)^\vee\) of \(E^\infty(K)\) is a \(\mathbb{Z}_p\)-module of finite type (i.e., \(E^\infty(K)\) is \(p\)-torsion of finite corank).

**Proof.** Let \(K' = \overline{K}\) if \(K\) is local and \(K = K^S\) if \(K\) is global. We have an exact sequence

\[ 0 \rightarrow E^\infty(K) \rightarrow H^1(K'/K, \hat{A}_r^\text{ord}) \rightarrow H^1(K'/K, \hat{J}_s^\text{ord}). \]

By [ADT, I.3.4], if \(K\) is local, \(H^1(K'/K, \hat{A}_r^\text{ord}) \cong \text{Pic}^0_{A/K}(K)^\vee\); so, we get the desired result. If \(K\) is global, \(\hat{A}_r^\text{ord}(K) \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H^1(K'/K, \hat{A}_r^\text{ord}[p]) \rightarrow H^1(K'/K, \hat{A}_r^\text{ord}[p])\) is exact, and the middle term is finite by Tate’s computation of the global cohomology (taking \(S\) to be finite); so, \(H^1(K'/K, \hat{A}_r^\text{ord})\) has Pontryagin dual finite type over \(\mathbb{Z}_p\). This finishes the proof. \(\square\)

**Proposition 9.2.** Assume (A) and (F). Let \(K\) be a finite extension of \(\mathbb{Q}_l\). Then we have

1. \(E^s(K) \cong \text{Ker}(\hat{A}_r^\text{ord}(K)^\vee \rightarrow J_s^\text{co-ord}(K)^\vee)\) which is under \(l \neq p\) in turn isomorphic to

\[ \text{Ker}(H_0(K, T_{p,\hat{A}_r^\text{ord}}(-1)) \rightarrow H_0(K, T_{p,\hat{J}_s^\text{ord}}(-1))) \]

for the negative Tate twist indicated by “\((-1)\)”.

2. If \(l \neq p\), the order \(|E^s(K)|\) is finite and is bounded for all \(s\).

3. If \(l \neq p\) and \(H_0(K, T_{p,\hat{A}_r^\text{ord}}(-1)) = 0\) (\(\Leftrightarrow A_r^\text{co-ord}(K) = 0\)), we have \(E^s(K) = 0\) for all \(s\).

4. Suppose \(l \neq p\). If \(\mathbb{T}\) is an integral domain and \(A_r\) has good reduction over the \(l\)-adic integer ring \(W\) of \(K\), then \(E^s(K) = 0\) for all \(s\).

We will prove the finiteness and boundedness of \(E^s(K)\) when \(l = p\) later in Section 17 under some extra assumptions (see Theorem 17.2).
Proof. We have an exact sequence

$$0 \to E^s(K) := \text{Coker}(\varpi_s : \hat{J}^\text{ord}_s(K) \to \varpi(\hat{J}^\text{ord}_s)(K)) \to H^1(K, \hat{A}^s_{\text{ord}}) \to H^1(K, \hat{J}^\text{ord}_s).$$

By [ADT, I.3.4], if $K$ is local, for an abelian variety $A$ over $K$, we have $H^1(K, A) = A'(K)^\dual$ for $A' = \text{Pic}^0_{A/K}$. Note that $\hat{A} \to A[p]$ if $l \neq p$ by (S) at the end of the introduction. From local Tate duality and Lemma 7.2 (combined with Weil pairing $A'[p\infty]$ and $T_pA(-1)$), we conclude

$$E^s(K) = \text{Ker}(\hat{A}^s_{\text{co-ord}}(K) \to \hat{J}^s_{\text{co-ord}}(K)^\dual) = \text{Ker}(H_0(K, T_p\hat{A}^s_{\text{ord}}(-1)) \to H_0(K, T_p\hat{J}^s_{\text{ord}}(-1)))$$

proving the first assertion.

We now claim that $\hat{A}^s_{\text{co-ord}}(K) \subset A_s[p\infty](K)$ is finite. If $A_s$ has good reduction over $W$, we have $A_s[p\infty](K) \cong A_s[p\infty](\mathbb{F})$ which is finite. Take the Néron model $A_{\text{\acute{e}t}, W}$ of $A_s$. By the universal property of the Néron model, we find $A_s(W) = A_{\text{\acute{e}t}}(K)$; so, $A_{\text{\acute{e}t}}[p\infty](W) = A_{\text{\acute{e}t}}[p\infty](K).$ Since multiplication by $p$ on $A_{\text{\acute{e}t}}/W$ is étale (as $l \neq p$; see [NMD, Lemma 7.3.2]), we find $A_{\text{\acute{e}t}}[p\infty](W) \cong A_{\text{\acute{e}t}}[p\infty](\mathbb{F})$, which is finite (as $A_{\text{\acute{e}t}}/W \otimes \mathbb{F}$ is of finite type over $\mathbb{F}$). This shows the claim. The claim proves the assertion (2) as $E^s(K) \to A_s[p\infty](K)$ by (1). The assertion (3) also follows from (1).

We now prove (4). If $A_s$ has good reduction over $W$, by Lemma 9.3 following this proof, $\varpi(\hat{J}^s_{\text{\acute{e}t}})$ and $\hat{J}^s_{\text{\acute{e}t}}$ have good reduction over the l-adic integer ring $W$ of $K$, we have $\varpi(\hat{J}^s_{\text{\acute{e}t}})(K) = \varpi(\hat{J}^s_{\text{\acute{e}t}})[p\infty](K) \cong \varpi(\hat{J}^s_{\text{\acute{e}t}})[p\infty](\mathbb{F})$ and similarly $\hat{J}^s_{\text{\acute{e}t}}[p\infty](K) \cong \hat{J}^s_{\text{\acute{e}t}}[p\infty](\mathbb{F})$ for the residue field $\mathbb{F}$ of $W$. From the sheaf exact sequence $A_s \to J_s \to \varpi(J_s)$, the corresponding sequence of their Néron models is exact by [NMD, Proposition 7.5.3 (a)]. Over finite field, by the vanishing of $H^1(\mathbb{F}, X)$ for an abelian variety $X/\mathbb{F}$ (Lang’s theorem [L56]), we find $\text{Coker}(\hat{J}^s_{\text{\acute{e}t}}(\mathbb{F}) \to \varpi(J_s)\text{\text{\acute{e}t}}(\mathbb{F})) = 0$; so, $E^s(K) = 0.$

Lemma 9.3. Suppose $l \neq p$, and let $W$ be the l-adic integer ring of $K$. Then if $A_s$ has good reduction (resp. additive, semi-stable) over $W$ and $\mathbb{T}$ is integral, then $\hat{J}^s_{\text{\acute{e}t}} = J_s[\mathbb{T}]^{\text{\acute{e}t}}$ and $\varpi(J_s)[\mathbb{T}]^{\text{\acute{e}t}}$ are contained in an abelian factor of $J_s$ with good reduction (resp. additive, semi-stable) over $W$ for all $s$.

We indicated this fact in the proof of Proposition 9.2. We now claim that $\hat{J}^s_{\text{\acute{e}t}}$ and $\varpi(J_s)^{\text{\acute{e}t}}$ have good reduction over $W$.

Proof. As is well known (e.g., [H11, Remark after Conjecture 3.4]), the l-type (i.e., the l-local representation $\pi_l$) of automorphic representation occurring in a given ordinary p-adic analytic family is independent of the member of the family. In other words, if l-type is a ramified principal series $\pi(\alpha, \beta)$ or a Steinberg representation $\sigma(\cdot, \cdot | \alpha, \alpha, \beta)$, and $\beta_1, \beta_2$ are independent of members, and if $\pi_l$ is super-cuspidal, the associated p-adic Galois representation restricted to the l-inertia group is independent of the members up to isomorphism. Then by the criterion of Néron-Ogg-Shafarevich, the reduction of any member $A_{\mathbb{T}}$ is independent of an arithmetic point $P$. □

10. CONTROL OF $\Lambda$-SELMER GROUPS

We start with a lemma.

Lemma 10.1. For a number field or an l-adic field $K$ and $\mathcal{G} = J^\text{\acute{e}t}[p\infty]$, the Pontryagin dual $\check{\mathcal{G}}(K)^\dual$ is a $\Lambda$-torsion module of finite type. For any arithmetic prime $P$, $\check{\mathcal{G}}(K)^\dual \otimes_{\mathcal{H}} \mathbb{F}/P^n$, $\mathcal{G}(K) \otimes_{\mathcal{H}} \mathbb{F}/P^n$ and $\mathcal{G}(K)[P^n]$ are all finite for any positive integer $n$.

Proof. We give a detailed argument when $K$ is a number field and briefly touch an l-adic field as the argument is essentially the same. Let $P \in \omega_\Lambda$, and suppose $K$ is a number field. Suppose that the Galois representation $\rho_P$ associated to $P$ contains an open subgroup $G$ of $SL_2(\mathbb{Z}_p)$. Let $L$ be the Pontryagin dual module of $\check{\mathcal{G}}(\mathbb{T})$. If the cuspid form $f_P$ associated to $P$ has conductor divisible by $N$, the localization $L_P$ is free of rank 2 over the valuation ring $V = \mathbb{F}/P^n$ finite over $\Lambda_P$ (e.g., [HMI, Proposition 3.78]). If not, by the theory of new form (e.g., [H86a, §3.3]), $L_P$ is free of rank 2 over a.
local ring of the form $V[X_1, \ldots, X_m]/(X_1^{e_1}, \ldots, X_m^{e_m}) = h_F$ with nilradical coming from old forms (e.g., [H13a, Corollary 1.2]). The contragredient $\bar{\rho}_F = \rho_F^{-1}$ of $\rho_F$ is realized by $L_F/PL_F$. Then $G$ is also contained in $\text{Im} (\rho_F)$, and $H_0(K, L_F/PL_F) \cong H(K, L_F)/PL_0(K, L_F)$ is a surjective image of $H_0(G, L_F/PL_F)$, which vanishes. Thus $H_0(K, L_F/PL_F) = 0$, which implies $H_0(K, L) = 0$ by Nakayama’s lemma. In particular, $H_0(K, L)$ is a $\Lambda$-torsion module whose support is outside $P$.

If $\rho_F$ does not contain an open subgroup of $SL_2(\mathbb{Z}_p)$, by Ribet [R85] (see also [GME, Theorem 4.3.18]), there exists an imaginary quadratic field $M$ such that $\bar{\rho}_F = \text{Ind}_M^Q \varphi$ for an infinite order Hecke character $\varphi$ of $\text{Gal}(\overline{Q}/M)$. Then it is easy to show that $H_0(K, L_F/PL_F) = 0$, and in the same way as above, we find $H_0(K, L) \otimes h/P^n$ is finite for all $n$. Thus for any arithmetic prime $P \in \omega_K$, $H_0(K, L_F) = 0$ and hence $\mathcal{G}(K)^{\psi} = H_0(K, L)$ is $\Lambda$-torsion with support outside $P$. Thus in any case, $\mathcal{G}(K)^{\psi} \otimes h/P^n = H_0(K, L) \otimes h/P^n$ is finite for all $n$ as $h$ is a semi-local ring of dimension 2 finite torsion-free over $\Lambda$. The module $\mathcal{G}(K)[P^n]$ is just the dual of $H_0(K, L) \otimes h/P^n$ and hence is finite. Then $(\mathcal{G}(K) \otimes h/P^n)^{\psi} = H_0(K, L)[P^n]$, which is finite by the above fact that $H_0(K, L)$ is $\Lambda$-torsion with support outside $P$.

If $K$ is $l$-adic, replacing $K$ by its finite extension, we may assume that $A_F$ has split semi-stable reduction. Write $F$ for the residue field of $P$. Then either $\bar{\rho}_F$ (Frob) for a Frobenius element Frob of $\text{Gal}(\overline{Q}/K)$ has infinite order without eigenvalue 1 or the space $V(\bar{\rho}_F)$ fits into a non-split extension $F \hookrightarrow V \twoheadrightarrow F(-1)$ for the Tate twist $F(-1)$ (by the degeneration theory of Mumford–Tate; cf., [DAV, Appendix]). Because of this description $H_0(K, L_F/PL_F) = 0$, and by the same argument above, the results follows.

Since $(\varpi)$ is supported by finitely many arithmetic primes, $E_{\text{Sel}}(K)^{\psi} := (\mathcal{G}(K) \otimes h/(\varpi))^\psi \cong \mathcal{G}(K)^{\psi}[(\varpi)]$ is finite by the above lemma; so, we get

**Corollary 10.2.** Assume (A). If $K$ is a number field, then $E_{\text{Sel}}(K)$ is finite.

Let $T$ be the local ring such that $\varpi \in m_T$. We define $\Omega_T$ to be the set of points $P \in \text{Spec}(T)(\overline{Q}_p)$ such that

$$ P \cap \Lambda \text{ contains } p^{\rho_P} - 1 \text{ for some } 0 < s \in Z \text{ and } P \text{ is principal as a prime ideal.} $$

So, $\Omega_T$ is a subset of $\text{Spec}(T) \cap \omega_K$ made of principal ideals. We see $E_{\text{Sel}}(K)_T = \text{Coker}(\varpi : \mathcal{G}(K)_T \rightarrow \mathcal{G}(K)_{T'})$, where $T' = T \otimes h T$ for an $h$-module $M$. Thus for the Galois representation $\rho_P$ acting on $T \mathcal{G}_T = \lim_{\rightarrow} T_p \mathcal{G}[\rho_P^{\rho_P} - 1]$, if $\rho_P$ modulo $m_T$ is absolutely irreducible over $\text{Gal}(\overline{Q}/K)$, we conclude $E_{\text{Sel}}(K) = 0$. Here $T' = (\rho_P \mod m_T)$ is the semi-simple two dimensional representation whose trace is given by $\text{Tr}(\rho_P)$ mod $m_T$. Indeed, the Galois module $\mathcal{G}[m_T]$ has Jordan-Hölder sequence whose sub-quotients are all isomorphic to $T'$. So, by Nakayama’s lemma, $\mathcal{G}(K) = 0$. Write $T'_T |_{\text{Gal}(\overline{Q}_p/K)} \cong \left( \frac{T'_T}{0_{T'}} \right)$ mod $m_T$ with the nearly ordinary character $\varphi$ (i.e., $\varphi([p, Q_p])$ is equal to the image modulo $m_T$ of $U(p)$). Here $\bar{\nu}_p = \nu_p \mod p$. Then it is plain that $\mathcal{G}(K_v) = 0$ for all place $v|p$ of $K$ if $\varphi_p \psi$ and $\varphi$ are both non-trivial over $\text{Gal}(\overline{Q}_p/K_v)$ for all $v|p$. We record this fact as

**Corollary 10.3.** Let $p > 2$, $K$ be a number field, and suppose one of the following two conditions:

1. $\varphi_p \psi$ is irreducible over $\text{Gal}(\overline{Q}/K)$;
2. $\varphi_p \psi$ and $\varphi$ are both non-trivial over $\text{Gal}(\overline{Q}_p/K_v)$ for all $p$-adic places $v|p$ of $K$.

Then we have $E_{\text{Sel}}(K) = 0$.

Recall $M_T = M \otimes h T$ for an $h$-module $M$.

**Theorem 10.4.** Let $K$ be a number field and $\text{Spec}(T)$ be the connected component such that $\varpi \in m_T$. Suppose $p > 2$, (A) and (F).

1. Assume one of the following two conditions
   - (1) $E^\infty(K_v)_{T'}$ is finite for all $v|p$,
   - (2) $A_v$ does not have split multiplicative reduction modulo $p$ at all primes $p|v$ of $K$.

Then the following sequence

$$ 0 \rightarrow \text{Sel}_K(A_v) \rightarrow \text{Sel}_K(J_v) \xrightarrow{\psi} \text{Sel}_K(J_v). $$


is exact up to finite error.

(2) Assume one of the following two conditions:

(E1) $E_{\text{Sel}}(K) = E^\infty(K_v)\mathbb{T} = 0$ for all $v|NP$,

(E2) $\mathbb{T}$ is an integral domain, $A_v$ has good reduction at all $v|NP$ and $|\varphi(\text{Frob}_v) - 1|_v = 1$ for all $v$. Here $\varphi(\text{Frob}_v)$ is a Frobenius element in $\text{Gal}(\overline{K}/K_v)$ acting trivially on $K[\mu_{p^\infty}]$.

By Proposition 9.2 and by the assumption (e1), we have an exact sequence:

\[ \pi(S_\infty) \to \pi(J_{\infty}) \to \pi(J_{\infty}^\text{ord}) \to \pi(J_{\infty}^\text{ord}) \]

Now we assume (e1). We need to prove the sequence (10.4) is exact up to finite error. By Corollary 10.2, (E2) implies $E_{\text{Sel}}(K) = 0$. Indeed, $\varphi(p^\infty) = 1$ as $A_v$ has good reduction modulo $p$. We prove the theorem under (E1) or (e1), since Theorem 17.2 combined with Proposition 9.2 (4) shows (e2)$\Rightarrow$(e1) and (E2)$\Rightarrow$(E1) for $v|p$.

**Proof.** Recall the following commutative diagram with two bottom exact rows and three right exact columns from (8.13) (tensored with $\mathbb{T}$ over $\mathfrak{h}$):

\[
\begin{array}{cccccc}
\text{Ker}(i_{\text{Sel},*}) & \xrightarrow{i_0} & \text{Sel}_K(A_{\text{ord}}) & \xrightarrow{i_{\text{Sel},*}} & \text{Sel}_K(J_{\infty}^\text{ord}) & \xrightarrow{\varpi_{\text{Sel},*}} & \text{Sel}_K(\varpi(J_{\infty}^\text{ord})) \\
\downarrow & & \downarrow a & \downarrow & \downarrow & & \\
E_{\text{Sel}}(K) & \xrightarrow{e} & H^1(A_{\text{ord}}|p^\infty) & \xrightarrow{i_{*}} & H^1(J_{\infty}^\text{ord}|p^\infty) & \xrightarrow{\varpi_{*}} & H^1(J_{\infty}^\text{ord}|p^\infty) \\
\downarrow & & \downarrow & & \downarrow & & \\
E_\infty^S(K) & \xrightarrow{e_0} & H^1_S(A_{\text{ord}}) & \xrightarrow{i_{\text{Sel},*}} & H^1_S(J_{\infty}^\text{ord}) & \xrightarrow{\varpi_{\text{Sel},*}} & H^1_S(\varpi(J_{\infty}^\text{ord})).
\end{array}
\]

By Proposition 9.2 and by the assumption (e1), $E_\infty^S(K)$ is finite. Since the middle two columns are exact, the left column is exact with injection $i$ (e.g., [BCM, I.1.4.2 (1)]). Since the bottom row is exact with injection $e_0$, the map $i_0$ is injective and $\text{Im}(i_0) = \text{Ker}(i_{\text{Sel},*})$. Suppose (E1). Then all the terms of the left column vanish. So $\text{Ker}(i_{\text{Sel},*}) = 0$ and the sequence:

\[
0 \to \text{Sel}_K(A_{\text{ord}}) \to \text{Sel}_K(J_{\infty}^\text{ord}) \to \text{Sel}_K(\varpi(J_{\infty}^\text{ord})) \xrightarrow{(8.11)} \text{Sel}_K(J_{\infty}^\text{ord})
\]

is exact. The cokernel $\text{Coker}(\text{Sel}_K(J_{\infty}^\text{ord}) \xrightarrow{\varpi} \text{Sel}_K(\varpi(J_{\infty}^\text{ord}))$ is global in nature and seems difficult to determine, although $\text{Coker}(\text{Sel}_K(\varpi(J_{\infty}^\text{ord})) \to \text{Sel}_K(J_{\infty}^\text{ord}))$ is local as in (8.11), and if $E_{MW}(K_v) = 0$ for all $v|p$, it vanishes.

Now we assume (e1). We need to prove the sequence (10.4) is exact up to finite error. By Corollary 10.2, $E_{\text{Sel}}(K)$ is finite. Since we know $E^\infty(K_v) = 0$ for $v$ prime to $p$ by Proposition 9.2, we conclude from (e1) that $E_\infty^S(K)$ is finite. Then the diagram (8.13) has two bottom rows exact up to finite error. Since the Pontryagin dual of all the modules in the above diagram are $\Lambda$-modules of finite type, we can work with the category of $\Lambda$-modules of finite type up to finite error (e.g., [BCM, VII.4.5]). Then in this new category, the bottom two rows are exact and the extreme left terms a pseudo-null. Thus the dual sequence of the theorem is exact up to finite error, and by taking dual back, the sequence in the theorem is exact up to finite error.

**Corollary 10.5.** Assume (F) and $p > 2$. Then we have

(1) The Pontryagin dual $\text{Sel}_K(J_{\infty}^\text{ord})^\vee$ of $\text{Sel}_K(J_{\infty}^\text{ord})$ is a $\Lambda$-module of finite type.

(2) If further $\text{Sel}_K(A_{\text{ord}}) = 0$ for a single element $\varpi \in \mathfrak{m}_\mathfrak{T}$ satisfying (A) and (E1), then $\text{Sel}_K(J_{\infty}^\text{ord}) = 0$ and $\text{Sel}_K(A_{\text{ord}}) = 0$ for every $\varpi \in \mathfrak{m}_\mathfrak{T}$ satisfying (A) and (E1).

(3) Suppose that $\mathbb{T}$ is an integral domain. If $\text{Sel}_K(A_{\text{ord}})$ is finite for some $\varpi$ satisfying (A) and (e1), then $\text{Sel}_K(J_{\infty}^\text{ord})^\vee$ is a torsion $\mathbb{T}$-module of finite type. Thus if $\mathbb{T}$ is a unique factorization domain, for almost all $P \in \Omega_\varpi$, $\text{Sel}_K(A_{P}^\text{ord})$ is finite.
Proof. The condition (A) and (e2) is satisfied by any non-trivial factor \( \varpi \) of \( (\gamma^p - 1)/(\gamma - 1) \). Thus \( \text{Sel}_K(J_{\infty}^{\text{ord}})^{\vee} / \varpi \cdot \text{Sel}_K(J_{\infty}^{\text{ord}})^{\vee} \) pseudo isomorphic to \( \text{Sel}_K(\hat{A}_{\text{ord}}^{\vee}) \) which is \( \mathbb{Z}_p \)-module of finite type; so, by the topological Nakayama’s lemma, we conclude that \( \text{Sel}_K(J_{\infty}^{\text{ord}})^{\vee} \) is a \( \Lambda \)-module of finite type. The last two assertions can be proven similarly. \( \square \)

Let \( K \) be a number field as in Theorem A. Recall the hypothesis of the assertion (4) of Theorem A:

1. \( \mathbb{T} \) is a unique factorization domain (so, (A) holds for every \( P \in \Omega_{\mathbb{T}} \)),
2. \( A_P/P_{\mathbb{T}} \) has good reduction at \( p \),
3. Writing \( \mathbb{F}_p \) for the residue field of a prime \( p \) of \( K \), \( A_P(\mathbb{F}_p) = 0 \) for all prime \( p \),
4. \( A_{P_v/K} \) has good reduction at all places \( v \mid N \) of \( K \).

Then (E1) is satisfied by Theorem 17.2 and Proposition 9.2 (4), hence the assertion (4) of Theorem A follows from Corollary 10.5 (2–3).

From the short exact sequence \( (J_{\infty}^{\text{ord}})^{\vee}(K) \otimes \mathbb{Q}_p / \mathbb{Z}_p) \rightarrow \text{Sel}_K(J_{\infty}^{\text{ord}})^{\vee} \rightarrow \text{Sel}_K(J_{\infty}^{\text{ord}})^{\vee} \rightarrow \text{II}_K(J_{\infty}^{\text{ord}})^{\vee} \), we get

Corollary 10.6. Assume (F). The limit Mordell–Weil group \( (J_{\infty}^{\text{ord}})^{\vee}(K) \otimes \mathbb{Q}_p / \mathbb{Z}_p) \) and the limit Tate–Shafarevich group \( \text{II}_K(J_{\infty})^{\vee} \) are \( \Lambda \)-module of finite type.

11. CONTROL OF \( \Lambda \)-BT GROUPS AND ITS COHOMOLOGY

Recall \( \mathcal{G} := \mathcal{G}_{\alpha, \delta, \xi} = J_{\infty}^{\text{ord}}[p^\infty] \) which is a \( \Lambda \)-BT group in the sense of [H14]. Here the set \( S \) is supposed to be finite. We study the control of the Tate–Shafarevich group of \( \mathcal{G} \).

Theorem 11.1. Let \( K \) be a number field. Suppose \( |S| < \infty \), (F) and (A) for \( \varpi \). Then the sequence \( 0 \rightarrow \text{II}(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty]) \rightarrow \text{II}(K^S/K, \mathcal{G}) \xrightarrow{\varpi} \text{II}(K^S/K, \mathcal{G}) \) is exact up to finite error.

Proof. From the exact sequence \( 0 \rightarrow \hat{A}_r^{\text{ord}}[p^\infty] \rightarrow \mathcal{G} \xrightarrow{\varpi} \mathcal{G} \rightarrow 0 \) of Corollary 6.5, we get a commutative diagram with exact bottom two rows and exact columns:

\[
\begin{array}{cccccc}
\text{Ker}(t_{\text{II},*}) & \longrightarrow & \text{II}(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty]) & \xrightarrow{\iota_{\text{II},*}} & \text{II}(K^S/K, \mathcal{G}) & \xrightarrow{\varpi_{\text{II},*}} & \text{II}(K^S/K, \mathcal{G}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_{v \in S} E_{\text{BT}}^\infty(K_v) & \longrightarrow & H^1(\hat{A}_r^{\text{ord}}[p^\infty]) & \longrightarrow & H^1(\mathcal{G}) & \longrightarrow & H^1(\mathcal{G}) \\
\end{array}
\]

where \( E_{\text{BT}}^\infty(k) = \text{Coker}(\varpi : \mathcal{G}(k) \rightarrow \mathcal{G}(k)) \).

By Lemma 10.1, \( E_{\text{BT}}^\infty(K) \) and \( E_{\text{BT}}^\infty(K_v) \) are finite. Thus as long as \( S \) is finite, \( \prod_{v \in S} E_{\text{BT}}^\infty(K_v) \) is finite. Then the above diagram proves the desired exactness. \( \square \)

Corollary 11.2. Let the notation and the assumption be as in the theorem. Assume that \( \mathbb{T} \) is an integral domain and that \( \text{II}(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty]) \) is finite for a principal arithmetic prime \( P_0 \in \Omega_{\mathbb{T}} \). Then \( \text{II}(K^S/K, \mathcal{G})^{\vee} \) is a torsion \( \mathbb{T} \)-module of finite type. In particular, for almost all principal arithmetic points \( P \in \text{Spec}(\mathbb{T}) \), \( \text{II}(K^S/K, \hat{A}_r^{\text{ord}}[p^\infty]) \) is finite.

12. CONTROL OF IND \( \Lambda \)-MW GROUPS

Put \( E_{MW}(k) := \text{Coker}(\varpi(J_{\infty}^{\text{ord}})^{\vee}(k) \rightarrow J_{\infty}^{\text{ord}}(k)) \). If \( P \in \text{Spec}(\mathbb{T})(\overline{\mathbb{Q}_p}) \) is an arithmetic point generated by \( \varpi \in \mathbb{T} \), we write \( E_{MW}^P(k) \) for \( E_{MW}(k) \). We start with a proposition showing that the limit Mordell–Weil group is of co-finite type over \( \Lambda \):

Proposition 12.1. Assume (F). Let \( P = (\varpi) \in \Omega_{\mathbb{T}} \). Let \( p > 2 \) and \( k \) be either a number field or an \( l \)-adic field. The Pontryagin dual of the following sequence

\[
0 \rightarrow \hat{A}_r^{\text{ord}}(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{\varpi} J_{\infty}^{\text{ord}}(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{\varpi} J_{\infty}^{\text{ord}}(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{\varpi} E_{MW}^P(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p \rightarrow 0
\]
is exact up to finite error except for $\text{Ker}(\varpi)/\text{Im}(\iota)$ which is at worst a $\Lambda$-torsion module of finite type.  In particular, if $\mathcal{T}$ is an integral domain, the Pontryagin dual $J := (J_{\infty,\mathcal{T}}^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a $\mathcal{T}$-module of finite type.

**Proof.** Since $\varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) = J_{\infty,\mathcal{T}}^{\text{ord}}(k)/\hat{A}_p^{\text{ord}}(k)$, we have the following three exact sequences:

$$0 \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \to E_p^{\infty}(k) \to 0,$$

(12.2)

$$0 \to \hat{A}_p^{\text{ord}}(k) \to J_{\infty,\mathcal{T}}^{\text{ord}}(k) \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \to 0,$$

$$0 \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \to J_{\infty,\mathcal{T}}^{\text{ord}}(k) \to E_{MW}^p(k) \to 0.$$

Tensoring with $\mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbb{Z}_p$, by $\text{Tor}_1^\mathbb{Z}(X, \mathbb{Q}_p/\mathbb{Z}_p) = X[p^\infty]$, we get the following exact sequences

$$\varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k))[p^\infty](k) \to E_p^{\infty}(k) \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0,$$

(12.3)

$$0 \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k))(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to J_{\infty,\mathcal{T}}^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0,$$

$$0 \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to J_{\infty,\mathcal{T}}^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to E_{MW}^p(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

The last sequence is exact since, by Corollary 6.3, $E_{MW}^p(k)$ is a flat $\mathbb{Z}_p$-module [BCM, I.2.5]. The image $\text{Im}(\varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k))[p^\infty]) \cong \hat{A}_p^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is killed by $\varpi$. Since $(\varpi)$ is arithmetic, by Lemma 10.1, this image is finite (i.e., factoring through the $K$-rational quotient of $(\varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k))[p^\infty]$ killed by $\varpi$). Thus for the sequence (12.3):

$$0 \to \hat{A}_p^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} J_{\infty,\mathcal{T}}^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to E_{MW}^p(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

$\iota$ has finite kernel, $\text{Ker}(\varpi)/\text{Im}(\iota)$ is the image of $E_{p}^{\infty}(k)$ with finite cokernel (by the first sequence combined with the second of (12.3)) and the last three right terms are exact.

Suppose that $\mathcal{T}$ is an integral domain. By Lemma 9.1, $E_{p}^{\infty}(k)$ has $p$-torsion with finite corank over $\mathbb{Z}_p$. Therefore $\text{Ker}(J_{\infty,\mathcal{T}}^{\text{ord}}(k) \xrightarrow{\varpi} J_{\infty,\mathcal{T}}^{\text{ord}}(k)^\vee)$ is a $\mathcal{T}$-torsion module of finite type. Then by Nakayama’s lemma, the Pontryagin dual $(J_{\infty,\mathcal{T}}^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a $\mathcal{T}$-module of finite type. □

**Corollary 12.2.** Suppose $\mathcal{T}$ is an integral domain and $k$ be a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_1$. Then, the error term $E_{MW}^p(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ vanishes for almost all $P \in \Omega_{\mathcal{T}}$. More precisely, $E_{MW}^p(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$ if and only if $P \in \Omega_{\mathcal{T}}$ is outside the support of the maximal $\mathcal{T}$-torsion submodule of $J$. Moreover $E_{MW}^p(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ vanishes if and only if $E_{MW}^p(k) = 0$, and otherwise, $\text{Coker}(J_{\infty,\mathcal{T}}^{\text{ord}}(k) \xrightarrow{\varpi} \hat{A}_p^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is isogenous to $\hat{A}_p^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

**Proof.** We look into the following exact sequence for an arithmetic point $P = (\varpi) \in \Omega_{\mathcal{T}}$:

$$0 \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \to J_{\infty,\mathcal{T}}^{\text{ord}}(k) \xrightarrow{\varpi} \hat{A}_p^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(k'/k, \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)))$$

where $k' = \overline{k}$ or $k^S$ according as $k$ is a finite extension of $\mathbb{Q}_1$ or $\mathbb{Q}$. Since $\text{Coker}(\rho_{\infty,\mathcal{T}})$ is $p$-divisible and $H^1(k'/k, \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)))$ is $p$-torsion, we conclude

$$\text{Coker}(\rho_{\infty,\mathcal{T}}) \cong \begin{cases} 0 & \text{call Case V,} \\ (\mathbb{Q}_p/\mathbb{Z}_p)^n & \text{call Case T.} \end{cases}$$

Here $n$ is given by $\text{dim}_{\mathbb{Q}_p} \hat{A}_p^{\text{ord}}(k) \otimes \mathbb{Q}_p$, and hence $\text{Coker}(\rho_{\infty,\mathcal{T}})$ is isogenous to $\hat{A}_p^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ which is in turn isogenous to $\hat{A}_p^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ in Case T.

Thus, in Case V, we have

$$\varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \cong J_{\infty,\mathcal{T}}^{\text{ord}}(k) \text{ and } \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong J_{\infty,\mathcal{T}}^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$ 

So $E_{MW}^p(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$.

We now assume that we are in Case T. Then we have the following short exact sequence:

$$0 \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \to J_{\infty,\mathcal{T}}^{\text{ord}}(k) \xrightarrow{\varpi} \text{Im}(\rho_{\infty,\mathcal{T}})(\cong E_{MW}^p(k)) \to 0$$

with $\mathbb{Z}_p$-free $\text{Im}(\rho_{\infty,\mathcal{T}})$. Then by [BCM, I.2.5], we still have a short exact sequence:

$$0 \to \varpi(J_{\infty,\mathcal{T}}^{\text{ord}}(k)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to J_{\infty,\mathcal{T}}^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} \text{Im}(\rho_{\infty,\mathcal{T}}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$
Note that $(\text{Im}(\rho_{\infty,T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)_v$ is a $P$-torsion $T$-module of finite type. Thus $P$ belongs to the support $Z \subset \text{Spec}(T)$ of the maximal $T$-torsion submodule of $(J_{\infty,T}^0(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p))_v$. Since $(J_{\infty,T}^1(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)_v$ is a $T$-module of finite type, $Z(\overline{\mathbb{Q}_p})$ is a finite set. By our proof, $E_{MW}^P(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \neq 0$ $\Leftrightarrow$ Case $T$. Thus $E_{MW}^P(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p = 0$ $\Leftrightarrow$ $E_{MW}^P(k) = 0$. \hfill $\Box$

**Proposition 12.3.** Let $k$ be either a number field or a finite extension of $\mathbb{Q}_l$ (including $l = p$). Take $P = (\varpi) \in \Omega_T$. If $E_{P}^\infty(k)$ is finite, then $E_{P}^\infty(k) := \frac{\varpi(J_{\infty,T}(k))}{\varpi(J_{\infty,T}(k))}$ is also finite, and moreover, if $l \nmid Np$ and $k$ is a finite extension of $\mathbb{Q}_l$, we have $E_{P}^\infty(k) = 0$.

**Proof.** We look into the following sheaf exact sequence for $A := A_P$:

$$0 \to \widehat{A}^\text{ord}[p^\infty] \to \varpi(J_{\infty,T}(k)) \xrightarrow{\varpi} \varpi(J_{\infty,T}(k)) \to 0.$$ 

The attached cohomology exact sequence gives rise to the following short exact sequence:

$$0 \to \varpi(J_{\infty,T}(k)) \to H^1(\widehat{A}^\text{ord}[p^\infty]) \to H^1(\varpi(J_{\infty,T}(k))) \to 0,$$

where $H^1(\varpi(J_{\infty,T}(k))) = H^1(\mathbb{Z}_p/(\varpi(J_{\infty,T}(k)))) = 0$. Thus we conclude for some non-negative $n' \leq n$ as in the proof of Corollary 12.2

$$E_{P}^\infty(k) = \frac{\varpi(J_{\infty,T}(k))}{\varpi(J_{\infty,T}(k))} = \varpi(J_{\infty,T}(k)) = \varpi(J_{\infty,T}(k)) = \varpi(E_{MW}^P(k)) \cong \mathbb{Z}_p^{n'}$$

up to finite error. Since $H^1(\widehat{A}^\text{ord}[p^\infty])$ is a $p$-torsion discrete module, the image of $\frac{\varpi(J_{\infty,T}(k))}{\varpi(J_{\infty,T}(k))}$ in $H^1(\widehat{A}^\text{ord}[p^\infty])$ has to be finite (i.e., $n' = 0$). Therefore $E_{P}^\infty(k)$ is finite.

Assume $l \nmid Np$ and that $k$ is a finite extension of $\mathbb{Q}_l$. Then $\varpi(J_s)$ has good reduction over $\mathbb{Z}_l$. Therefore $\varpi(J_{s,T}(k)) \cong \varpi(J_{s,T}(k))$ for the residue field $\mathbb{F}$ of $k$. Passing to the limit, we find $\varpi(J_{s,T}(k)) \cong \varpi(J_{\infty,T}(k))$. Therefore $E_{P}^\infty(\mathbb{F}) = E_{P}^\infty(k)$. Taking to the Néron model and for each level $Np^s$ and passing to the limit, we have an exact sequence

$$0 \to \widehat{A}^\text{ord}[p^\infty]/p \to \varpi(J_{\infty,T}(k))/\varpi \cong \varpi(J_{\infty,T}(k))/\varpi \to 0.$$

Thus we have an inclusion $E_{P}^\infty(\mathbb{F}) \hookrightarrow H^1(\mathbb{F}, \widehat{A}^\text{ord}[p^\infty])$. Since $H^1(\mathbb{F}, \widehat{A}^\text{ord}[p^\infty])$ is a factor of $H^1(\mathbb{F}, \widehat{A}[p^\infty])$, we need to prove $H^1(\mathbb{F}, \widehat{A}[p^\infty]) = 0$. From the exact sequence

$$0 \to \widehat{A}[p^\infty] \to \widehat{A} \times_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$$

we have an exact sequence

$$0 \cong A[p^\infty] \to H^1(\mathbb{F}, \widehat{A}[p^\infty]) \to H^1(\mathbb{F}, \widehat{A}) = H^1(\mathbb{F}, \widehat{A}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$$

by Lang’s theorem [L56]. Thus we conclude $E_{P}^\infty(k) = 0$. \hfill $\Box$

**Corollary 12.4.** Let $K$ be a number field and $S = \{v|Np\}$ for finite places $v$ of $K$. Suppose that $A_P$ satisfies (e2) for a number field $K$. If the group $\text{III}(K^S/K, \widehat{A}^\text{ord}[p^\infty])$ is finite, then $E_{P}^\infty(K)$ and $E_{P}^\infty(K)$ are both finite.

**Proof.** By Proposition 9.2 and Theorem 17.2, we have $E_{P}^\infty(K)$ is finite for all finite places $v$ of $K$ and hence $E_{P}^\infty(K_v)$ is finite for all $v$ and vanishes except for $v|Np$ by Proposition 12.3. By the commutative diagram with exact rows for $S := \{v|Np\}$ and $A = A_P$:

$$\begin{array}{cccccc}
E_{P}^\infty(K) & \xrightarrow{i_0} & H^1(K^S/K, \widehat{A}^\text{ord}[p^\infty]) & \xrightarrow{i} & H^1(K^S/K, \varpi(J_{\infty,T})) \\
\Pi_{v \in S} E_{P}^\infty(K_v) & \xrightarrow{i_v} & H^1_{\delta}(\widehat{A}^\text{ord}[p^\infty]) & \xrightarrow{i_1} & H^1_{\delta}(\varpi(J_{\infty,T}))
\end{array}$$

(12.4)
The assumption of the existence of $P_0$ as in Theorem A implies the assumption (b) (see Proposition 13.1); so, this corollary proves Theorem A (3).

Proof. By Assumption (b), Proposition 12.1 and Corollary 12.4 tell us that we have the following exact sequence up to finite error:

$$
\hat{A}_{P_0}(K) \otimes_{Z_p} Q_p/Z_p \rightarrow T^{\text{ord}}_{\text{max}}(K) \otimes_{Z_p} Q_p/Z_p \rightarrow T^{\text{ord}}_{\text{max}}(K) \otimes_{Z_p} Q_p/Z_p \rightarrow \hat{A}_{P_0}(K) \otimes_{Z_p} Q_p/Z_p.
$$

Write the Pontryagin dual exact sequence as $0 \rightarrow E \rightarrow J \rightarrow A \rightarrow 0$. Write $X \sim Y$ if $T$-modules $X$ and $Y$ are pseudo isomorphic. Then by the classification of $T$-modules, we have $J \sim Y := T' \oplus T$ for a torsion $T$-module $T$ of finite type; i.e., rank $J = r$.

We first suppose that $E^{P_0}_{\text{MW}}(K)$ is non-zero compact (i.e., $E^{P_0}_{\text{MW}}(K) \otimes_{Z_p} Q_p/Z_p = 0$); so, $E^{P_0}_{\text{MW}}(K)$ and $\hat{A}_{P_0}(K)$ are pseudo isomorphic to $(T/P_0)^m$ for $m = \dim_{H_{P_0}} A_{P_0}(K) \otimes Q$. Since $J$ is pseudo isomorphic to $X := T' \oplus T$ for a torsion $T$-module $T$ of finite type, we find that $(T/P_0)^n \oplus T/P_0 T \sim X/\omega X \sim A \sim (T/P_0)^m$ and $E \sim T/P_0 T \sim (T/P_0)^m$. Thus $r + m = m'$, and we conclude $r = 0$. Thus $J$ is a torsion $T$-module. By Corollary 11.2, the assumption (b) implies that $\text{III}(K^S/K, \hat{A}_{P_0}(K))^{\text{ord}}_{\text{max}}[p^{\infty}]$ is finite for almost all $P \in \Omega_T$ (i.e., finite for all $P \in \Omega_T$ outside $Z_g := \text{Supp}(\text{III}(K^S/K, G_T)^\nu)$). Thus the sequence

$$
\hat{A}_{P_0}(K) \otimes_{Z_p} Q_p/Z_p \rightarrow J^{\text{ord}}_{\text{max}}(K) \otimes_{Z_p} Q_p/Z_p \rightarrow J^{\text{ord}}_{\text{max}}(K) \otimes_{Z_p} Q_p/Z_p \rightarrow E^{P_0}_{\text{MW}}(K) \otimes_{Z_p} Q_p/Z_p
$$

is exact up to finite error for $P \in \Omega_T - Z_g$. We write the Pontryagin dual sequence as $0 \rightarrow E' \rightarrow J \rightarrow A' \rightarrow 0$. The module $E'$ is $0$ for all $P \in \Omega_T - Z$ for the support $Z$ of the maximal $T$-torsion submodule of $J$; so, we conclude (2) for $P \in \Omega_T - (Z \cup Z_g)$.

We now suppose that $E^{P_0}_{\text{MW}}(K)$ vanishes (i.e., in Case V); so, $E^{P_0}_{\text{MW}}(K) \otimes_{Z_p} Q_p/Z_p = 0$. Thus $P_0$ is outside the support of the maximal $T$-torsion submodule of $J$ by Corollary 12.2. Hence Ker$(J \rightarrow J)$ is finite. Since $0 \rightarrow J \rightarrow A \rightarrow 0$ is exact up to finite error, we conclude $r = m$ and the assertion (1) in the same way as above.

Corollary 12.6. Let $K$ be a number field. Assume that $T$ is a normal integral domain and to have $P_0 = (\infty) \in \Omega_T$ satisfying (e2) such that $\text{III}(K^S/K, \hat{A}_{P_0}(K))^{\text{ord}}_{\text{max}}[p^{\infty}]$ is finite. Then dim$_{H_P} A_P(K) \otimes_{Z} Q$ is constant over $\Omega_T - (Z \cup Z_1)$ for proper closed sets $Z_2$ and $Z$, where $Z$ is the support of the maximal $T$-torsion submodule of $J$ and $Z_2 = \text{Supp}(\text{III}(K^S/K, G_T)^\nu)$.

Proof. By the same argument in the proof of Corollary 12.5, we get dim$_{H_P} A_P(K) \otimes_{Z} Q = \text{rank}_J$ for all $P \in \Omega_T - (Z \cup Z_2)$. 

For our later use, we record

**Corollary 12.7.** Let $K$ be a number field. Assume that $T$ is a normal integral domain and to have $P = (\omega) \in \Omega_T$ such that $A_P$ satisfies (e2) and that $\Pi(K^S/K, \hat{A}_P^{\text{ord}}[p^\infty])$ is finite. Then the sequence

$$0 \to \hat{A}_P^{\text{ord}}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to J_\infty^{\text{ord}}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \omega(J_\infty^{\text{ord}})(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

is exact up to finite error.

**Proof.** By Corollary 12.4, $E_\infty^P(K)$ is finite. Therefore by the first and the second exact sequence of (12.3) combined, we conclude the desired assertion. □

**Theorem 12.8.** Assume (F). Suppose that $T$ is an integral domain with infinite $\Omega_T$. Let $p > 2$ and $k$ be either a number field or an $l$-adic field. If $k$ is a number field, suppose to have a point $P_0 \in \Omega_T$ satisfying (e2) such that $\Pi(k^S/k, \hat{A}_P^{\text{ord}}[p^\infty])$ is finite. Then for almost all $P \in \Omega_T$, the following sequence

$$(12.5) \quad 0 \to \hat{A}_P^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to J_\infty^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\omega} J_\infty^{\text{ord}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

is exact up to finite error.

The exceptional finite set of $P \in \Omega_T$ (not satisfying the assertion of the theorem) is contained in the union of the following three finite sets

(a) the support of the maximal $T$-torsion submodule of $J_\infty^{\text{ord}}(k)^\vee$ whose $\mathbb{T}_p$-points are finite by Corollary 12.2 (this set may not contain $P_0$ in the theorem if $k$ is a number field);

(b) the support of $\Pi(k^S/k, \hat{G}_T)^\vee$ whose $\mathbb{Q}_p$-points are finite by Theorem 11.1 (this only applies to the case when $k$ is a number field);

(c) $P \in \Omega_T$ not satisfying (e2) (this applies to the case where $k$ is either a number field or a $p$-adic field).

**Proof.** By Proposition 12.1, we need to show finiteness of $E_{MW}^P(k)$ and $E_\infty^P(k)$ for almost all $P \in \Omega_T$. If $P \in \Omega_T$ is outside the set defined by (b) and (c) as above, $\Pi(k^S/k, \hat{A}_P^{\text{ord}}[p^\infty])$ is finite; so, by Corollary 12.4, $E_\infty^P(k)$ is finite. Therefore the sequence (12.5) is exact except for the extreme end term.

If $P \in \Omega_T$ is outside the set defined by (a), $E_{MW}^P(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ vanishes by Corollary 12.2, therefore the extreme end term of (12.5) is exact, as desired. □

### 13. Control of $\Lambda$-TS Groups

We now study control of the limit Tate–Shafarevich groups from which Theorem A (2) and (3) in the introduction follows.

**Proposition 13.1.** Suppose that $T$ is an integral domain flat over $\Lambda$. Let $K$ be a number field and pick an arithmetic point $P \in \text{Spec}(T)$. Assume $|S| < \infty$ and that $P$ is principal with $P = (\omega)$. Let $\text{Ker}_{MW}^P$ be the kernel of the natural diagonal map: $\hat{A}_P^{\text{ord}}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \prod_{v|p} \hat{A}_P^{\text{ord}}(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Then we have the following exact sequence

$$0 \to \text{Ker}_{MW}^P \to \Pi(K^S/K, \hat{A}_P^{\text{ord}}[p^\infty]) \xrightarrow{\Pi_P} \Pi_K(\hat{A}_P^{\text{ord}}).$$

In particular, if $\Pi(K^S/K, \hat{A}_P^{\text{ord}}[p^\infty])$ vanishes (resp. is finite), the error term $\text{Ker}_{MW}^P$ vanishes (resp. is finite). Similarly if $|\Pi_K(\hat{A}_P^{\text{ord}})| < \infty$ and $\dim_{H_P} A_P(K) \otimes \mathbb{Q} = 1$, the two modules $\text{Ker}_{MW}^P$ and $\Pi(K^S/K, \hat{A}_P^{\text{ord}}[p^\infty])$ are finite.

By principality of $P$, the conditions (A) is satisfied for $(\omega, P, A_P, T)$, and as we remarked after stating the condition (F) in Section 4, we actually need in this section is flatness of $T$ over $\Lambda$ (though we could have supposed the stronger condition (F)).
Proof. For $K' = K^S$ and $\overline{K}_v$, $\hat{A}_v(K')$ (and hence $\hat{A}_v^{\text{ord}}(K')$) is $p$-divisible $\mathbb{Z}_p$-modules; so, the $\mathbb{Z}_p$-module $\hat{A}_v^{\text{ord}}(K') / \hat{A}_v^{\text{ord}}[p\infty](K')$ is a $\mathbb{Q}_p$-vector space (i.e., it is isomorphic to $\hat{A}_v^{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$). From the short exact sequence $\hat{A}_v^{\text{ord}}[p\infty](K') \hookrightarrow \hat{A}_v^{\text{ord}}(K') \rightarrow \hat{A}_v^{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ ($K' = K^S, \overline{K}_v$) of Galois modules, we get the following commutative diagram with the bottom two exact rows:

$$
\begin{array}{cccc}
\text{Ker}_P^{MW} & \longrightarrow & \mathbb{III}(K^S / K, \hat{A}_v^{\text{ord}}[p\infty])) & \longrightarrow & \mathbb{III}_K(\hat{A}_v^{\text{ord}}) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{A}_v^{\text{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p & \overset{\rho}{\longrightarrow} & H^1(\hat{A}_v^{\text{ord}}[p\infty])) & \overset{\iota}{\longrightarrow} & H^1(\hat{A}_v^{\text{ord}}) \longrightarrow 0 \\
\delta & & \downarrow \text{Res}[p\infty] & & \text{Res} \\
\prod_{v|p}(\hat{A}_v^{\text{ord}}(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p) & \overset{i_S}{\longrightarrow} & H^1_S(\hat{A}_v^{\text{ord}}[p\infty])) & \overset{i_S\ast}{\longrightarrow} & H^1_S(\hat{A}_v^{\text{ord}}) \longrightarrow 0.
\end{array}
$$

(13.1)

The injectivity of $I_S$ and exactness of the bottom sequence prove the exact sequence in the proposition by [BCM, I.1.4.2 (1)].

If dim$_H$ $A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$, then $A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \overset{i}{\longrightarrow} \prod_{v|p} A_P(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has non-trivial image which is an $H_P$-vector space. This it has dimension 1 over $H_P$, and hence $i$ is an injection. Thus $\delta_0 : \hat{A}_v^{\text{ord}}(K) \rightarrow \prod_{v|p} \hat{A}_v^{\text{ord}}(K_v)$ is a morphism of $\mathbb{Z}_p$-module of finite type with finite kernel. Then it is clear after tensoring $\mathbb{Q}_p / \mathbb{Z}_p$ over $\mathbb{Z}_p$, $\delta$ has finite kernel; i.e., Ker$_P^{MW}$ is finite, and hence finiteness of $\mathbb{III}_K(\hat{A}_v^{\text{ord}})$ implies that of $\mathbb{III}(K^S / K, \hat{A}_v^{\text{ord}}[p\infty]))$.

\begin{proposition}
Suppose that $\mathbb{T}$ is an integral domain flat over $\Lambda$. Let $K$ be a number field, and pick an arithmetic point $P \in \text{Spec}(\mathbb{T})$ with $A_P$ satisfying (e2). Assume $|S| < \infty$ and that $P = (\varpi)$ is principal. If $\mathbb{III}(K^S / K, \hat{A}_v^{\text{ord}}[p\infty]))$ is finite, then we have an exact sequence

$$0 \rightarrow \mathbb{III}(K^S / K, \hat{A}_v^{\text{ord}}[p\infty])) \rightarrow \mathbb{III}_K(\varpi(\hat{J}_v^{\text{ord}})) \overset{\varpi}{\longrightarrow} \mathbb{III}_K(\varpi(\hat{J}_v^{\text{ord}}))$$

up to finite error, and $\mathbb{III}_K(\varpi(\hat{J}_v^{\text{ord}}))$ and $\mathbb{III}_K(\varpi(\hat{J}_v^{\text{ord}}))$ are both torsion $\mathbb{T}$-modules of finite type.

This proposition does not guarantee the finiteness of $\mathbb{III}_K(\hat{A}_v^{\text{ord}})$.

Proof. We look into the sheaf exact sequence of Corollary 6.4:

$$0 \rightarrow \hat{A}_v^{\text{ord}}[p\infty)] \rightarrow \varpi(\hat{J}_v^{\text{ord}}) \overset{\varpi}{\longrightarrow} \varpi(\hat{J}_v^{\text{ord}}) \rightarrow 0.$$

This produces a commutative diagram with two bottom exact rows:

$$
\begin{array}{cccc}
\mathbb{III}(K^S / K, \hat{A}_v^{\text{ord}}[p\infty])) & \longrightarrow & \mathbb{III}_K(\varpi(\hat{J}_v^{\text{ord}})) & \overset{\varpi}{\longrightarrow} & \mathbb{III}_K(\varpi(\hat{J}_v^{\text{ord}})) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(K^S / K, \hat{A}_v^{\text{ord}}[p\infty])) & \overset{i}{\longrightarrow} & H^1(K^S / K, \varpi(\hat{J}_v^{\text{ord}})) & \overset{\varpi}{\longrightarrow} & H^1(K^S / K, \varpi(\hat{J}_v^{\text{ord}})) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_S(\hat{A}_v^{\text{ord}}[p\infty])) & \overset{j}{\longrightarrow} & H^1_S(\varpi(\hat{J}_v^{\text{ord}})) & \overset{\varpi}{\longrightarrow} & H^1_S(\varpi(\hat{J}_v^{\text{ord}})).
\end{array}
$$

The kernel Ker$(j)$ is covered by $\prod_{v|p} \mathbb{E}_v^\infty(K_v)$ which is finite by Proposition 12.3 (as $E_v^\infty(K_v)$ for $v|p$ is finite by Theorem 17.2), and Ker$(i)$ is covered by $\mathbb{E}_v^\infty(K)$ which is finite by Corollary 12.4. Thus the top sequence is exact up to finite error by [BCM, Proposition I.1.4.2 (1)]. Then finiteness of $\mathbb{III}(K^S / K, \hat{A}_v^{\text{ord}}[p\infty]))$ tells us that $\mathbb{III}_K(\varpi(\hat{J}_v^{\text{ord}}))$ is a $\mathbb{T}$-torsion module of finite type.

We now look into the following sheaf exact sequence to relate $\mathbb{III}_K(\varpi(\hat{J}_v^{\text{ord}}))$ and $\mathbb{III}_K(\hat{J}_v^{\text{ord}})$:

$$0 \rightarrow \hat{A}_v^{\text{ord}} \rightarrow J_v^{\text{ord}} \rightarrow \varpi(\hat{J}_v^{\text{ord}}) \rightarrow 0.$$

Writing $X$ for one of the terms of the above exact sequence, we have a sheaf exact sequence $0 \rightarrow X[p^n] \rightarrow X \overset{p^n}{\longrightarrow} X \rightarrow 0$, which in turn produces the following commutative diagram with exact rows.
after passing to the limit with respect to $n$:

$$X(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1(K^S/K, X[p^\infty]) \longrightarrow H^1(K^S/K, X)$$

By the snake lemma, we get a short exact sequence:

$$0 \rightarrow X(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \text{Sel}_K(X) \rightarrow \text{III}_K(X) \rightarrow 0.$$ 

This produces the following diagram with exact rows:

$$\begin{align*}
\hat{A}^\text{ord}_{P}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow \text{Sel}_K(\hat{A}^\text{ord}_{P}) \longrightarrow \text{III}_K(\hat{A}^\text{ord}_{P}) \\
\cap & \quad \cap \\
J^\text{ord}_{\infty,T}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow \text{Sel}_K(J^\text{ord}_{\infty,T}) \longrightarrow \text{III}_K(J^\text{ord}_{\infty,T}) \\
\text{onto} & \quad \text{onto} \\
\varpi(J^\text{ord}_{\infty,T}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow \text{Sel}_K(\varpi(J^\text{ord}_{\infty,T})) \longrightarrow \text{III}_K(\varpi(J^\text{ord}_{\infty,T})).
\end{align*}$$

Up to finite error, the first column is exact by Corollary 12.7, and the second column is exact by Theorem 10.4. Thus by the snake lemma applied to first two rows, we conclude that

$$0 \rightarrow \text{III}_K(\hat{A}^\text{ord}_{P}) \rightarrow \text{III}_K(J^\text{ord}_{\infty,T}) \rightarrow \text{III}_K(\varpi(J^\text{ord}_{\infty,T}))$$

is exact up to finite error. Since $\text{III}_K(\hat{A}^\text{ord}_{P})$ and $\text{III}_K(\varpi(J^\text{ord}_{\infty,T}))$ are $\mathbb{T}$-torsion modules of finite type, we conclude that $\text{III}_K(J^\text{ord}_{\infty,T})$ is a $\mathbb{T}$-torsion module of finite type. 

**Theorem 13.3.** Suppose that $\mathbb{T}$ is a normal domain with infinite $\Omega_T$. Let $K$ be a number field. Assume $|S| < \infty$ and one of the following conditions for an arithmetic point $P_0 \in \text{Spec}(\mathbb{T})$ with $A_{P_0}$ satisfying (e2):

1. $\dim_{H_{P_0}}(A_{P_0}(K) \otimes \mathbb{Q}) \leq 1$ and $|\text{III}_K(\hat{A}^\text{ord}_{P_0})| < \infty$,
2. $\text{III}(K^S/K, A^\text{ord}_{P_0}[p^\infty])$ is finite.

Then for almost all $P = (\varpi) \in \Omega_T$, we have an exact sequence

$$0 \rightarrow \text{III}_K(\hat{A}^\text{ord}_{P}) \rightarrow \text{III}_K(J^\text{ord}_{\infty,T}) \rightarrow \text{III}_K(\varpi(J^\text{ord}_{\infty,T}))$$

up to finite error.

A description of the finite subset of points $P \in \Omega_T$ which does not give exact sequence as in the theorem is given after Theorem 12.8. This set could contain $P_0$ (and the sequence for $P_0$ may not be exact).

**Proof.** We have the following commutative diagram with exact columns:

$$\begin{align*}
\hat{A}^\text{ord}_{P}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow J^\text{ord}_{\infty,T}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow J^\text{ord}_{\infty,T}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \\
\cap & \quad \cap \\
\text{Sel}_K(\hat{A}^\text{ord}_{P}) & \longrightarrow \text{Sel}_K(J^\text{ord}_{\infty,T}) \longrightarrow \text{Sel}_K(J^\text{ord}_{\infty,T}) \\
\text{onto} & \quad \text{onto} \\
\text{III}_K(\hat{A}^\text{ord}_{P}) & \longrightarrow \text{III}_K(J^\text{ord}_{\infty,T}) \longrightarrow \text{III}_K(J^\text{ord}_{\infty,T}).
\end{align*}$$

Since the middle horizontal sequence is exact up to finite error by Theorem 10.4 (as long as $A_P$ satisfies (e2)), the exactness in the theorem is equivalent by the snake lemma to the exactness of the top sequence:

$$0 \rightarrow \hat{A}^\text{ord}_{P}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J^\text{ord}_{\infty,T}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow J^\text{ord}_{\infty,T}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0,$$
which follows from Theorem 12.8.

The combination of Theorem 13.3 and Proposition 13.2 proves

**Corollary 13.4.** Let the assumptions and the notation be as in the theorem. Then for almost all \( P \in \Omega_\tau \), \( \mathcal{W}_K(A_{\text{an}}^P) \) is finite.

### 14. Parameterization of congruent abelian varieties

Let \( B/\mathbb{Q} \) be a \( \mathbb{Q} \)-simple abelian variety of GL(2)-type (as in the introduction). We assume that \( O_B = \text{End}(B/\mathbb{Q}) \cap H_B \) is the integer ring of its quotient field \( H_B \). Then the compatible system of two-dimensional Galois representations \( \rho_B = \{ \rho_B, l \} \) realized on the Tate module of \( B \) has its \( L \)-function \( L(s, B) \) equal to \( L(s, f) \) for a primitive form \( f \in S_2(\Gamma_1(C)) \) for the conductor \( C = C_B \) of \( \rho_B \) (see [KW09, Theorem 10.1]). Thus \( B \) is isogenous to \( A_f \) over \( \mathbb{Q} \) (by a theorem of Faltings). The abelian variety \( A_f \) is known to be \( \mathbb{Q} \)-simple as \( A_H \) is generated by \( \text{Tr}(\rho_B(\text{Frob})) \) for primes \( l \) outside \( Np \). Let \( \pi_f \) be the automorphic representation of \( \text{GL}_2(\mathbb{A}) \) associated to \( f \).

Fix a connected component \( \text{Spec}(\mathcal{T}) \) of \( \text{Spec}(\mathcal{H}, \mathfrak{A}, \delta, \xi) \). If \( (\alpha, \delta, \xi) \neq (0, 1, \omega_\nu) \), for \( P \in \Omega_\tau \), the minimal (nearly ordinary) form \( f := f_P \) (in the sense of [H90, L1–3] and [H10, §1.1]) in \( \pi_f \) may not be primitive. We use the notation introduced in [H10, §1.1] for adelic automorphic forms without recalling its definition. Assume that \( P \) is principal (i.e., (A)) and \( f_P \) is on \( \hat{\Gamma}_r \). Then we define \( A_f = J_1(\mathcal{A}, \mathfrak{a}) \) as in (A). If \( H_{A_f} = H_A \), \( A_f = \mathbb{Q} \)-simple and is isogenous to \( A_f \).

**Lemma 14.1.** Let the notation be as above. If the conductor of \( f \) is divisible by \( Np \), the abelian variety \( A_f \) is isogenous to \( B \) over \( \mathbb{Q} \) and \( H_{A_f} = H_A = H_B \). If the conductor of \( f \) is equal to \( N \) prime to \( p \) and \( f(U(p)) = \varphi(p)f \), \( A_f \) is isogenous to \( B \otimes_{\mathbb{Q}} \text{Ob}[\varphi(p)] \) as abelian varieties of GL(2)-type, which is turn \( \mathbb{Q} \)-isogenous to \( B \times \mathbb{Q} \) just as abelian varieties.

**Proof.** Since \( a_t := \text{Tr}(\rho_B(\text{Frob})) \in H_{A_f} \) for all \( t \mid Np \), we have \( H_B \subset H_{A_f} \). Write \( \pi_f = \otimes_v \varphi_v \) and \( \varphi_p = \varphi(\varphi, \beta) \) or \( \sigma(\varphi, \beta) \) with \( p \)-adic unit \( i_p(\varphi(p)) \). Note that the \( f \) is characterized by

\[
(14.1) \quad f \in H^0(\hat{\Gamma}_1(Np^\nu), \pi) \subset S_2(\hat{I}_1(Np^\nu)), \quad f|T(l) = a_l f \quad \text{for all } l \mid Np, \quad f|U(p) = \varphi(p)f
\]

and

\[
(14.2) \quad \rho_f|_{I_p} \simeq \begin{pmatrix} \varphi_p & \ast \\ 0 & \varphi \end{pmatrix} \quad \text{with } \beta = |\cdot|^{-1}(i_p^{-1} \circ \psi) \quad (\psi \text{ has finite order over } I_p)
\]

for the inertia subgroup \( I_p \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), regarding \( \varphi, \psi \) as characters of \( I_p \) by local class field theory. Then \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/H_B) \) implies \( \rho_B \simeq \rho_B \simeq (\pi^{\infty})^\sigma \simeq \pi^{\infty} \), where \( \pi^{\infty} = \otimes_{l \leq \infty} \pi_l \). This shows the minimal field of definition of \( \pi^{\infty} \) is \( H_B \) (a result of Waldspurger), and by (14.2), \( H_B \) contains the values of \( \varphi |_{I_p} \). Thus \( H_{A_f} = H_B(\varphi) \) generated over \( H_B \) by the values of \( \varphi \), as the central character \( \varphi_p \) of \( \pi \) has values in \( H_B \) over \( \mathfrak{A}^{\infty} \) (which follows from the fact that \( \text{det} \rho_B = \psi \rho \nu \) for the compatible system \( \nu \) of the cyclotomic characters). Let \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). If \( \varphi \) or \( \beta \) is non-trivial over \( \mathbb{Z}_p^* \) or \( A_f \) is potentially multiplicative at \( p \) (i.e., the conductor of \( f \) is divisible by \( p \)), the nearly ordinary vector \( f \) is characterized by the above properties (14.1) without \( f|U(p) = \varphi(p)f \). Thus in this case, \( f^\nu \in \mathcal{P}_\nu = \pi^{\nu} \) for \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/H_B) \) implies \( f^\nu = f \). In particular, \( H_{A_f} = H_B \) as desired. If \( f \) has conductor \( N \), \( f \) is \( p \)-stabilized (i.e., \( f(z) = f(z) - \beta(p) f(pz) \)), then \( H_{A_f} = H_B(\varphi(p)) \). Since \( \varphi(p) \) satisfies \( X^2 - a_p X + \psi(p)(p) = 0 \) for the \( T(p) \) eigenvalue \( a_p \) of \( f \), we have \( |H_{A_f} : H_B| \leq 2 \), and \( A_f \) is isogenous to \( B \otimes_{\mathbb{Q}} \text{Ob}[\varphi(p)] \) (as an abelian variety of GL(2)-type).

If the central character \( \psi_p \) is trivial, \( H_B \) is totally real, and \( H_B(\varphi(p)) \) is totally imaginary; so, \( A_f \) is isogenous to \( B \times B \) if the conductor of \( B \) is prime to \( p \). Even if the central character is not trivial, choosing a square root \( \zeta := \sqrt{\psi(p)} \), \( T(p) \zeta^{-1} \) is self adjoint on \( S_2(\Gamma_0(N), \psi_p) \) (e.g., [MFM, Theorem 4.5.4]), and hence \( a_p \zeta^{-1} \) is totally real, but for the root \( \varphi(p) \zeta^{-1} \) of \( X^2 - a_p \zeta^{-1} X + p \),
Let $A$ be another $\mathbb{Q}$-simple abelian variety of $GL(2)$-type. Thus $A$ is isogenous to $A_q$ for a primitive form $g \in S_2(\Gamma_1(CA))$ of conductor $CA$. Let $\pi_q$ be the automorphic representation of $g$, and write $\mathfrak{g}$ for the minimal nearly $p$-ordinary form in $\pi_q$. Without losing generality, we may (and do) assume that $O_A = \text{End}(A/\mathbb{Q}) \cap H_A$ is the integer ring of $H_A$. Note that $H_B \cong \mathbb{Q}(f) \subset \mathbb{Q}$ and $H_A \cong \mathbb{Q}(g)$. Suppose $A$ is congruent to $B$ modulo $p$ with $(B[\mathbb{Z}_p] \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^{ss} \cong (A[\mathbb{Z}_p] \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^{ss}$ as $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. Here, for any ring $R$ and a prime ideal $p$ of $R$, $\kappa(p)$ is the residue field of $p$.

Write $O_{p,a}$ for the $p_a$-adic completion of $O_A$, and let $T_{p,a} = \lim_{\leftarrow, p} A[p_a^r](\overline{\mathbb{Q}}_p)$ (the $p_a$-adic Tate module of $A$). We call that $A$ is of $p_a$-type $(\alpha, \delta, \xi)$ if we have an exact sequence of $I_p$-modules

$$0 \to V(\nu e^{-\delta}, \xi^{-1}) \to T_{p,a} \to V(e^\alpha, \xi^{-1}) \to 0$$

with $V(\nu e^{-\delta}, \xi^{-1}) \cong V(e^\alpha, \xi^{-1}) \cong O_{p,a}$ as $O_{p,a}$-modules, where $\epsilon$ is a character of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$ with values in $\mu_{p\infty}$, $u \in \mathbb{Z}_p^\times$ (resp. $[\zeta, \mathbb{Q}_p]$ for $\zeta \in \mu$) acts on $V(\nu e^{-\delta}, \xi^{-1})$ by $u^{-1} \cdot \epsilon^{-u}(u)$ (resp. by $\xi^{-1}(1, \zeta)$) and on $V(e^\alpha, \xi^{-1})$ by $\epsilon(u)^\alpha$ (resp. by $\xi^{-1}(1, \zeta)$). Here $[\zeta, \mathbb{Q}_p]$ is the local Artin symbol. If $\xi(\zeta, \zeta') = \xi(\xi)$ for $(\zeta, \zeta') \in \mu^2$ and $\alpha = 0$, this is just a $p_a$-ordinarity.

Choosing $g$ (resp. $f$) well in the Galois conjugacy class of $g$ (resp. $f$), we may assume that $\mathfrak{p}_A$ and $\mathfrak{p}_B$ are both induced by the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

**Lemma 14.2.** Let the notation be as above. Suppose that $CA/CB$ is in $\mathbb{Z}[\frac{1}{\delta}]^\times$ and that $B$ (resp. $A$) is of $\mathfrak{p}_B$-type (resp. $\mathfrak{p}_A$-type) $(\alpha, \delta, \xi)$. Write $CB = \mathbb{N}p^\infty$. Then there exists a connected component $\text{Spec}(\mathbb{C}[\alpha, \delta, \xi](N))$ such that for some primes $P, Q \in \text{Spec}(\mathbb{T})$, $f = f_p$ and $g = f_q$.

**Proof.** Let $T$ be the two dimensional Galois representation into $GL_2(\mathbb{F})$ realized on $B[\mathbb{Z}_p]$ for $\mathbb{F} = Ob/\mathfrak{p}_B$. Write $N$ for the prime-to-$p$ part of $CA$ (and hence of $CA$). Replacing $T$ by its semi-simplification, we may assume that $T$ is semi-simple. Since $(B[\mathbb{Z}_p] \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^{ss} \cong (A[\mathbb{Z}_p] \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^{ss}$, $L(s, A) = L(s, g)$ and $L(s, B) = L(s, f)$ imply $f$ mod $\mathfrak{p}_B = g$ mod $\mathfrak{p}_B$. Since $f_B := f$ is nearly $p$-ordinary with nearly ordinary character given by $[u_\zeta, \mathbb{Q}_p] \mapsto \epsilon_B(u)^\zeta\xi^{-1}(1, \zeta)$ ($u \in \Gamma$ and $\zeta \in \mu$) for a character $\epsilon_B : \mathbb{Z}_p^\times \to \mu_{p\infty}$, $u \in \mathbb{Z}_p^\times$ (resp. $[\zeta, \mathbb{Q}_p]$ for $\zeta \in \mu$) acts on $V(\nu e^{-\delta}, \xi^{-1})$ by $u^{-1} \cdot \epsilon^{-u}(u)$ (resp. by $\xi^{-1}(1, \zeta)$) and on $V(e^\alpha, \xi^{-1})$ by $\epsilon(u)^\alpha$ (resp. by $\xi^{-1}(1, \zeta)$). Here $[\zeta, \mathbb{Q}_p]$ is the local Artin symbol. If $\xi(\zeta, \zeta') = \xi(\xi)$ for $(\zeta, \zeta') \in \mu^2$ and $\alpha = 0$, this is just a $p_a$-ordinarity.

Choosing $g$ (resp. $f$) well in the Galois conjugacy class of $g$ (resp. $f$), we may assume that $\mathfrak{p}_A$ and $\mathfrak{p}_B$ are both induced by the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

**Corollary 14.3.** Let the notation and the assumptions be as in Lemma 14.2 and Theorem 5.3 (in particular, we assume (F)). Assume that the abelian variety $B$ has conductor $N$ prime to $p$. Let $f \in S_2(\Gamma_0(N), \chi)$ be the primitive form with conductor $N$ prime to $p$ (so, $\xi = 1$) whose $L$-function is totally imaginary as with $|\alpha| \leq 2\sqrt{p}$ combined with $|\beta| < |\varphi(p)| = 1$. This shows that $H_A$ is a quadratic extension of $H_B$, and hence $A_f$ is isogenous to $B \times B$. \qed
gives \( L(s, B) \). Write \( \chi | \cdot |^{-1} \) for the central character of the automorphic representation generated by \( f \). Write \( f(T(p)) = a_p f \). If \( p \nmid 6D_N \varphi(N) \) and \( (a_p \mod \mathfrak{p}_B) \notin \Omega_{B,p} := \{0, 1, \pm \sqrt{\chi(p)}\} \), then \( \mathfrak{T} \) is a regular integral domain and \( \mathfrak{g} \) and \( \mathfrak{g} \) belongs to \( \text{Spec}(\mathfrak{T}) \).

Again we can replace the condition: \( p \nmid 6D_N \varphi(N) \) by \( p \nmid 2D_N \varphi(N) \) in the case where \( h_{\alpha, \delta, \xi}(N) \) is \( \Lambda \)-free (see Proposition 18.2 for such cases).

15. A GENERALIZED VERSION OF THEOREM B INCLUDING EXOTIC TOWERS

Let \( B_{\mathbb{Q}} \) be a \( \mathbb{Q} \)-simple abelian variety of \( \text{GL}(2) \)-type of conductor \( N \) such that \( O_B = \text{End}(B_{\mathbb{Q}}) \cap H_B \) is the integer ring of its quotient field \( H_B \). Let \( \rho_B = \{ \rho_B, \iota \} \) be the two dimensional compatible system of Galois representations associated to \( B \). Then \( \rho_B \) comes from a Hecke eigenform \( f = \sum_{n=1} a_n q^n \in S_2(\Gamma_0(N), \chi) \) by [KW09, Theorem I.10.1]: so, \( L(s, B) = L(s, \rho_B) = L(s, f) \). Fix an embedding \( \mathfrak{O}_B \hookrightarrow \mathbb{Q} \) and write \( \mathfrak{p}_B \) for the prime ideal of \( O_B \) induced by \( \iota_p : \mathbb{Q} \hookrightarrow \mathfrak{O}_p \). Then we realize the Hecke algebra \( h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) \) inside \( \text{End}_\mathbb{C}(S_2(\Gamma_0(N), \chi)) \) which is generated over \( \mathbb{Z}[\chi] \) by all Hecke operators \( T(n) \) and \( U(l) \). Then this Hecke algebra is free of finite rank over \( \mathbb{Z} \), and hence its reduced part (modulo the nilradical) has a well defined discriminant \( D_\chi \) over \( \mathbb{Z} \).

**Definition 15.1.** Let \( S = S_B \) be the set of prime factors of \( 6D_N \varphi(N) \) for the conductor \( N \) of \( \rho_B \), where \( D_\chi \) is the discriminant of the reduced part of \( h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) \).

We could include \( p = 3 \) defining \( S = S_B \) to be the set of prime factors of \( 2D_N \varphi(N) \) if \( h_{\alpha, \delta, \xi} \) is \( \Lambda \)-free (see remarks after Proposition 4.1 and see also Proposition 18.2).

The prime \( p \) is admissible for \( B \) if \( B \) has good reduction modulo \( p \) (so, \( p | N \) and \( (a_p \mod \mathfrak{p}_B) \notin \Omega_{B,p} := \{0, 1, \pm \sqrt{\chi(p)}\} \) (so, \( B \) has potential partially \( \mathfrak{p}_B \)-ordinary reduction modulo \( p \)) and write \( \mathfrak{p}_B \)-type of \( B \) as \( (\alpha, \delta, 1) \). Since \( B \) has conductor prime to \( p \), \( \rho_B \) is unramified at \( p \), and \( \xi \) has to be the identity character \( 1 \) of \( \mu \times \mu \) (on the other hand, \( (\alpha, \delta) \) can be freely chosen). We prove the following result more general than Theorem B including abelian varieties \( A \) of \( \mathfrak{p}_A \)-type \( (\alpha, \delta, 1) \) (not just those \( \mathfrak{p}_A \)-ordinary ones):

**Theorem 15.2.** Assume (F) for \( (\alpha, \delta, 1) \), and let \( K \) be a number field. Let \( p \notin S_B \) be a prime admissible for \( B \) and \( N \) be the conductor of \( B \). Suppose that \( B \) is isogenous to \( A_{P_0} \), and \( |\text{End}_K(B)^{\text{ord}}_p| < \infty \) and \( \text{dim}_{H_B} B(K) \otimes \mathbb{Q} \leq 1 \). Consider the set \( A_{B,p} \) made up of all \( \mathbb{Q} \)-isogeny classes of \( \mathbb{Q} \)-simple abelian varieties \( A_{\mathbb{Q}} \) of \( \mathfrak{p}_A \)-type \( (\alpha, \delta, 1) \) congruent to \( B \) modulo \( p \) over \( \mathbb{Q} \) with prime-to-\( p \) conductor \( N \). Then, almost all members \( A \in A_{B,p} \) have finite \( \text{End}_K(A)^{\mathfrak{p}_A} \) and \( \text{dim}_{H_B} A(K) \otimes \mathbb{Q} \) is a constant independent of \( A \) given by \( 0 \) or \( 1 \). If further \( B^{\text{ord}} = A_{P_0}^{\text{ord}} \) for \( P_0 \in \Omega_T \) with \( \text{Sel}_K(B)^{\mathfrak{p}_B} = 0 \) and all prime factors of \( p \) in \( K \) has residual degree \( 1 \), then \( \text{Sel}_K(A)^{\mathfrak{p}_A} \) is finite for all \( A \in A_{B,p} \) without exception.

As is well known, there are density one (partially) ordinary admissible primes in \( O_B \) if \( B \) does not have complex multiplication (e.g., [H13b, Section 7])

**Proof.** Suppose that \( p \) is outside \( S_B \), by Theorem 5.3, \( T \) is a regular integral domain \( \mathbb{I} \). Thus for any \( P \in \Omega_T \), we have \( P = (\varnothing) \) for \( \varnothing \in \mathbb{I} \) and \( (\varnothing, A_P) \) satisfies (A).

Since \( B[p_B]^{\infty} \) is an ordinary Barsotti–Tate group by our assumption, \( A[p_B^{\infty}] \) is potentially ordinary by the congruence modulo \( p \) between \( A \) and \( B \). Here we say \( A[p_B^{\infty}] \) “potentially ordinary” if \( H_0(k, A[p_B^{\infty}]((-\mathfrak{p}_B^\infty))) \) has non-trivial \( p \)-divisible rank and \( A[p_B^{\infty}] \) over \( \mathbb{Q}_p \) extends to a Barsotti–Tate group with non-trivial étale quotient over the integer ring of a finite extension \( k \) of \( \mathbb{Q}_p \). Choosing the embedding \( O_A \hookrightarrow \overline{\mathbb{Q}} \) well, we may assume that \( \mathfrak{p}_A \) is induced by \( i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p} \). Then by Lemma 14.2, \( A \) is isogenous to a modular abelian variety \( A_P \) for \( P \in \Omega_T \) of a connected component \( \text{Spec}(\mathbb{T}) \) of \( \text{Spec}(h_{\alpha, \delta, \xi}(N)) \) for the big \( p \)-adic Hecke algebra \( h_{\alpha, \delta, \xi}(N) \). Since \( B \) is of \( \text{GL}(2) \)-type, we have \( B \cong A_{P_0} \) (an isogeny) for \( P_0 \in \Omega_T \) with \( P_0 = (\varnothing_0) \). Thus we conclude, up to isogeny,

\[
A_{B,p} = \{ A_\mathbb{Q} | Q \in \Omega_T \}
\]

by the theorem of Khare–Wintenberger [KW09, Theorem I.10.1] (combined with the proof of the Tate conjecture for abelian varieties by Faltings). Since \( O_A \) is the integer ring of \( H_A \), we can factor
$O_{A,p} = O_A \otimes \mathbb{Z} \mathbb{Z}_p$ into the product $O_{A,p} = O_{A,p}^{\text{ord}} \oplus O_{A,p}^\circ$ so that for the idempotent $e$ of the factor $O_{A,p}^{\text{ord}}$, $eA[p^{\infty}]$ is the maximal $p$-ordinary Barsotti–Tate group which becomes étale and multiplicative after étale extension. Since $O_{A,p}$ acts on $\hat{A}$, we can define $\hat{A}^{\text{ord}} := e(\hat{A})$. Since $A$ is isogenous to $A_p$, $A^{\text{ord}}$ is isogenous to $A_{p}^{\text{ord}}$, so, $\Pi_K(A^{\text{ord}})$ is isogenous to $\Pi_K(A_{p}^{\text{ord}})$. Then if $\Pi_K(\hat{A}^{\text{ord}})$ is finite, $\Pi_K(\hat{A}_{p}^{\text{ord}})$ is finite as it is isogenous to the finite $\Pi_K(\hat{A}^{\text{ord}})$. Since $\Pi_K(\hat{A}_{p}^{\text{ord}}) \subset \Pi_K(\hat{A}_{p}^{\text{ord}})$, by Corollary 13.4, we conclude finiteness of $\Pi_K(\hat{A}_{p}^{\text{ord}})$ for almost all $A \in A_{B,p} \equiv \Omega$. The assertion for the Mordell–Weil rank follows from Corollary 12.5.

Suppose $\text{Sel}_K(\hat{A}^{\text{ord}}) = 0$ and $K_r$ for all $v|p$ has residue field $\mathbb{F}_p$. Then $|\varphi(Frob_v) - 1|_p = |a_p - 1|_p = 1$ as $p \not\in \Omega_{B,p}$. Thus by Schneider [Sc83, Proposition 2, Lemma 3] (see also [Sc82, Proposition 2]), we have, for all $v|p$,

$$|H^1(K_v[\mu_{p^\infty}]/K_v, \hat{A}_{p}^{\text{ord}}(K_v[\mu_{p^\infty}]| = |\hat{A}_{p}^{\text{ord}}(\mathbb{F}_p)|^2 = |\hat{A}_v(\mathbb{F}_p)|^2. \quad (15.1)$$

Note that $|\hat{A}_v(\mathbb{F}_p)|^2 = 1$ by our assumption. Strictly speaking, Schneider assumes in [Sc83, §7] that $A_r$ has ordinary good reduction, but his argument works well without change replacing $(A_r(p) := A_r[p^{\infty}], A_r)$ there by $(A_r[p^{\infty}], \hat{A}_{r}^{\text{ord}})$. Indeed, he later takes care of the general case of formal Lie groups in [Sc87, Theorem 1] (including the case of the ordinary part of the formal group of $A_r$). So, $E^{\infty}(K_v)[\tau] = E_{\text{Sel}}(K_v)[\tau] = 0$ for all $v|p$ (see Theorem 17.2 and Corollary 10.3 for more details of this fact). Then by the same argument as above, using Corollary 10.5 (2) in place of Corollaries 12.5 and 13.4, we conclude $\text{Sel}_K(A^{\text{ord}})_{p} \equiv \Omega$ for all $P \in \Omega$.

\begin{remark}
If we start with an elliptic curve $E$ as in Theorem B, by its modularity, we find a modular factor $B \subset J_1(Np^r)$ isogenous to $E$. Choose $(\alpha, \delta, 1) = (0, 1, 1)$. The finiteness of $\Pi_K(E)_p$ implies the finiteness of $\Pi_K(B)$; so, the above theorem implies the statement of Theorem B. The rational elliptic curves listed in Corollary C give examples for such curves with Mordell–Weil $\mathbb{Q}$-rank 0 and finite Tate–Shafarevich group, and the elliptic curve factor of $J_0(37)$ with root number $-1$ give a Mordell–Weil $\mathbb{Q}$-rank 1 example with finite $\Pi_{K}(E)$.

Here is a conjecture:

\begin{conjecture}
Suppose and $\xi(a, a) = 1$ for all $a \in \mathbb{Z}_p^\times$. Fix a totally real field $K$. Let $\text{Spec}(\mathbb{I})$ be a primitive irreducible component of $\text{Spec}(\mathbb{H})$. If $\alpha/\delta = 1$, we assume that the root number is $\varepsilon := \pm 1$ for $K$. Then,

1. if $\alpha/\delta = 1$, we have $\dim H^r_{\text{et}} A_p(K) \otimes \mathbb{Q} \equiv \frac{1}{2} \xi(a, a)$ for almost all $P \in \Omega$,
2. if $\alpha/\delta \not= 1$, we have $\dim H^r_{\text{et}} A_p(K) \otimes \mathbb{Q} \equiv 0$ for almost all $P \in \Omega$.

As we remarked after stating Theorem A, if we could prove $\dim H^r_{\text{et}} A_p(K) \otimes \mathbb{Q} \equiv \frac{1}{2} \xi(a, a) \mod 2$ for almost all $P \in \Omega$, Conjecture 15.4 (1) holds once we find a good point $P_0$ with $A_{P_0}$ satisfying the assumptions of Theorem 15.2.

\end{conjecture}

16. $p$-Local cohomology of formal Lie groups

We prove a technical lemma on Galois cohomology for proving vanishing of the error terms when $l = p$ in Theorem 17.2. Just for finiteness of the error term, as will be explained in the proof of the theorem, it follows from the computation of the universal norm by P. Schneider in [Sc83, Proposition 2 and Lemma 3, §7] and [Sc87, Theorem 1], and therefore, perhaps, for the first reading, the reader may want to skip this section.

Let $K$ be a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. Write $K_s = K[\mu_{p^r}]$ and $X^{ur}$ for the maximal unramified extension of $X = K, K_s$ and $\hat{X}^{ur}$ is the completion of $X^{ur}$. Let $A$ be an abelian variety defined over $K$. Suppose that $\text{End}(A/K)$ contains a reduced commutative algebra $O_A$. Assume

1. $A/K$ has semi-stable reduction over the integer ring $W_r$ of $K_r$;
2. The formal Lie group of the Néron model of $A$ over $W_r$ has a maximal multiplicative factor $A$ (see [Sc87, §1] for the maximal multiplicative factor);
(A3) Writing $O_A$ for the $p$-adic closure of the image of $O_A$ in $\text{End}(A_{/W_r})$, we have $A \cong \hat G_m \otimes_{\Bbb Z_p} \frak A$ over $\hat W_r^{ur}$ as formal $O_A$-modules, where $\frak A$ is an $O_A$-lattice in $O_A \otimes_{\Bbb Z_p} \Bbb Q_p$ (i.e., $\frak A \otimes_{\Bbb Z_p} \Bbb Q_p \cong O_A \otimes_{\Bbb Z_p} \Bbb Q_p$) and $\hat W_r^{ur}$ is the $p$-adic completion of the integer ring $W_r^{ur}$ of $K^{ur}$.

We now study the $\text{Gal}(K_s/K)$-module structure and the cohomology of $A(W_s)$. The Barsotti-Tate group $\hat A^{\text{ord}}[p^{\infty}] / O_p$ has a filtration $A[p^{\infty}] \hookrightarrow A^{\text{ord}}[p^{\infty}] \rightarrow A^{\text{ord}}[p^{\infty}]_{\text{pet}}$, where $A^{\text{ord}}[p^{\infty}]_{\text{pet}}$ becomes unramified over $\Bbb Q_p[\mu_p]$. On $T_p A[p^{\infty}]$, $\text{Gal}(K^{ur}/K^{ur})$ acts by a character $\nu_p \psi$ with values in $O_{\hat A}$, where $\nu_p$ is the $p$-adic cyclotomic character. The character $\psi$ factors through $\text{Gal} (K^{ur}/K^{ur}) \cong \text{Gal} (\Bbb Q_p[\mu_p]/\Bbb Q_p)$. Identifying $\psi$ with the corresponding character of $\text{Gal}(\Bbb Q_p[\mu_p]/\Bbb Q_p)$, we twist the Galois action on the group functor $R \mapsto A(R)$ so that

$$\sigma \cdot x := \psi^{-1}(\sigma[\Bbb Q_p[\mu_p]]) \sigma(x) \quad (16.1)$$

for $\Bbb Q_p[\mu_p]$-algebras $R$, where $\sigma \in \text{Aut}(R/\Bbb Q_p)$. Since $\psi(\sigma)^{-1} \in \text{Aut}(A[\Bbb Q_p[\mu_p]])$ gives a descent datum (see [GME, §1.11.3, (DS2)]), we can twist $A$ by this cocycle, and get another abelian variety $A_{\mu/\Bbb Q_p}$ (see [Mi72, (a)]).

Similarly, on $T_p A[p^{\infty}]_{\text{pet}}$, $\text{Gal}(K^{ur}/K^{ur}) \cong \text{Gal}(\Bbb Q_p[\mu_p]/\Bbb Q_p)$ acts by a character $\nu_p \psi$ with values in $O_{\hat A}$. Identifying $\nu_p \psi$ with the corresponding character of $\text{Gal}(\Bbb Q_p[\mu_p]/\Bbb Q_p)$, via the new action $\sigma \cdot x := \varphi^{-1}(\sigma[\Bbb Q_p[\mu_p]]) \varphi(x)$, we get another abelian variety $A_{\nu/\Bbb Q_p}$. Thus the Galois action on $A_{\nu/\Bbb Q_p}[p^{\infty}]_{\text{pet}}$ is unramified over $\Bbb Q_p$.

For a scheme $X_s / S'$ and finite flat morphism $S' \rightarrow S$, we write $\text{Res}_{S'/S} X$ for the Weil restriction of scalars; so, $\text{Res}_{S'/S} X$ is a scheme over $S$ such that $\text{Res}_{S'/S} X(T) = X(S' \times S T)$ for all $S$-schemes $T$. We describe the twisted abelian variety $A_?^{(\nu)}$ (?= et) as a factor of $\text{Res}_{K_s/K} A$. Here is a known facts from [NMD, §7.6]:

(Res1) If $S' / S$ is finite flat, $\text{Res}_{S'/S} X$ exists [NMD, Theorem 4],

(Res2) If $X$ is a separated scheme over $S$, the natural map $X \rightarrow \text{Res}_{S'/S}(X \times S S')$ corresponding to the projection $T \times S S' \rightarrow T$ is a closed immersion [NMD, page 197],

(Res3) If $X \hookrightarrow Y$ is a closed immersion, then $\text{Res}_{S'/S} X \rightarrow \text{Res}_{S'/S} Y$ is a closed immersion,

(Res4) Let $k'/k$ be a finite extension of fields. If $X_{k'}$ for a field $k'$ is an abelian scheme with Néron model $\tilde X_{O'}$ for a discrete valuation ring $O'$ with quotient field $k$, $\text{Res}_{O'/O} \tilde X$ is the Néron model of $\text{Res}_{k'/k} X$ [NMD, Proposition 6].

Let $\text{Res}_{K_s/K} A$ be the restriction of scalars. Since $A_? \cong A \cong A_{\nu/\Bbb Q_p}$ over $W_r$, we find $\text{Res}_{K_s/K} A \cong \text{Res}_{K_s/K} A_{\nu/\Bbb Q_p} \cong \text{Res}_{K_s/K} A_{\nu/\Bbb Q_p}$. Since $\text{Res}_{K_s/K} A(R) = A(R \otimes_K K_r^*)$ for each $K$-algebra $R$, the inclusion $R \hookrightarrow R \otimes_K K_r^*$ given by $x \mapsto x \otimes 1$ produces a monomorphism of covariant functors $A(R) \rightarrow \text{Res}_{K_s/K} A(R)$; so, we have a morphism of schemes (by Yoneda's lemma), $A \rightarrow \text{Res}_{K_s/K} A$. Since $A$ and $\text{Res}_{K_s/K} A$ are projective, we find that $A \rightarrow \text{Res}_{K_s/K} A$ is a closed immersion. In the same way, we have another closed immersion $A_{\nu/\Bbb Q_p} \rightarrow \text{Res}_{K_s/K} A_{\nu/\Bbb Q_p} \cong \text{Res}_{K_s/K} A$.

Since $K_r^* \otimes_K K_r^* \cong \prod_{\sigma \in \text{Gal}(K_r/K)} K_r^*$ by sending $x \otimes y$ to $(x \sigma(y))_\sigma$, for any variety $X$ defined over $K_r^*$, we have $\text{Res}_{K_s/K} X \cong \prod_{\sigma \in \text{Gal}(K_r/K)} X^\sigma$, where $X^\sigma = X \otimes_K \sigma K_r^*$. Thus $\tau \in \text{Gal}(K_r/K)$ acts on $\text{Res}_{K_s/K} X$ by a permutation: $x \mapsto (x_\sigma)_\sigma \mapsto \tau \cdot x := (x_{\sigma \tau})_\sigma$, and $\text{Gal}(K_r/K) \hookrightarrow \text{Aut}(\text{Res}_{K_s/K} X)$. Thus $O_A[\text{Gal}(K_r/K)] \subset \text{End}(\text{Res}_{K_s/K} A_{\nu/\Bbb Q_p})$ by embedding $\text{Gal}(K_r/K)$ in this way. For $x = (x_\sigma)_\sigma \in \text{Res}_{K_s/K} X(\overline{\Bbb Q}_p)$, we have $x^\tau = \tau [K_r^*/(x_\sigma^\tau)]$. Then the image of $A$ in $\text{Res}_{K_s/K} A_{\nu/\Bbb Q_p}$ is given by $1_\psi(\text{Res}_{K_s/K} A_{\nu/\Bbb Q_p})$, where $1_\psi = [K_r^*/K_r^*]^{-1} \sum_\sigma \psi^{-1}(\sigma) \sigma \in O_A[\text{Gal}(K_r/K)]$. Since $\tau \in \text{Gal}(\overline{\Bbb Q}_p/K)$ acts on $x \in \text{Res}_{K_s/K} A_{\nu/\Bbb Q_p}$ by $x \mapsto (x_\sigma^\tau)_\sigma$, writing the Galois action on $A_{\nu/\Bbb Q_p}$ as $x \mapsto x^\tau$ the action of $\sigma \in \text{Gal}(\overline{\Bbb Q}_p/K)$ on $x \in A(\overline{\Bbb Q}_p)$ is $x \mapsto \psi^\tau(\sigma[K_r^*/(x_\sigma^\tau)])$, where $\psi(\sigma[K_r^*])$ is regarded as an automorphism of $A_{\nu/\Bbb Q_p}$. By the same argument, writing the Galois action on $A_{\mu/\Bbb Q_p}$ as $x \mapsto x^\mu$, the action of $\sigma \in \text{Gal}(\overline{\Bbb Q}_p/K)$ on $x \in A(\overline{\Bbb Q}_p)$ is $x \mapsto \psi^\mu(\sigma[K_r^*/(x_\sigma^\mu)])$. Since $\text{Gal}(\overline{\Bbb Q}_p/K)$ unramified, and the action of $\text{Gal}(K_\infty/K)$ on $A_{\nu/\Bbb Q_p}$ is via the $p$-adic cyclotomic character. Here $A_{
u/\Bbb Q_p}^{\text{ord}}$ is the formal Lie group whose Barsotti–Tate group is the potentially connected part of the Barsotti–Tate group of $A_{\nu/\Bbb Q_p}$. This formal Lie group descends to $W_r$ and is isomorphic $\hat G_m \otimes_{\Bbb Z_p} \frak A$ over $\hat W_r^{ur}$ for the integer ring $W_r$ of $K$. Thus we have an identity $A \cong \hat G_m \otimes_{\Bbb Z_p} \frak A(\psi) \otimes \hat W_r^{ur}$ and $A[p^{\infty}] \cong \mu_{p^{\infty}} \otimes_{\Bbb Z_p} \frak A(\psi) \otimes \hat W_r^{ur}$, where $\frak A(\psi) \cong \frak A$ as $O_A$-modules on which $\text{Gal}(K_\infty/K)$ acts by
ψ. Note that the second identity is valid over $W_r^{nr}$ as this is the identity of Barsotti–Tate groups. From this, we get

**Lemma 16.1.** Assume $p > 2$. Let $a \in O_A$ be given by the action of Frob. Then we have, for $s \geq r$,

$$H^1(\text{Gal}(K_s/K), \mathcal{A}[p^s](W_s)) \cong (\mathcal{A}/(p^{s-1}, \nu_p\psi(\sigma) - 1)\mathcal{A})[a - 1]$$

which is finite and bounded independent of $s \geq r$.

*Proof.* The Frobenius element Frob acts on $\mathcal{A}[p^s]$ via multiplication by $a$. Note that, for $s \geq r$

$$\mathcal{A}[p^s](W_s) = (\mu_{p^{s-r}}(W_s^{nr}) \otimes \mathbb{Z} \mathcal{A}(\psi))[a - 1]$$

as $\text{Gal}(K_\infty/K)$-modules. Since $\text{Gal}(K_\infty/K)$ acts on $\mu_{p^{s-r}}$ by $\nu_p$, we conclude

$$H^1(\text{Gal}(K_s/K), \mathcal{A}[p^s](W_s)) \cong (\mathcal{A}/(p^{s-1}, \nu_p\psi(\sigma) - 1)\mathcal{A})[a - 1]$$

as desired. \qed

17. **Finiteness of the $p$-local error term**

We assume (F) and $p > 2$. Here $K_{/\mathbb{Q}_p}$ is a finite extension with $p$-adic integer ring $W$. Put $K_s = K[\mu_{p^s}]$ with integer ring $W_s$.

We studied the $\Lambda$-BT group $G_{1,0,\omega_d}$ associated to the tower $\{X_1(Np^r)\}_r$ in [H14, §5], which is defined over $\mathbb{Z}_p[\mu_{p^\infty}]$. Here $\omega_d(a, d) = \omega(d)$. For the general tower $\{X_r\}_r$ determined by the fixed data $(\alpha, \delta, \xi)$, $J_r$ is a factor of $\text{Res}_{\mathbb{Q}_p/\mathbb{Q}}J_1(Np^r)$ again over $\mathbb{Q}[\mu_{p^r}]$ if $r \geq s$, since $F_\xi \subset \mathbb{Q}[\mu_{p^r}]$. Thus taking the tower of regular model $X_r/\mathbb{Z}_p[\mu_{p^s}]$ made out of the regular model $X_1(Np^r)/\mathbb{Z}_p[\mu_{p^s}]$ (via the corresponding Weil restriction of scalars) and considering $J_r/\mathbb{Z}_p[\mu_{p^s}] := \text{Pic}_K^{1,\infty}(X_r/\mathbb{Z}_p[\mu_{p^s}])$, over $\mathbb{Z}_p[\mu_{p^s}]$, $G = G_{\alpha, \delta, \xi, \omega_d}[\mathbb{Z}_p[\mu_{p^s}]] := J_{\infty}^{\text{ord}}(\mathbb{Z}_p[\mu_{p^s}])$ is a $\Lambda$-direct factor of $G_{1,0,\omega_d}^{[F_\xi: \mathbb{Q}_p]}$. Thus $G_{\alpha, \delta, \xi, \omega_d}[\mathbb{Z}_p[\mu_{p^s}]]$ is a $\Lambda$-BT group in the sense of [H14, §3] (replacing (CT) by (ct) in [H14, Remark 5.5] if $\xi = 1$). Though it is assumed that $p > 3$ in [H14, §3], the result there is valid for $p = 2, 3$. This is because the ordinary or nearly ordinary part is trivial if $N_p \leq 3$ (and the assumption $p > 3$ is imposed to have $N_p \geq 4$ for the representability of the elliptic moduli problem). We take its connected component $G_{\alpha, \delta, \xi, \omega_d}^{\circ}[\mathbb{Z}_p[\mu_{p^s}]]$ and put $G_{\alpha, \delta, \xi, \omega_d}^{\circ}[\mathbb{Z}_p[\mu_{p^s}]] = G_{\alpha, \delta, \xi, \omega_d}[\mathbb{Z}_p[\mu_{p^s}]] - 1$ which is a connected Barsotti–Tate group defined over $\mathbb{Z}_p[\mu_{p^s}]$. Write $G_{\alpha, \delta, \xi, \omega_d}^{\circ}[\mathbb{Z}_p[\mu_{p^s}]]$ for the formal Lie group associated to the connected Barsotti–Tate group $G_{\alpha, \delta, \xi, \omega_d}^{\circ}[\mathbb{Z}_p[\mu_{p^s}]]$ [GME, 1.13.5].

We put $G_{\alpha, \delta, \xi, \omega_d}^{\circ}[\mathbb{Z}_p[\mu_{p^s}]] = \lim_{\longrightarrow \bigg\rarr} G_{\alpha, \delta, \xi, \omega_d}^{\circ}$, where the projection $G_{\alpha, \delta, \xi, \omega_d}^{\circ} \rightarrow G_{\alpha, \delta, \xi, \omega_d}^{\circ}$ is induced by the natural trace map $\pi_{s^r}^* : G_{s^r} \rightarrow G_{s^r}$ for $s' > s$. We study $\text{Coker}(F_{\text{ord}}(K) \rightarrow F_{\text{ord}}(K))$. Identify $F_{\text{ord}}(K) = \mathbb{Z}_p[\mu_{p^s}]$ with exact rows:

\[
0 \rightarrow A_s \rightarrow G_s \rightarrow G_s/A_s \rightarrow \mathcal{A}(G_s) \rightarrow 0
\]

This produces the following commutative diagram of formal Lie groups over $\mathbb{Z}_p[\mu_{p^{s-r}}]$ with exact rows:

\[
A_s \overset{\pi_{s^r}^*}{\longrightarrow} G_s \overset{\pi_{s^r}^*}{\longrightarrow} G_s/A_s \rightarrow \mathcal{A}(G_s)
\]

Since $G_s/A_s$ is a Barsotti–Tate group over $W_s$ by [H14, Theorem 5.4], $G_s/A_s$ is a smooth formal group over $W_s$ (e.g., [Sc87, Lemma 1]). Thus $G_s \cong (G_s/A_s) \times W_s A_s$ as formal schemes (but not necessarily as formal groups). Anyway, this shows that $G_s(W_s') \rightarrow \mathcal{A}(G_s(W_s'))$ is surjective for all $s' \geq s$. Therefore, we get an exact sequence

\[
0 \rightarrow A_s(W_s') \rightarrow G_s(W_s') \rightarrow \mathcal{A}(G_s)(W_s') \rightarrow 0\]

for all $s' \geq s \geq r$ including $s' = \infty$.\]
Since $\text{Gal}(\overline{K}/K^{ur})$ acts on $A_s$ by the $p$-adic cyclotomic character, we find $A_s \cong \hat{\mathbb{G}}_m^d$ over $\hat{W}^{ur}$ for $d = \dim A_s$. In Corollary in the introduction of [000] (see also [H13a, Lemma 4.2]), Ohta shows that $T G^\sigma := \lim_s T_p G_s^\sigma \cong h$ (and hence $T G^\sigma_T \cong T$) canonically as $h$-modules. Assuming (F), we have $T_s G_s^\sigma \cong h_s$; so, $G_s \cong \hat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} h_s$ over $\hat{W}^{ur}$. Define $\mathfrak{A} \subset h_s$ by the annihilator of $G_s/A_s$ and $\mathfrak{B} := \text{Ker}(h_s \to \text{End}(A_{\ell}/K))$. Hence we have an exact sequence of formal groups:

$$(17.3) \quad 0 \to \hat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{A} \to G_s \xrightarrow{\varpi} G_s \to \hat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} h_s/\mathfrak{B} \to 0$$

since $0 \to \mathfrak{A} \to h_s \xrightarrow{\varpi} h_s/\mathfrak{B} \to 0$ is an exact sequence of $\mathbb{Z}_p$-free modules. Thus we have $A_s \cong \hat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{A}$ over $\hat{W}^{ur}$, and $\mathfrak{A}$ is an $h_s$-ideal and is an $O_{A_s}$-module. This shows that $A_s$, $B_s$, $\varpi(J_s^\text{ord}) := J_s^\text{ord}/\mathfrak{A}^\text{ord}$ and $J_s$ all satisfy (A1–3) in Section 16.

The action of the Frobenius $[p : \mathbb{Q}_p]$ on $A_s[p^\infty](\overline{\mathbb{Q}}_p)$ is the multiplication by $\alpha_p^{-1} \in O_{A_s}$ (where $\alpha_p$ is the image of $U(p)$ in $O_{A_s}$). Thus $A_s(\hat{W}^{ur}) = \mathfrak{A} \otimes \hat{\mathbb{G}}_m(\hat{W}^{ur}) \cong \mathfrak{A} \otimes_{\mathbb{Z}_p}(1 + m_{p^\infty})$ on which the natural Galois action on $\hat{\mathbb{G}}_m(\hat{W}^{ur})$ is twisted by a character $\psi : \text{Gal}(K_\ell/K) \cong \text{Gal}(K_\ell^{ur}/K^{ur}) \to O_{A_s}$ induced by the nearly ordinary character $\psi$ sending $[z, \mathbb{Q}_p]$ ($z \in \mathbb{Z}_p$) to the image in $O_{A_s}$ of the Hecke operator in $h_s$ of the class of $(1, 0, 0)$ in $T^\infty_1/\mathbb{T}_1$. Write simply $A$ for the abelian variety $A_s$. Let $O_A := \text{End}(A_{\mathbb{Q}})$, which is an order of the Hecke algebra generated over $\mathbb{Q}$ by Hecke operators $T(n)$ in $\text{End}^0(A_{\mathbb{Q}}) = \text{End}(A_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Recall the Galois representation $\rho_A$ of $\text{Gal}(\overline{\mathbb{Q}}_p/K)$ realized on $T_p A^\text{ord}$. Take the connected component $\text{Spec}(T)$ of $\text{Spec}(h)$ such that $h/\varpi h = T/\varpi T$. Write symbolically $\rho_A|_{\text{Gal}(\overline{\mathbb{Q}}_p/K)} \cong \left( \begin{array}{cc} \nu_p \psi & * \\ 0 & \varphi \end{array} \right)$ and $\rho_t = \left( \begin{array}{cc} \nu_p \psi_t & * \\ 0 & \varphi_t \end{array} \right)$ for a deformation $\psi : \text{Gal}(K_\ell^{ur}/K) \to T^\infty$ of $\psi$. Here $\nu_p \psi_t$ and $\nu_p \psi$ act on $T_p A_s[p^\infty]$ and on $T G^\sigma_T$, respectively. Thus $\psi$ (resp. $\varphi$) gives the action on $T_p A^\text{ord}/T_p A_s$ (resp. on $T G^\sigma_T$). Note that $T G^\sigma_T \cong \text{Hom}_{\mathbb{A}}(T, A)$ as $T$-modules (so, if $T$ is Gorenstein, the above form $\rho_t$ of 2 × 2 matrix is literally true). We write $\text{Frob} \in \text{Gal}(\overline{K}^{ur}/K)$ for the Frobenius element inducing the generator of $\text{Gal}(\overline{\mathbb{F}}_p/F)$ and an appropriate power of the identity id = $[p, \mathbb{Q}_p]$ on $\overline{K}/K$.

**Proposition 17.1.** Suppose (F). Let $G^\sigma$ be the connected component of $G = G_{a, \delta, \xi}$, and take a generator $\sigma$ of $\text{Gal}(K_\ell/K)$. Then we have $H^1(K_\ell/K, G^\sigma_T(W_{\infty})) = (T_s/(\nu_p \psi(\sigma) - 1)T_s)[\varphi(\text{Frob}) - 1]$. If either $|\nu_p \psi(\sigma) - 1|_p = 1$ or $|\varphi(\text{Frob}) - 1|_p = 1$, then we have the vanishing $H^1(K_\ell/K, G^\sigma_T(W_{\infty})) = 0$.

**Proof.** As we saw, under (F), we have $G^\sigma_T(W_{\infty}) \cong \mu_{p^\infty}(\hat{W}^{ur}) \otimes_{\mathbb{Z}_p} T_s(\psi)$ as $G_{K_\ell^{ur}/K}$-modules, where $G_{K_\ell^{ur}/K}$ acts on $T_s(\psi)$ by $\psi$. We apply Lemma 16.1 to the formal Lie group $A$ with $A[p^\infty] = G^\sigma_T$. Note that $a$ in the Lemma is the image of $\varphi(\text{Frob})$ in $O_A$ by [H14, (6-1)]. From this, the cohomology of $G_s$ vanishes if either $|\nu_p \psi(\sigma) - 1|_p = 1$ or $|\varphi(\text{Frob}) - 1|_p = 1$. We have then $H^1(K_\ell/K, G^\sigma(\hat{W}_{\infty})) = \lim_s H^1(K_\ell/K, G^\sigma_s(\hat{W}_{\infty})) = 0$. \(\square\)

**Theorem 17.2.** Let the notation be as in Theorem 10.4. Let $K$ be a finite extension of $\mathbb{Q}_p$ for $p > 2$, and put $K_s = K[\mu_p]$ (s = 1, 2, . . ., ∞). If $A_s$ does not have split multiplicative reduction over $W_\tau$, then the error term $E^\infty(K)_T$ is finite. If further $A_s$ has good reduction over $W_1 = W[\mu_p]$ with $|\varphi(\text{Frob}) - 1|_p = 1$, then $E^\infty(K)_T$ vanishes.

**Proof.** Let us first sketch the proof. As before, we write symbolically $\varpi(J_s)$ for the abelian variety quotient $J_s/A_s$, since $J_s/A^\text{ord}_s = J_s^\text{ord}/A^\text{ord} = \varpi(J_s^\text{ord})$ by definition. Thus $A_s(F) \hookrightarrow J_s(F) \xrightarrow{\varpi} \varpi(J_s(F))$ is exact for any algebraic extension $F/K$, and hence $A^\text{ord}_s(F) \hookrightarrow J^\text{ord}_s(F) \xrightarrow{\varpi} \varpi(J^\text{ord}_s(F))$ is exact. We first assume that $J^\text{ord}_s(F)$ is contained in an abelian subvariety of $J_s$ having good reduction over $W_\tau$ (so, we may assume that the subabelian variety has good reduction over $W_\tau$). Then the sequence

$$(17.4) \quad 0 \to A_s[p^\infty] \to G_s \to \varpi(G_s) \to 0$$

is exact as Barsotti–Tate groups over $W_\tau$ (see [H14, §5] and a remark after Corollary 6.5). Since the complex of Néron models $A_s/W_\tau \to J_s/W_\tau \to \varpi(J_s/W_\tau)$ is exact up to $p$-finite errors [NMD,
Proposition 7.5.3, the exactness of (17.4) shows the sequence $\tilde{A}^{\text{ord}}_{s/W_s} \to \tilde{J}^{\text{ord}}_{s/W_s} \to \varpi(j^{\text{ord}}_{s/W_s})$ is exact as fppf sheaves over $W_s$. Since $0 \to \tilde{A}^{\text{ord}}_{s/W_s}(\mathbb{F}) \to \tilde{J}^{\text{ord}}_{s/W_s}(\mathbb{F}) \to \varpi(j^{\text{ord}}_{s/W_s})(\mathbb{F}) \to H^1(\mathbb{F}, \tilde{A}^{\text{ord}}) = 0$ is exact, (17.2) shows that $\tilde{J}^{\text{ord}}_s(K_{\infty}) \to \varpi(\tilde{J}^{\text{ord}}_s)(K_{\infty})$ is onto.

By (15.1), we have

$$|H^1(K_{\infty}/K_r, \tilde{A}^{\text{ord}}_r(K_{\infty}))| = |A_r[p^\infty]\text{ord}(\mathbb{F})|^2.$$ 

Since we have an exact sequence $\tilde{A}^{\text{ord}}_r(K_{\infty}) \to \tilde{J}^{\text{ord}}_s(K_{\infty}) \to \varpi(\tilde{J}^{\text{ord}}_s(K_{\infty}))$, by cohomology sequence of this short exact sequence, we have the claimed finiteness.

Let us now sketch the proof in the non-split multiplicative case (over $W$). We have a similar exact sequence of the formal Lie groups, and applying the formal version [Sc87, Theorem 1] (particularly in the non-split multiplicative case), we get the finiteness for the connected part. The surjectivity (up to finite error) for the special fiber (of Néron models) will be shown below. Thus if $A_r$ has either good or non-split multiplicative reduction over $W_r$, we still have finiteness of $H^1(K_{\infty}/K_r, \tilde{A}^{\text{ord}}_r(K_{\infty}))$ as above. Then by the inflation-restriction exact sequence:

$$H^1(K_r/K, \tilde{A}^{\text{ord}}_r(K_r)) \to H^1(K_{\infty}/K, \tilde{A}^{\text{ord}}_r(K_{\infty})) \to H^0(K_r/K, H^1(K_{\infty}/K_r, \tilde{A}^{\text{ord}}_r(K_{\infty}))) \to H^2(K_r/K, \tilde{A}^{\text{ord}}_r(K_r)),$$

finiteness of $H^1(K_r/K, \tilde{A}^{\text{ord}}_r(K_r))$ $(j = 1, 2)$ and $H^1(K_{\infty}/K_r, \tilde{A}^{\text{ord}}_r(K_{\infty}))$ tells us finiteness of the cohomology $H^1(K_{\infty}/K, \tilde{A}^{\text{ord}}_r(K_{\infty}))$, from which we conclude the finiteness of $E^\infty(K)_{\infty}$. If $r = 1$, $p \mid [K_1 : K]$ and

$$H^q(K_1/K, \tilde{A}^{\text{ord}}_r(K_1)) = 0 \text{ for } q > 0.$$ 

Then, still assuming $r = 1$, we conclude

$$(17.5) \quad H^1(K_{\infty}/K, \tilde{A}^{\text{ord}}_r(K_{\infty})) \cong H^0(K_r/K, H^1(K_{\infty}/K_r, \tilde{A}^{\text{ord}}_r(K_{\infty})))$$

If in addition $|\varphi(\text{Frob})|_p = 1$ and $A_r$ has good reduction over $W_r$, from $|H^1(K_{\infty}/K_r, \tilde{A}^{\text{ord}}_r(K_{\infty}))| = |A_r[p^\infty]\text{ord}(\mathbb{F})|^2 = 0$, the groups in (17.5) vanish so, $E^\infty(K)_{\infty} = 0$. In any case, $H^1(K_{\infty}/K, \tilde{A}^{\text{ord}}_r(K_{\infty}))$ is finite. Similarly, by the formal group version [Sc87, Theorem 1], we conclude the finiteness of $H^1(K_{\infty}/K, A_r(W_{\infty}))$.

We now give details of the proof in the general case. We first look into the identity connected components over $W_{\infty}$. By (17.2),

$$0 \to A_r(W_{\infty}) \to G_{\infty}(W_{\infty}) \to \varpi(G_{\infty})(W_{\infty}) \to 0$$

is exact. Taking its Galois cohomology sequence, we get another exact sequence

$$0 \to A_r(W) \to G_{\infty}(W) \xrightarrow{\varpi_{\infty}} \varpi(G_{\infty})(W) \to H^1(K_{\infty}/K, A_r(W_{\infty})).$$

Since the cohomology group $H^1(K_{\infty}/K, A_r(W_{\infty}))$ is finite (cf. [Sc87, Theorem 1]), we find that

$$\text{Coker}(G_{\infty}(W_{\infty})_{\text{Gal}(K_{\infty}/K)} \xrightarrow{\varpi_{\infty}} \varpi(G_{\infty})(W_{\infty})_{\text{Gal}(K_{\infty}/K)})$$

is finite.

As for the special fiber (of the Néron models), we have the exact sequence:

$$0 \to \tilde{A}^{\text{ord}}(\mathbb{F}) \to \tilde{J}^{\text{ord}}_{s}(\mathbb{F}) \to \varpi(\tilde{J}^{\text{ord}}_{s})(\mathbb{F}) \to H^1(\mathbb{F}, \tilde{A}^{\text{ord}}).$$

If $\varphi(\text{Frob}) \neq \pm 1$, $A_r$ has good reduction (not just semi-stable one) over $W_r$; so, by Lang’s theorem [L56], $H^1(\mathbb{F}, \tilde{A}^{\text{ord}}) \subset H^1(\mathbb{F}, A_r) = 0$. Even if $\varphi(\text{Frob}) = \pm 1$, from the exact sequence

$$0 \to A_r(\overline{\mathbb{F}}_p) \to A_r(\mathbb{F}_p) \to \pi_0(A_r/\mathbb{F}_p) \to 0$$

for the connected component $A^0$ of $A_r/\mathbb{F}_p$, we find $H^1(\mathbb{F}, A_r) \cong H^1(\mathbb{F}, \pi_0(A_r))$ as $\text{Gal}(\mathbb{F}_p/\mathbb{F})$ has cohomological dimension 1. Thus $H^1(\mathbb{F}, \tilde{A}^{\text{ord}})$ is finite. After passing to the limit, we find

$$|\text{Coker}(J^{\text{ord}}_s(\mathbb{F}) \xrightarrow{\varpi_{\infty}} \varpi(J^{\text{ord}}_s)(\mathbb{F}))| \leq |\pi_0(A_r/\mathbb{F}_p)| < \infty.$$

We have the following exact sequence:

$$0 \to G^q(W_{\infty})_T \to G(K_{\infty})_T \xrightarrow{\text{red}} J^{\text{ord}}_s[p^\infty]_T \to 0.$$
Indeed, the maximal étale quotient $G_{et}$ of $G_{W_{\infty}}$ is a $\Lambda$-BT group by [H14, Proposition 6.3]; so, its closed points lift to a $W_{\infty}$-point as $W_{\infty}$ is henselian. (Note that $G_{et,W_{\infty}}$ may not be an étale Barsotti-Tate group for finite $s$.) Taking the fixed point of $\text{Gal}(K_{\infty}/W_{\infty})$, we have

$$0 \rightarrow G^\circ(W_{\infty})^\text{red} \rightarrow G(K)_T \rightarrow J_{\infty}(F)[p^\infty]^\text{ord} \rightarrow H^1(K_{\infty}/W_{\infty}, G^\circ(W_{\infty})_T).$$

Then by Proposition 17.1, $\text{Coker}(G(K)_T)^{\text{red}} = J_{\infty}(F)[p^\infty]^\text{ord} = 0$ (assuming either $\lvert \varphi(\text{Frob}) - 1 \rvert_p = 1$ or $\lvert \psi_\nu(\sigma) - 1 \rvert_p = 1$), and in particular, $\text{Coker}(J_{\infty}^\text{ord}(K)_T)^{\text{red}} = J_{\infty}(F)[p^\infty]^\text{ord} = 0$.

We have the following commutative diagram with exact rows and columns:

(17.6) $\begin{array}{cccc}
G_{\infty}(W_{\infty})^{\text{Gal}(K_{\infty}/K)} & \overset{\varphi}{\longrightarrow} & J_{\infty}(K)^{\text{ord}} & \overset{\text{red}}{\longrightarrow} & J_{\infty}(F)[p^\infty]^\text{ord} \\
\varpi(\infty) & \downarrow & \varpi & \downarrow & \varpi(\infty) \\
\text{Coker}(\varpi(\infty)) & \longrightarrow & \text{Coker}(\varpi(\infty)) & \longrightarrow & \text{Coker}(\varpi(\infty)).
\end{array}$

For the Frobenius endomorphism $\phi (\varphi(\text{Frob}))$, we have

$$J_{\infty}(F)[p^\infty]^\text{ord} = J_{\infty}(F)(p^\infty)^{\text{ord}}[\phi - 1].$$

Since $\phi \equiv \varphi(\text{Frob}) \mod m_T$, if $\varphi(\text{Frob}) \neq 1 \mod m_W$ ($\Leftrightarrow \lvert \varphi(\text{Frob}) - 1 \rvert_p = 1$), $J_{\infty}(F)[p^\infty]^\text{ord} = 0$, and $\text{Coker}(\varpi(\infty)) \rightarrow \text{Coker}(\varpi(\infty))$ is exact; so, $\text{Coker}(\varpi(\infty))$ is finite.

We need to argue more if $\lvert \varphi(\text{Frob}) - 1 \rvert_p < 1$. We apply $X \mapsto X^\vee := \text{Hom}(X, Q_p/Z_p)$ to the above diagram. Since $Q_p/Z_p$ is $Z_p$-injective, $X \mapsto X^\vee$ is an exact contravariant functor; all arrows of (17.6) are reversed, but exactness is kept. Since $\text{Coker}(\varpi(\infty)) \hookrightarrow H^1(K, A_{\text{et}})$, its Pontryagin dual module $\text{Coker}(\varpi(\infty))^\vee$ is a $Z_p$-module of finite type. Since this module killed by arithmetic prime (\varpi), we need to show the vanishing of the (\varpi)-localization $\text{Coker}(\varpi(\infty))^{\varpi} = 0$. Note that we have a surjective morphism of $\Lambda$-module: $\text{Coker}(\text{red}) \rightarrow \text{Coker}(\text{red}_{\text{ord}})$ and that $\text{Coker}(\text{red})$ is killed by $(\nu_p(\psi(\sigma) - 1))(\gamma_t - 1)$ by Proposition 17.1. Since (\varpi) is prime to $\gamma_t - 1$, we have the vanishing of the localization $\text{Coker}(\text{red}_{\text{ord}})^{\varpi} = 0$. From the diagram obtained by applying $X \mapsto X^\vee$, the localized sequence

$0 = \text{Coker}(\varpi(\infty))^{\varpi} \rightarrow \text{Coker}(\varpi(\infty))^{\varpi} \rightarrow \text{Coker}(\varpi(\infty))^{\varpi}$

is exact. Since $\text{Coker}(\varpi(\infty))^{\varpi}$ is finite under $\psi(\sigma) \neq 1$ and $\varphi(\text{Frob}) \neq 1$, we conclude $\text{Coker}(\varpi(\infty))^{\varpi} = 0$; so, $\text{Coker}(\varpi(\infty))^{\varpi} = 0$ by the finiteness of $\text{Coker}(\varpi(\infty))$. Since $\text{Coker}(\varpi(\infty))^{\varpi} = 0$ is finite $Z_p$-module, dualizing back, this shows finiteness of $\text{Coker}(\varpi(\infty))$ as desired.

\section{Twisted family}

We describe, in down-to-earth terms, how to create $p$-adic analytic family of modular form associated to an irreducible component of $\text{Spec}(\mathbf{h}_{\phi,d})$ from a $p$-ordinary family coming from an irreducible component of $\text{Spec}(\mathbf{h}_{0,1,\phi,d})$ for $\phi_{ord}(a,d)$ only dependent on $d$; i.e., $\phi_{ord}(a,d) = \phi(d)$ for some character $\phi$ of $(\mathbb{Z}/p^\infty)^\times$. We show that as a $\Lambda$-algebra $\mathbf{h}_{\phi,d}^{\text{ord}} := \mathbf{h}_{0,1,\phi,d}(\xi)$ for a specific choice $\xi$ depending on $\phi$ by $T(l) \mapsto (l)^{-1}T(l)$ regarding $l$ as an idele $l$ in $(A^{(p^\infty)})^\times$ supported on $Q^\times$. Here $\kappa$ is a suitably chosen character of $(\mathbb{A}^{(\infty)})^\times/Q^\times$ with values in $\Lambda$.

Let $S_k(\Gamma_0(Np^\delta), \varepsilon(\phi))$ for a character $\varepsilon : Z_p^\times / \mu_{p^\infty}, \chi : (Z/NZ)^\times \rightarrow \mathbb{Q}^\times$ and $\phi : (Z/p^\delta Z)^\times \rightarrow \mathbb{Q}^\times$ be the space of cusp forms in $S_k(\Gamma_1(Np^\delta))$ satisfying the following identity

$$f(ab + cd) = \varepsilon(a_p^{-\alpha}d_p^\delta \phi(a \bmod p^\delta) \chi(a \bmod N)f(x)(cz + d)^k$$

for all $(a b \ c d) \in \Gamma_0(Np^\delta)$. Write $\psi$ for the character of $\Gamma_0(Np^\delta)$ given by

$$\gamma = \left(\begin{array}{cc}a & b \\c & d \end{array}\right) \mapsto \varepsilon(a_p^{-\alpha}d_p^\delta \phi(a \bmod p^\delta) \chi(a \bmod N).$$
We lift \( f \) to \( f : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \hat{\Gamma}_{H,s} \rightarrow \mathbb{C} \) as in [H10, page 779]. Here we lift a Dirichlet character \( \eta : (\mathbb{Z}/M\mathbb{Z})^\times \rightarrow \mathbb{Q}^\times \) to a character of \( \eta_{\lambda} : \mathbb{A}^\times / \mathbb{Q}^\times \mathbb{R}_+^\times \rightarrow \mathbb{Q}^\times \) so that \( \eta_{\lambda}(l) = \eta(l) \) for all primes \( l \nmid M \), where \( l_i \in \mathbb{A}^\times \) has \( l \)-component \( l \) and outside \( l \), it is trivial. Then \( f \) satisfies

\[
\mathbf{f}(zxu) = \phi_{\lambda}\chi_{\lambda}e^{\alpha-\delta}(z)e_{\lambda}(a_p^n d_p^{-\delta})\phi_{\lambda,p}(d_p)\chi_{\lambda,N}(d_N)f(x)
\]

for \( u = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \hat{\Gamma}_0(Np^r) \) and \( z \) in the center \( Z(\mathbb{A}) \) of \( GL_2(\mathbb{A}) \). Here we regard \( zu \mapsto \phi_{\lambda}\chi_{\lambda}e^{\alpha-\delta}(z)e_{\lambda}(a_p^n d_p^{-\delta})\phi_{\lambda,p}(d_p)\chi_{\lambda,N}(d_N) \) as a character of \( Z(\mathbb{A})\hat{\Gamma}_0(Np^r) \), and write it as \( \psi_{\lambda} : Z(\mathbb{A})\hat{\Gamma}_0(Np^r) \rightarrow \mathbb{C}^\times \). Then we consider the space \( \mathcal{S}_k(\hat{\Gamma}_0(Np^r), \psi_{\lambda}) \) defined in [H10, page 779]. By the correspondence described in [H10, §1.1],

\[
\mathcal{S}_k(\hat{\Gamma}_0(Np^r), \psi) \cong \mathcal{S}_k(\hat{\Gamma}_0(Np^r), \psi_{\lambda}).
\]

Restart with the classical space \( \mathcal{S}_k(\hat{\Gamma}_0(Np^r), \Phi_{\lambda}) \) with usual Neben character \( \Phi_{\lambda} \) for \( \Phi : (\mathbb{Z}/p^r\mathbb{Z})^\times \rightarrow \mathbb{Q}^\times \) and \( \chi \) as above. Thus \( f \in \mathcal{S}_k(\hat{\Gamma}_0(Np^r), \Phi_{\lambda}) \) satisfies

\[
f\left( \frac{az + b}{cz + d} \right) = \Phi((a \mod p^r)\chi(a \mod N))f(x)(cz + d)^k
\]

for all \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \hat{\Gamma}_0(Np^r) \). Lift \( f \) to \( GL_2(\mathbb{A}) \) in the same manner as above, we get \( f \in \mathcal{S}_k(\hat{\Gamma}_0(Np^r), \Phi_{\lambda}) \) (for \( \Phi_{\lambda} = \Phi_{\lambda,\chi} \)) satisfying \( f(zxu) = \Phi_{\lambda,\chi}(z)\Phi_{\lambda,p}(d_p)\chi_{\lambda,N}(d_N)f(x) \). Then, for a character \( \varphi : \mathbb{Z}_p^\times \rightarrow \mathbb{Q}^\times \), take a unique character \( \varphi_{\lambda} \) of \( \mathbb{A}^\times / \mathbb{Q}^\times \mathbb{R}^\times \) with \( \varphi_{\lambda}\mid_{\mathbb{Z}_p^\times} = \varphi^{-1} \), and define \( f \otimes \varphi \in \mathcal{S}(\hat{\Gamma}_0(Np^r), \psi_{\varphi,\lambda}) \) by \( f \otimes \varphi(g) = \varphi_{\lambda}(\det(g))f(g) \). We then go back to \( f \otimes \varphi \in \mathcal{S}_k(\hat{\Gamma}_0(Np^r), \psi_{\varphi,\lambda}) \) by the isomorphism \( \mathcal{S}_k(\hat{\Gamma}_0(Np^r), \psi_{\varphi,\lambda}) \cong \mathcal{S}_k(\hat{\Gamma}_0(Np^r), \psi_{\varphi,\lambda}) \) in [H10, §1.1]. Here we have \( \psi_{\varphi,\lambda}(zu) = \varphi_{\lambda}(z)^2\varphi_{\lambda}(a_p)\varphi_{\lambda}(d_p)\Phi_{\lambda}(d_p)\chi_{\lambda,N}(d_N) \). By definition, we get

**Lemma 18.1.** If \( f \) as above satisfies \( f(T(n)) = \lambda(T(n))f \) (a Hecke eigenform) with \( T(l) = U(l) \) if \( l \nmid Np \), we have \( f \otimes \varphi | T(l) = \varphi_{\lambda}(l)\lambda(T(l))f \otimes \varphi \) for all primes \( l \) prime to \( p \), and for \( U(p) \), we have \( f \otimes \varphi | U(p) = \lambda(U(p))f \otimes \varphi \). Thus this operation \( f \mapsto f \otimes \varphi \) preserves “ordinarity”.

Here we have used the well known fact that \( \varphi \) factors through the \( p \)-adic cyclotomic character whose value at \( p \) is equal to \( 1 \); thus, \( \varphi_{\lambda}(p) = 1 \) and the formula \( (f \otimes \varphi)(U(p)) = \lambda(U(p))(f \otimes \varphi) \) is consistent with \( (f \otimes \varphi)(U(l)) = \varphi_{\lambda}(l)\lambda(T(l))(f \otimes \varphi) \).

By the lemma, if \( f \) has weight \( 2 \), from the abelian subvariety \( A_f \otimes \varphi \) attached to \( f \), we get an abelian variety \( A_f \otimes \varphi \) of \( J_s \) (for a suitable choice of \( H \)) which is the \( \varphi \)-twist of \( A_f \) (see (16.1): i.e., we have an identity of \( L \)-adic Tate modules \( T_{A_f \otimes \varphi} \cong (T_{A_f}) \otimes \varphi \) as Galois modules regarding \( \varphi \) as a Galois character via \( \mathbb{Z}_p^\times = \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \).

If \( f \) (or starting \( f \)) is a Hecke eigenform, the modular form \( f : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \rightarrow \mathbb{C} \) and its right translations \( R(g)(f)(x) = f(xg) \) for \( g \in GL_2(\mathbb{A}) \) generate an irreducible automorphic representation \( \pi = \pi_{\lambda} \) of \( GL_2(\mathbb{A}) \). Similarly, the modular form \( f \otimes \varphi : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \rightarrow \mathbb{C} \) and its right translation \( R(g)(f \otimes \varphi)(x) = (f \otimes \varphi)(xg) = f(xg)\varphi_{\lambda}(\det(xg)) \) for \( g \in GL_2(\mathbb{A}) \) generate an irreducible automorphic representation \( \pi_{f \otimes \varphi} \). Plainly, we have \( \pi_{f \otimes \varphi} \cong \pi \otimes \varphi \). Inside \( \pi_{f \otimes \varphi} \), we find a unique new vector which corresponds to a primitive Hecke eigenform \( f_{\varphi} \in \mathcal{S}_k(\hat{\Gamma}_0(C(\pi \otimes \varphi)), \varphi \chi \varphi^2) \) for the conductor \( C(\pi \otimes \varphi) \). The form \( f_{\varphi} \) is usually not equal to \( f \otimes \varphi \) even if \( f \) is primitive (as their Neben types are plainly different). As explained in [H09, §3.1], \( f \otimes \varphi \) often has level smaller than the primitive form \( f_{\varphi} \). Unless the \( p \)-component \( \pi_p \) is super-cuspidal, \( \pi_p \otimes \varphi \) has non-zero \( U(p) \)-eigenvector with non-zero eigenvalue. Indeed, if \( \pi_p = \pi(\alpha, \beta) \), there are non-zero eigenspaces in \( \pi \otimes \varphi \) on which \( U(p) \) acts by \( \alpha_{\varphi_{\lambda}}(p_p) \) and \( \beta_{\varphi_{\lambda}}(p_p) \) (if \( \alpha(p) \neq \beta(p) \), the eigenspaces of each of the above value is one-dimensional). If \( \pi_p \) is special, we have one dimensional eigenspace with non-zero eigenvalue. Even if \( \varphi \) is highly ramified at \( p \), the eigenvalues of \( U(p) \) for \( f \otimes \varphi \) and \( f \) are equal.

Suppose that

\[
\mathcal{F}_1 := \{ f_p \in \mathcal{S}_2(\hat{\Gamma}_0(Np^r), \phi \chi \varepsilon_p) \mid p \in \text{Spec}(\mathbb{Q}) \}
\]

is the ordinary \( p \)-adic analytic family for \( \chi \) modulo \( N \), \( \phi \) modulo \( p^r \), \( \varepsilon_p : \mathbb{Z}_p^\times / \mu \rightarrow \mu_{p^\infty}(\mathbb{Q}) \). Pick a positive integer \( b \) prime to \( p \) and a character \( \varphi : (\mathbb{Z}/p^b\mathbb{Z})^\times \rightarrow \mathbb{Z}_p^\times \). Then we consider the
twisted family $\mathcal{F}(b) = \{ f_p \otimes \varepsilon_p^{-1/h} \varphi \}$. Since $f_p \in S_2(\Gamma_0(Np^r(P)), \phi \chi \varepsilon_p)$, we have $f_p \otimes \varepsilon_p^{-1/h} \varphi \in S_2(\Gamma_0(Np^r(P), \psi_p)$, where $\psi_p((a, b)_d^e) = \phi(a \mod p) \chi(a \mod N) \varepsilon_p^{-1/h}(a_p) \varepsilon_p^{-1/h}(d_p) \varphi^{-1}(d_p)$. Thus in this case, $\psi_{\varepsilon_p^{-1/h}} \varphi$ factors through $G/H$ for $H$ defined for $(\alpha, \delta) = (1, b - 1)$ and $\xi(a, d) = \varphi(a) \phi \varphi(d)$. If one starts with $f_{p_0} \in S_2(\Gamma_0(Np))$ whose $L$-function has root number $\pm 1$, the $L$-function $f_p \otimes \varepsilon_p^{-1/2}$ has the same root number. Therefore, the most interesting case is when $b = 2$ (so, $p > 2$), $(\alpha, \delta) = (1, 1)$ and $\phi \chi = 1$. This process can be reversed by tensoring back $\varepsilon_p^{-1/h} \varphi^{-1}$. Thus we have one-to-one onto correspondence of families of modular forms of $h_{0,1,\xi_{ord}}$ and $h_{1, b - 1, \xi_{ord}}$, where $\xi(a, d) = \phi(d) \varphi(a) \varphi(d)$. This shows, writing $\Lambda + Z_p[[T]]$ with $t = 1 + T$,

**Proposition 18.2.** Let the notation as above. Then the $\mathbb{Z}_p$-algebra $h_{1, b - 1, \xi_{ord}}$ is isomorphic to $h_{0,1,\xi_{ord}}(\xi_{ord}(a, d) = \phi(d))$ by $T(l^n) \mapsto t^{-\log_p(l^n)} \log_p(\gamma) \varphi_l(l^n) T(l^n)$ for primes $l$ as $\mathbb{Z}_p$-algebras, where $\gamma = 1 + p^e$ and $\log_p$ is the $p$-adic logarithm and we have written $T(l^n) = U(l^n)$ for $l | Np$. The $\Lambda$-algebra structure of $h_{1, b - 1, \xi_{ord}}$ obtained twisting the $\Lambda$-algebra structure of $h_{0,1,\xi_{ord}}$ by the character $\kappa : Z_p^+ / \mu \rightarrow \Lambda^\times$, which is given by $\kappa^s(\gamma) = t^e$ for $s \in Z_p$. In particular, the algebra $h_{1, b - 1, \xi_{ord}}$ is free of finite rank over $\Lambda$ for all primes $p$.

**References**

**Books**


**Articles**


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