ANALYTIC VARIATION OF TATE–SHAFAREVICH GROUPS

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Abstract. We study a tower $\{X_r\}_{r=0,1,2,...}$ of modular curves obtained systematically twisting the standard tower associated to $\{X_1(Np^r)\}_r$ for a prime $p \geq 5$ and N prime to p. A given $\mathbb{Q}$-simple factor A_{P_0} of the Jacobian J_{r_0} of X_{r_0} associated to a p-ordinary Hecke eigenform is a member of a p-adic analytic family $\{A_P\}_P$ of $\mathbb{Q}$-simple factors of J_r $(r = 0, 1, 2, \dots)$, where P runs over arithmetic points of a local ring $\mathbb{T}$ of the big Hecke algebra of the tower. Supposing a minimalist condition on the arithmetic cohomology of A_{P_0} in addition to regularity of $\mathbb{T}$, we study a sufficient condition for infinitely many A_P having Mordell–Weil rank ≤ 1 and finite Tate–Shafarevich group over a number field K. If we choose the twist well and if A_{P_0} has root number $\epsilon = \pm 1$, we can make A_P to have the equal root number all over the family, which may be an interesting case.

1. Introduction

Fix a prime p and a positive integer N prime to p throughout the paper. Let $\mathrm{Spec}(\mathbb{I})$ be an irreducible component of (the spectrum of) the p-ordinary big Hecke algebra $\mathbf{h}$. Attached to $\mathbb{I}$ is the Mazur–Kitagawa p-adic L-function $L(k, s)$ for the weight variable k and the cyclotomic variable s. The function L is an element of the affine ring of the irreducible component of $\mathrm{Spec}(\mathbf{h}^{\mathrm{n.ord}})$ covering $\mathrm{Spec}(\mathbb{I})$ for the two variable nearly p-ordinary big Hecke algebra $\mathbf{h}^{\mathrm{n.ord}}$. We study in this paper the tower of modular curves $\{X_r\}_r$ whose jacobians (or more precisely their p-ordinary part) correspond to the one variable p-adic L-function $k \mapsto L((\alpha + \delta)k + 2, \delta k + 1)$ heuristically for a fixed pair of p-adic integers $\alpha, \delta \in \mathbb{Z}_p$. More precisely, choosing a primitive root $\zeta_r \in \mu_{p^r}$ compatibly with $r > 0$ and picking a weight 2 Hecke eigenform f belonging to $\mathbb{I}$ whose Neben character restricted to $\mathbb{Z}_p^\times$ sends $1 + p \in \mathbb{Z}_p^\times$ to $\zeta_r^{\alpha+\delta}$, the p-adic L-function L interpolates the complex L-value $L(1, f \otimes \epsilon)$ for the p-power order order character ϵ of $\mathbb{Z}_p^\times$ with $\epsilon(1 + p) = \zeta_r^{-\delta}$. In this introduction, for simplicity, we assume that $\alpha = \delta = 1$; so the corresponding p-adic L-function $k \mapsto L(2k + 2, k + 1)$ interpolates the central critical values (so the function $k \mapsto L(2k + 2, k + 1)$ could be identically zero). We call the tower $\{X_r\}_r$ with $(\alpha, \delta) = (1, 1)$ the self-dual tower (of prime-to-p level N), and here we sketch the results for the self-dual tower. The general case of an arbitrary (α, δ) will be taken care of in the main text (see Section 3 for a precise definition of $\{X_r\}_r$). The standard tower $\{X_1(Np^r)\}_r$ corresponds to $(\alpha, \delta) = (0, 1)$. Let $J_{r/\mathbb{Q}}$ for the Jacobian variety of X_r. Since X_r is essentially a Galois twist of the modular curve sitting between $X_1(Np^r)$ and $X_0(Np^r)$, we may assume that $H^0(J_r, \Omega_{J_r/\mathbb{C}}) \cong S_2(\Gamma_r)$ for a congruence subgroup Γ_r with $\Gamma_1(Np^r) \subset \Gamma_r \subset \Gamma_0(Np^r)$.

For a set of places S of a number field K, write K^S/K for the maximal extension unramified outside S. For a topological $\mathrm{Gal}(K^S/K)$-module M and $v \in S$, we write $H^\bullet(K^S/K, M)$ (resp. $H^\bullet(K_v, M)$ for the v-completion K_v of K) for the continuous cohomology of the profinite group $\mathrm{Gal}(K^S/K)$ (resp. $\mathrm{Gal}(\overline{K}_v/K_v)$ for an algebraic closure $\overline{K}_v$ of K_v) giving the **discrete** topology to the coefficients M (so, $H^q(K^S/K, M)$ and $H^q(K, M)$ is a torsion module if $q > 0$). Define

$$Ш^j(K^S/K, M) = \mathrm{Ker}(H^j(K^S/K, M) \to \prod_{v \in S} H^j(K_v, M)) \ \text{ for } j = 1, 2$$

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and put $Ш^j(K^S/K,M)_p := Ш^j(K^S/K,M)\otimes_{\mathbb{Z}}\mathbb{Z}_p$. Often we simply write Ш for $Ш^1$. More generally, for a module M, we define M_p by $M\otimes_{\mathbb{Z}}\mathbb{Z}_p$ (so, M_p is the maximal p-power torsion submodule $M[p^\infty]$ of M if M is torsion, and the maximal p-profinite quotient if M is profinite). Throughout the paper, when M is related to an abelian variety, we always assume that S contains all finite places at which the abelian variety has bad reduction in addition to all p-adic and archimedean places of K. Unless otherwise mentioned, we assume S to be chosen finite.

In addition to the divisible Mordell–Weil group $J_r(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$, we study the Tate–Shafarevich group $Ш_K(J_r)$, $Ш_K(K^S/K,J_r[p^\infty])$ and the Selmer group

$$\mathrm{Sel}_K(J_r)=\mathrm{Ker}(H^1(K^S/K,J_r[p^\infty])\to\prod_{v\in S}H^1(K_v,J_r)).$$

The Tate–Shafarevich group and the Selmer group of an abelian variety are independent of S; so, we omitted "K^S/K" from the notation. The Hecke operator $U(p)$ and its dual $U^*(p)$ acts on $Ш_K(J_r)$ and their p-adic limit $e=\lim_{n\to\infty}U(p)^{n!}$ and $e^*=\lim_{n\to\infty}U^*(p)^{n!}$ are well defined on the above groups H. We write $H^{\mathrm{ord}}:=e(H)$. More generally, adding superscript or subscript "ord" (resp. "co-ord"), we indicate the image of e (resp. e^*) depending on the situation.

By Picard functoriality, we have injective limits $\mathcal{G}:=\varinjlim_r\mathcal{G}_r$ with $\mathcal{G}_r:=J_r[p^\infty]^{\mathrm{ord}}$ (a Λ-BT group in the sense of [H14]), $R\mapsto J_\infty^{\mathrm{ord}}(R)=\varinjlim_r\widehat{J}_r^{\mathrm{ord}}(R)$ for $\widehat{J}_r(R)=\varprojlim_n J_r(R)/p^nJ_r(R)$ as an fppf sheaf over K, $Ш_K(J_\infty^{\mathrm{ord}})=\varinjlim_r Ш_K(J_r)_p^{\mathrm{ord}}$, $Ш_K(K^S/K,\mathcal{G})=\varinjlim_r Ш_K(K^S/K,J_r[p^\infty]^{\mathrm{ord}})$, and $\mathrm{Sel}_K(J_\infty^{\mathrm{ord}})=\varinjlim_r\mathrm{Sel}_K(J_r)_p^{\mathrm{ord}}$. We study control under Hecke operators acting on these arithmetic cohomology groups and $J_\infty^{\mathrm{ord}}(R)\otimes_{\mathbb{Z}}\mathbb{Q}_p/\mathbb{Z}_p$ for number fields $R=K$. These groups are discrete modules over the corresponding big Hecke algebra $\mathbf{h}$, and we call them Λ-*BT groups*, *ind* Λ-*MW groups*, *ind* Λ-*TS groups* and Λ-*Selmer groups* in order. By adding the superscript "$\vee$", we indicate their Pontryagin dual which are p-profinite $\mathbf{h}$-modules. For a local ring $\mathbb{T}$ of $\mathbf{h}$, adding subscript $\mathbb{T}$, we indicate the module cut out by $\mathbb{T}$; e.g., $J_{\infty,\mathbb{T}}^{\mathrm{ord}}(R)=J_\infty^{\mathrm{ord}}(R)\otimes_{\mathbf{h}}\mathbb{T}$. For each Shimura's abelian subvariety $A_f\subset J_r$ associated to a Hecke eigenform $f\in S_2(\Gamma_r)$ (e.g., [IAT, Theorem 7.14]), we can think of the ordinary part of the Tate–Shafarevich group $Ш_K(A_f)_p^{\mathrm{ord}}$ and the Selmer group $\mathrm{Sel}_K(A_f)_p^{\mathrm{ord}}$ (see (1.2) and Section 8 of the text or [ADT, page 74] for the definition of these groups). Let $\mathbf{h}$ be a big ordinary Hecke algebra with respect to the tower, and pick a primitive connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ in the sense of [H86a, §3]. Then points $P\in\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ correspond one-to-one to p-adic Hecke eigenforms f_P in a slope 0 analytic family. Assuming for example that $\mathbb{T}$ is a unique factorization domain, in a densely populated subset $\Omega_{\mathbb{T}}\subset\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ of principal primes (indexed by $(\zeta^\delta,\zeta^\alpha)=(\zeta,\zeta)$ for $\zeta\in\mu_{p^\infty}$), f_P is classical, new at all prime factors of N and of weight 2 (a definition of $\Omega_{\mathbb{T}}$ will be given in (10.1)). Write $Np^{r(P)}$ for the minimal level of f_P. Let $A_{P/\mathbb{Q}}$ (resp. $B_{P/\mathbb{Q}}$) be Shimura's abelian subvariety (resp. abelian variety quotient) of $J_{r(P)}$ associated to f_P (see Definition 5.3). Write H_P for the subfield of $\mathrm{End}(A_{P/\mathbb{Q}})\otimes_{\mathbb{Z}}\mathbb{Q}$ generated by the Hecke operators. In this introduction, for simplicity, we assume that A_P for every $P\in\Omega_{\mathbb{T}}$ has potentially good reduction at p and that A_P for some $P\in\Omega_{\mathbb{T}}$ has good reduction over $\mathbb{Z}_p$. We prove that $\mathrm{rank}_{\mathbb{T}}M^\vee=\dim_{\mathrm{Frac}(\mathbb{T})}M^\vee\otimes_{\mathbb{T}}\mathrm{Frac}(\mathbb{T})$ is finite for $M=J_{\infty,\mathbb{T}}^{\mathrm{ord}}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$, $\mathrm{Sel}_K(J_{\infty,\mathbb{T}}^{\mathrm{ord}})$, $Ш_K(J_{\infty,\mathbb{T}}^{\mathrm{ord}})$ and $Ш_K(K^S/K,\mathcal{G}_{\mathbb{T}})$ if $\mathbb{T}$ is a domain. We then prove partial control result relating the above M with the corresponding classical arithmetic cohomology. Our control results implies

Theorem A. *Let K be a number field (i.e., a finite extension of $\mathbb{Q}$). Suppose that $\mathbb{T}$ is a unique factorization domain and that there exists $P_0\in\Omega_{\mathbb{T}}$ such that $\dim_{H_{P_0}}A_{P_0}(K)\otimes_{\mathbb{Z}}\mathbb{Q}\le 1$ with $|Ш_K(A_{P_0})_p^{\mathrm{ord}}|<\infty$ and A_{P_0} has potentially good reduction at p. Then we have*

1. $\mathrm{rank}_{\mathbb{T}}J_{\infty,\mathbb{T}}^{\mathrm{ord}}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)^\vee\le\dim_{H_{P_0}}A_{P_0}(K)\otimes_{\mathbb{Z}}\mathbb{Q}\le 1$*;*
2. *if* $\dim_{H_{P_0}}A_{P_0}(K)\otimes_{\mathbb{Z}}\mathbb{Q}=0$, *then* $\mathrm{rank}_{\mathbb{T}}(J_{\infty,\mathbb{T}}^{\mathrm{ord}}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)^\vee=0$, $Ш_K(A_P)_p^{\mathrm{ord}}$ *and* $A_P(\mathbb{Q})$ *are finite for almost all* $P\in\Omega_{\mathbb{T}}$*;*
3. *if* $\mathrm{rank}_{\mathbb{T}}(J_{\infty,\mathbb{T}}^{\mathrm{ord}}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)^\vee=\dim_{H_{P_0}}A_{P_0}(K)\otimes_{\mathbb{Z}}\mathbb{Q}=1$, *then there exists an infinite subset* $Ct_{\mathbb{T}}\subset\Omega_{\mathbb{T}}$ *such that* $\dim_{H_P}A_P(K)\otimes_{\mathbb{Z}}\mathbb{Q}=1$ *for all* $P\in Ct_{\mathbb{T}}$ *and* $Ш_K(A_P)_p^{\mathrm{ord}}$ *is finite for almost all* $P\in Ct_{\mathbb{T}}$.

Here the words "almost all" means "except for finitely many". We say that A_{P_0} (or P_0) satisfy the minimalist condition over K if $\dim_{H_{P_0}} A_{P_0}(K)\otimes_{\mathbb{Z}}\mathbb{Q}\le 1$ with $|\text{Ш}_K(A_{P_0})_p^{\mathrm{ord}}|<\infty$. This theorem is a special case of Theorem 13.4, and the assertion (2) should be known at least for the standard tower via control of Λ-Selmer groups done by earlier authors (though we give a proof of this in our own way). Thus the new point in this paper is the assertion (3). Since we took the self-dual tower here in the introduction, we may choose $\mathbb{T}$ so that the root number ϵ is constant ± 1 in the family. In the setting of the assertion (3), if we assume $\epsilon=-1$ and a weak form of the Birch–Swinnerton Dyer conjecture asserting $A_P(\mathbb{Q})$ is infinite for all $P\in\Omega_{\mathbb{T}}$, we have $Ct_{\mathbb{T}}=\Omega_{\mathbb{T}}$ by the definition of $Ct_{\mathbb{T}}$. Without assuming the weak form of the Birch–Swinnerton Dyer conjecture, though unlikely, we could have $\dim_{H_P} A_P(K)\otimes_{\mathbb{Z}}\mathbb{Q}=0$ over an infinite set $Ct_{\mathbb{T}}^{\perp}$ outside $Ct_{\mathbb{T}}$. The set $Ct_{\mathbb{T}}^{\perp}$ is made of arithmetic points where the control for the Mordell–Weil group fails (i.e., by p-adically interpolating infinite order points in $A_P(K)$ for $P\in Ct_{\mathbb{T}}$, we have non-triviality of $(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)^{\vee}$, but for $P\in Ct_{\mathbb{T}}^{\perp}$ such point specializes a transcendental object not descending to K). Since our method does not construct a concrete point in $A_P(K)$, it is difficult by our way to show unconditionally that $Ct_{\mathbb{T}}^{\perp}$ is finite (in an earlier version of this paper, it was claimed that $Ct_{\mathbb{T}}^{\perp}$ is finite, but the argument for finiteness of $Ct_{\mathbb{T}}^{\perp}$ is still incomplete). If we can determine the parity of $\dim_{H_P} A_P(K)\otimes_{\mathbb{Z}}\mathbb{Q}$ in terms of ϵ independently of the point P over the entire $\Omega_{\mathbb{T}}$ outside a finite subset E, we would have

$$\dim_{H_P} A_P(K)\otimes_{\mathbb{Z}}\mathbb{Q}\equiv \dim_{H_{P_0}} A_{P_0}(K)\otimes_{\mathbb{Z}}\mathbb{Q} \mod 2,$$

and Theorem A would imply the identity $\dim_{H_P} A_P(K)\otimes_{\mathbb{Z}}\mathbb{Q}=\dim_{H_{P_0}} A_{P_0}(K)\otimes_{\mathbb{Z}}\mathbb{Q}$ for all $P\in\Omega_{\mathbb{T}}-E$. See Conjecture 15.4 and a remark after the conjecture. Thus determining the parity is important, though we do not touch this topic in this paper.

The parity conjecture for p-Selmer groups (for the self-dual tower) holds true under good circumstances by the results of Nekovář [N06, Theorem 12.2.8], [N07] and [N09] (particularly, the result in [N07] is valid over any number field K). Thus, by modifying A_P by an isogeny so that the integer ring O_P of H_P is embedded into $\mathrm{End}(A_{P/\mathbb{Q}})$, if $\mathrm{corank}_{O_P}\mathrm{Sel}_K(A_P)^{\mathrm{ord}}\equiv\dim_{H_P} A_P(K)\otimes_{\mathbb{Z}}\mathbb{Q} \mod 2$ hold (i.e., $\mathrm{corank}_{O_P}\text{Ш}_K(A_P)_p^{\mathrm{ord}}$ is even), there is some hope of getting the parity of $\dim_{H_P} A_P(K)\otimes_{\mathbb{Z}}\mathbb{Q}$ and definitely know the generic rank in Theorem A.

The remaining case: $0=\mathrm{rank}_{\mathbb{T}} J^{\mathrm{ord}}_{\infty,\mathbb{T}}(\mathbb{Q})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)^{\vee}\ne\dim_{H_{P_0}} A_{P_0}(K)\otimes_{\mathbb{Z}}\mathbb{Q}=1$ is mysterious, though in this non-matching dimension case, if we find another point P_1 satisfying the minimalist condition with $\dim_{H_{P_1}} A_{P_1}(K)\otimes_{\mathbb{Z}}\mathbb{Q}=0$, we can bring the case into the situation in (2) (otherwise if such points do not exist, we would have unthinkable $\mathrm{rank}_{\mathbb{T}}\text{Ш}_K(J^{\mathrm{ord}}_{\infty,\mathbb{T}})^{\vee}=1$).

General statements covering all modular twisted towers (including the standard tower) will be given in the main text. The ring $\mathbb{T}$ is often a power series ring of one variable over a discrete valuation ring (and hence a unique factorization domain; see Theorem 5.6).

Let us now describe one technical idea and a most important tool for the proof. The technical idea is how to separate the p-primary part of the arithmetic cohomology groups by "(partially) completing p-adically" the coefficients, and the important ingredient is the control by $\mathrm{Gal}(X/Y)$ of rational points of Jacobians of a Galois covering $X\to Y$ of curves. Fix a base field $k=\mathbb{Q}$ or $\mathbb{Q}_l$. For an abelian variety A over k, we consider the following Galois module

$$\begin{aligned}\widehat{A}(\kappa)&=\varprojlim_n A(\kappa)/p^nA(\kappa)\quad\text{for a finite Galois extension }\kappa/k,\\ \widehat{A}(\kappa)&=\varinjlim_F \widehat{A}(F)\quad\text{for an infinite Galois extension }\kappa/k\end{aligned}\tag{1.1}$$

with F running over all finite Galois extensions k inside κ. An explicit description of $\widehat{A}(F)$ for a finite extension F/k is given at the end of this introduction as Statement (S), and only when $\kappa/k/\mathbb{Q}$ are finite extensions, we have the identity $\widehat{A}(\kappa)=A(\kappa)\otimes_{\mathbb{Z}}\mathbb{Z}_p$. The following fact plays a key role to separate the p-primary part (and also the ordinary part of it):

(P) *Though* $\mathrm{End}(A_{/\mathbb{Q}})\otimes_{\mathbb{Z}}\mathbb{Z}_p=\mathrm{End}(A[p^\infty]_{/\mathbb{Q}})$ *does not act on the abelian variety* $A_{/\mathbb{Q}}$*, it acts on the fppf/étale abelian sheaf* $\widehat{A}_{/\mathbb{Q}}$.

Thus we have $A^{\mathrm{ord}}(R)$ if A is modular abelian variety (i.e., $A = A_P$ or J_r). We thus assume now $A = A_P$. We consider the (continuous) Galois cohomology groups $H^q(K^S/K, \widehat{A}(K^S))$ for a number field K and $H^q(K, \widehat{A}(\overline{K}))$ for $k = \mathbb{Q}_l$ putting discrete topology on $\widehat{A}(\kappa)$ for $\kappa = K^S, \overline{K}$ and profinite topology on the Galois group. Here a number field means a finite extension of $\mathbb{Q}$. We write these cohomology groups as $H^q(\widehat{A})$ for a statement valid globally and locally. Recall $M_p := M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for a p-torsion module M. Then we prove, as Lemma 7.2, $H^1(\widehat{A}) \cong H^1(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p =: H^1(A)_p$, where $H^1(A)$ stands for $H^q(K^S/K, A(K^S))$ if K is global and $H^q(K, \widehat{A}(\overline{K}))$ if K is local. Thus we conclude

$$
\begin{aligned}
&Ш_K(\widehat{A}^{\mathrm{ord}}) := \mathrm{Ker}(H^1(K^S/K, \widehat{A}^{\mathrm{ord}}(K^S)) \to \prod_{v \in S} H^1(K_v, \widehat{A}^{\mathrm{ord}}(\overline{K}_v))) \cong Ш_K(A)_p^{\mathrm{ord}}, \\
&\mathrm{Sel}_K(\widehat{A}^{\mathrm{ord}}) := \mathrm{Ker}(H^1(K^S/K, \widehat{A}[p^\infty]^{\mathrm{ord}}(K^S)) \to \prod_{v \in S} H^1(K_v, \widehat{A}^{\mathrm{ord}}(\overline{K}_v))) \cong \mathrm{Sel}_K(A)_p^{\mathrm{ord}}.
\end{aligned} \tag{1.2}
$$

Anyway by (P), the p-part of the algebro-geometric Ш and Sel are translated into the sheaf theoretic counterparts. Since $\widehat{A}^{\mathrm{ord}}[p^\infty](k) \hookrightarrow \widehat{A}^{\mathrm{ord}}(k) \twoheadrightarrow \widehat{A}^{\mathrm{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an exact sequence of $\mathrm{Gal}(k/\mathbb{Q})$-modules for $k = \mathbb{Q}^S, \overline{\mathbb{Q}}_l$, assuming $N = 1$ for simplicity (so, taking $S = \{p, \infty\}$), we have a commutative diagram with exact rows:

$$
\begin{array}{ccccc}
\widehat{A}_P^{\mathrm{ord}}(\mathbb{Q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\hookrightarrow} & H^1(\mathbb{Q}^S/\mathbb{Q}, \widehat{A}_P^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\twoheadrightarrow} & H^1(\mathbb{Q}^S/\mathbb{Q}, \widehat{A}_P^{\mathrm{ord}}) \\
{\scriptstyle\delta}\downarrow & & {\scriptstyle\mathrm{Res}}\downarrow & & {\scriptstyle\mathrm{Res}}\downarrow \\
\widehat{A}_P^{\mathrm{ord}}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\hookrightarrow} & H^1(\mathbb{Q}_p, \widehat{A}_P^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\twoheadrightarrow} & H^1(\mathbb{Q}_p, \widehat{A}_P^{\mathrm{ord}}).
\end{array}
$$

Since $\widehat{A}_P^{\mathrm{ord}}(k) \otimes \mathbb{Q}_p$ for $k = \mathbb{Q}, \mathbb{Q}_p$ is a vector space over $\mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of dimension 1 if $\dim_{H_P} A_P(\mathbb{Q}) \otimes \mathbb{Q} = 1$ (as easily seen; see Lemmas 5.4 and 5.5), δ is surjective for al,most all P if the generic rank of the family is equal to 1, showing finiteness of $Ш(\mathbb{Q}^S/\mathbb{Q}, \widehat{A}_P^{\mathrm{ord}}[p^\infty])$ implies finiteness of $Ш_{\mathbb{Q}}(\widehat{A}_P^{\mathrm{ord}})$. The existence of P_0 implies the $\mathbb{T}$-torsion property of $Ш(\mathbb{Q}^S/\mathbb{Q}, \mathcal{G}_{\mathbb{T}})^\vee$ (see Corollary 11.2) and hence the assertion (3) of the theorem follows since $A_P(\mathbb{Q})$ is infinite for infinitely many $P \in \Omega_{\mathbb{T}}$ under the non-triviality condition $\mathrm{rank}_{\mathbb{T}}(J_{\infty,\mathbb{T}}^{\mathrm{ord}}(\mathbb{Q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee = 1$. Because of the use of this ad hoc fact that $A_P(\mathbb{Q}_p) \otimes \mathbb{Q}_p$ is $H_P \otimes \mathbb{Q}_p$-free of rank 1, the proof of Theorem A (3) is simpler when $K = \mathbb{Q}$. When $K \neq \mathbb{Q}$, we need to go through more technical arguments described in Section 13.

To show infinity of P having infinite $A_P(K)$ out of non-triviality of $\mathrm{rank}_{\mathbb{T}}(J_{\infty,\mathbb{T}}^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee$, we need to find a way of how to recover $A_P^{\mathrm{ord}}(K)$ from the module $J_{\infty,\mathbb{T}}^{\mathrm{ord}}(K)$ (this type of results is called control theorems relating an interpolated object to its specialization with small error terms). The control is therefore an important ingredient in showing Theorem A. Since $U(p)$ acts on $H^1(\widehat{J}_r)$, by (P), the limit idempotent e acts also on the coefficients $\widehat{J}_r$. Though e acts on the outer tensor product $H^1(J_r)_p$ without completing the coefficients, it is essential to have the action of e on the coefficients to have a control result of $\widehat{J}_r$ for $J_r = \mathrm{Pic}^0_{X_r/K}$ under the action of $\Gamma = 1 + p\mathbb{Z}_p$ (through diamond operators acting on the covering X_∞/X_1). Let us describe this in some details. The control stems from the following two facts:

(i) contraction property of the $U(p)$ operator (e.g., (u1) in Section 3), and
(ii) a high power of $U(p)$ kills the p-primary part of the kernel and cokernel of the natural (pull-back) morphism: $J_s^r(K) := \mathrm{Pic}^0_{X_s^r/\mathbb{Q}}(K) \to J_s(K)^{\Gamma^{p^{r-1}}}$ $(s > r)$ of K-rational points of Jacobians for the modular curve X_s^r of $\Gamma_s^r := \Gamma_0(p^s) \cap \Gamma_r$ (see (u) and (u2)).

The author applied in [H86b] (and [H14]) the correspondence action of $U(p)$ to the functor $X \mapsto H^1(X, \mathbb{Q}_p/\mathbb{Z}_p)$ and in [H86a] to the functor $X \mapsto H^1(X, \omega^k)$ for modular curves X and the sheaf ω^k of modular forms of weight k in order to prove the facts corresponding to (i) and (ii) in these cases which result the modular p-adic deformation theory of ordinary modular forms and the Barsotti–Tate groups of the ordinary part of the Jacobian of X (this includes the p-adic deformation theory of modular Galois representations). Though it was clear at the time that if we had a well behaving contravariant functor $X \mapsto H(X)$ with a correspondence action, we would have deformation theory of the ordinary part of $H(X)$. However there was not (at least to the author) a clear choice (other than $H^1(X,\mathbb{Q}_p/\mathbb{Z}_p)$ and $H^1(X,\omega^k)$) of the functor at the beginning. A few years after the publication of [H86a] and [H86b], the author realized that the functor $X_{/K}\mapsto H^1_{\mathrm{fppf}}(X_{/K},\mathbb{G}_m)=\mathrm{Pic}_{X/K}(K)$ would possibly work (though the application to Ш is a more recent development). This paper in conjunction with [H15] and [H16] represents the endeavour for achieving the control properties (i) and (ii) for this functor (although the work should have been done earlier). In this way, we get the control of the ind Λ-MW group, and out of the control of the ind Λ-MW group, we pull out the control of the ind Λ-Selmer group and the ind Λ-TS groups. Similarly out of the control of $\mathcal{G}$ done in [H86b] and [H14], we get the control of $Ш_K(K^S/K,\mathcal{G}_{\mathbb{T}})$ which shows the $\mathbb{T}$-torsion property of $Ш_{\mathbb{Q}}(\mathbb{Q}^S/\mathbb{Q},\mathcal{G}_{\mathbb{T}})^\vee$ under the existence of P_0 as in Theorem A. Since $J^{\mathrm{ord}}_\infty(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$ is not really studied in [H15] and [H16], this paper is essentially self-contained independent of [H15] and [H16].

We may reformulate our result via congruence among abelian varieties. For such reformulation, we recall first the definition of the congruence. An F-simple abelian variety (with a polarization) defined over a number field F is called, in this paper, "*of* $\mathrm{GL}(2)$*-type*" if we have a subfield $H_A\subset \mathrm{End}^0(A_{/F})=\mathrm{End}(A_{/F})\otimes_{\mathbb{Z}}\mathbb{Q}$ of degree $\dim A$ (stable under Rosati-involution). If $F=\mathbb{Q}$ (or more generally F has a real place), for the two-dimensional compatible system ρ_A of Galois representation of A with coefficients in H_A, H_A is generated by traces $\mathrm{Tr}(\rho_A(\mathrm{Frob}_{\mathfrak{l}}))$ of Frobenius elements $\mathrm{Frob}_{\mathfrak{l}}$ for F-primes $\mathfrak{l}$ of good reduction (i.e., the field H_A is uniquely determined by A; see [GME, §5.3.1] and [Sh75, Theorem 0]). We always regard F as a subfield of the algebraic closure $\overline{\mathbb{Q}}$. Thus $O'_A:=\mathrm{End}(A_{/F})\cap H_A$ is an order of H_A. Write O_A for the integer ring of H_A. Replacing A by the abelian variety representing the group functor $R\mapsto A(R)\otimes_{O'_A}O_A$, we may choose A so that $O'_A=O_A$ in the F-isogeny class of A. Since finiteness of the Tate–Shafarevich group of A (not necessarily its exact size) is determined by the F-isogeny class of A, we hereafter assume that $\mathrm{End}(A_{/F})\cap H_A=O_A$ for any abelian variety of GL(2)-type over F. For two abelian varieties A and B of GL(2)-type over F, we say that A is *congruent to B modulo a prime p over F* if we have a prime factor $\mathfrak{p}_A$ (resp. $\mathfrak{p}_B$) of p in O_A (resp. O_B) and field embeddings $\sigma_A:O_A/\mathfrak{p}_A\hookrightarrow\overline{\mathbb{F}}_p$ and $\sigma_B:O_B/\mathfrak{p}_B\hookrightarrow\overline{\mathbb{F}}_p$ such that $(A[\mathfrak{p}_A]\otimes_{O_A/\mathfrak{p}_A,\sigma_A}\overline{\mathbb{F}}_p)^{ss}\cong(B[\mathfrak{p}_B]\otimes_{O_B/\mathfrak{p}_B,\sigma_B}\overline{\mathbb{F}}_p)^{ss}$ and $\det T_{\mathfrak{p}_A}A=\det T_{\mathfrak{p}_B}B$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-characters, where the superscript "*ss*" indicates the semi-simplification and $T_{\mathfrak{p}_A}A$ is the $\mathfrak{p}_A$-adic Tate module of the abelian variety A. Hereafter, we assume that the field F is equal to $\mathbb{Q}$, but the base field K is any number field.

Let $E_{/\mathbb{Q}}$ be an elliptic curve. Writing the Hasse–Weil L-function $L(s,E)$ as a Dirichlet series $\sum_{n=1}a_nn^{-s}$ with $a_n\in\mathbb{Z}$ (i.e., $1+p-a_p=|E(\mathbb{F}_p)|$ for each prime p of good reduction for E), we call p *admissible* for E if E has good reduction at p, the self-dual p-adic analytic family including E has **generic rank equal to** $\mathrm{rank}\,E(\mathbb{Q})$ and (a_p mod p) is not in $\Omega_E:=\{\pm1,0\}$. Therefore, the maximal étale quotient of $E[p]$ over $\mathbb{Z}_p$ is **not** isomorphic to $\mathbb{Z}/p\mathbb{Z}$ up to unramified quadratic twists. By the Hasse bound $|a_p|\le 2\sqrt{p}$, $p\ge7$ is not admissible if and only if $a_p\in\Omega_E$ (so, 2 and 3 are not admissible). Thus if E does not have complex multiplication, the Dirichlet density of non-admissible primes is zero by a theorem of Serre as $L(s,E)=L(s,f)$ for a rational Hecke eigenform f. A proto-typical fact in this reformulation (which follows from Theorem A) is

Theorem B. *Let $E_{/\mathbb{Q}}$ be a p-ordinary elliptic curve with $|Ш_K(E)|<\infty$ and $\dim_{\mathbb{Q}}E(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Q}=1$. Let N be the conductor of E, and pick an admissible prime p for E. Consider the set $\mathcal{A}_{E,p}$ made up of all $\mathbb{Q}$-isogeny classes of $\mathbb{Q}$-simple abelian varieties $A_{/\mathbb{Q}}$ of* $\mathrm{GL}(2)$*-type with prime-to-p conductor N congruent to E modulo p over $\mathbb{Q}$. Then there exists an explicit (computable) finite set S_E of primes depending on N but independent of K such that if $p\notin S_E$, infinite members $A\in\mathcal{A}_{E,p}$ have finite* $Ш_{\mathbb{Q}}(A)_{\mathfrak{p}_A}$ *and constant dimension* $\dim_{H_A}A(\mathbb{Q})\otimes\mathbb{Q}=1$.

Here for the prime $\mathfrak{p}_A|p$, we have $(A[\mathfrak{p}_A]\otimes_{O_A/\mathfrak{p}_A,\sigma_A}\overline{\mathbb{F}}_p)^{ss}\cong(E[p]\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p)^{ss}$, and $Ш_K(A)_{\mathfrak{p}_A}$ (resp. $\mathrm{Sel}_K(A)_{\mathfrak{p}_A}$) is the $\mathfrak{p}_A$-primary part of $Ш_K(A)_p$ (resp. $\mathrm{Sel}_K(A)_p$). We stated this theorem for $K=\mathbb{Q}$ and an elliptic curve E as it is difficult to verify the minimalist condition numerically for general K and a higher dimensional abelian variety, though we state and prove this theorem as Theorem 15.2 in the general setting. Also the definition of the set S_E will be given in Definition 15.1.

If instead assuming finiteness of $E(\mathbb{Q})$ and $Ш_{\mathbb{Q}}(E)$ (i.e., finiteness of $\mathrm{Sel}_{\mathbb{Q}}(E)$), this type of results is known at least for the standard tower under possibly different assumptions by the control theorem of the Λ-Selmer group. The new point of the above theorem is that we allow $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1$. The optimal expectation would be to have the assertion of the theorem for almost all members of $\mathcal{A}_{E,p}$ but the theorem is short of this expectation. When the ϵ-factor is -1 for E in the theorem, as already discussed, this optimal expectation follows from the weak form of the Birch–Swinnerton Dyer conjecture.

For p outside S_E, the local ring $\mathbb{T}$ containing $E = A_{P_0}$ is unique and is a regular ring (so, UFD). The set S_E is usually very small (and for example, for the rank 1 elliptic curve of conductor 37, S_E is empty). If we assume the Birch-Swinnerton Dyer conjecture for abelian varieties of GL(2)-type and we start with E having epsilon factor -1, the generic rank condition for the family should be valid always in the definition of admissible primes.

For an extension X of an abelian variety by a finite group scheme defined either over a number field K or a local field K of characteristic 0, we define the fppf abelian sheaf $\widehat{X}$ explicitly as follows:

$$\text{(S)} \qquad \widehat{X}(R) = \begin{cases} X(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \text{if } [K:\mathbb{Q}] < \infty, \\ X[p^\infty](R) & \text{if } [K:\mathbb{Q}_l] < \infty \ (l \neq p) \text{ or } [K:\mathbb{R}] < \infty, \\ (X/X^{(p)})(R) \text{ as a sheaf quotient} & \text{if } [K:\mathbb{Q}_p] < \infty \end{cases}$$

for fppf algebras $R_{/K}$, where $X^{(p)}$ is the maximal prime-to-p torsion subgroup of X. If R is a finite extension field of K (except for the case of $K = \mathbb{R}, \mathbb{C}$), $\widehat{X}(R) = \varprojlim_n X(R)/p^n X(R)$ as already mentioned. Therefore, we could have defined $\widehat{X}(R) := \varprojlim_n X(R)/p^n X(R)$ except in the case where $K = \mathbb{R}, \mathbb{C}$ (and using this definition, the value $\widehat{X}(R)$ is computed in [H15, (S) in page 228] as specified in (S) above). For $K = \mathbb{R}, \mathbb{C}$, this is just a convention as $H^q(K, ?)$ with coefficients in a $\mathbb{Z}_p$-module ? just vanishes if $p > 2$. Throughout the paper, we write $M^\vee$ for the Pontryagin dual module $\mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ for a $\mathbb{Z}_p$-module M.

Contents

2. $U(p)$-ISOMORPHISMS

Replacing fppf cohomology we described in [H15, §3] by étale cohomology, we reproduce the results and proofs in [H15, §3] as it gives the foundation of our control result, though we need later to adjust technically the method described here to get precise control of the limit Tate–Shafarevich group. Let $S = \mathrm{Spec}(K)$ for a field K. Let $X \to Y \to S$ be proper morphisms of noetherian schemes. We study

$$H^0_{\mathrm{fppf}}(T, R^1 f_*\mathbb{G}_m) = H^0_{\text{ét}}(T, R^1 f_*\mathbb{G}_m) = R^1 f_* O_X^\times(T) = \mathrm{Pic}_{X/S}(T)$$

for S-scheme T and the structure morphism $f : X \to S$. Write the morphisms as $X \xrightarrow{\pi} Y \xrightarrow{g} S$ with $f = g \circ \pi$. We note the following general fact:

Lemma 2.1. *Assume that π is finite flat. Then the pull-back of line bundles: $\mathrm{Pic}_{Y/S}(T) \ni \mathcal{L} \mapsto \pi^*\mathcal{L} \in \mathrm{Pic}_{X/S}(T)$ induces the Picard functoriality which is a natural transformation $\pi^* : \mathrm{Pic}_{Y/S} \to \mathrm{Pic}_{X/S}$ contravariant with respect to π. Similarly, we have the Albanese functoriality sending $\mathcal{L} \in \mathrm{Pic}_{X/S}(T)$ to $\bigwedge^{\deg(X_{/Y})} \pi_*\mathcal{L} \in \mathrm{Pic}_{Y/S}(T)$ as long as X has constant degree over Y. This map $\pi_* : \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{Y/S}$ is a natural transformation covariant with respect to π.*

Hereafter we always assume that π is finite flat with constant degree.

In [H15, §3], we assumed that f and g have compatible sections $S \xrightarrow{s_g} Y$ and $S \xrightarrow{s_f} X$ so that $\pi \circ s_f = s_g$. However in this paper, we do not assume the existence of compatible sections, but we limit ourselves to $T = \mathrm{Spec}(\kappa)$ for an étale extension κ of the base field K. Then we get (e.g., [NMD, Section 8.1] and [ECH, Chapter 3]), writing $X_T = X \times_S T$ and $Y_T = Y \times_S T$,

$$\mathrm{Pic}_{X/S}(T) = H^0_{\mathrm{fppf}}(T, R^1 f_*\mathbb{G}_m) \overset{(*)}{=} H^1_{\mathrm{fppf}}(X_T, O_{X_T}^\times) = H^1_{\text{ét}}(X_T, O_{X_T}^\times)$$
$$\mathrm{Pic}_{Y/S}(T) = H^0_{\mathrm{fppf}}(T, R^1 g_*\mathbb{G}_m) \overset{(*)}{=} H^1_{\mathrm{fppf}}(Y_T, O_{Y_T}^\times) = H^1_{\text{ét}}(Y_T, O_{Y_T}^\times)$$

for any S-scheme T. The identity at $(*)$ follows from the fact: $\mathrm{Pic}_T = 0$, since T is a union of points (i.e., $\kappa = k_1 \oplus \cdots \oplus k_m$ for finite separable field extensions k_j/K). We suppose that the functors $\mathrm{Pic}_{X/S}$ and $\mathrm{Pic}_{Y/S}$ are representable by group schemes whose connected components are smooth (for example, if X, Y are smooth proper and geometrically reduced (and $S = \mathrm{Spec}(K)$ for a field K); see [NMD, 8.2.3, 8.4.2–3]). We then write $J_? = \mathrm{Pic}^0_{?/S}$ $(? = X, Y)$ for the identity connected component of $\mathrm{Pic}_{?/S}$. Anyway we suppose hereafter also that X, Y, S are varieties (i.e., geometrically reduced separated schemes of finite type over a field).

For an fppf covering $\mathcal{U} \to Y$ and a presheaf $P = P_Y$ on the fppf site over Y, we define via Čech cohomology theory an fppf presheaf $\mathcal{U} \mapsto \check{H}^q(\mathcal{U}, P)$ denoted by $\underline{\check{H}}^q(P_Y)$ (see [ECH, III.2.2 (b)]). The inclusion functor from the category of fppf sheaves over Y into the category of fppf presheaves over Y is left exact. The derived functor of this inclusion of an fppf sheaf $F = F_Y$ is denoted by $\underline{H}^\bullet(F_Y)$ (see [ECH, III.1.5 (c)]). Thus $\underline{H}^\bullet(\mathbb{G}_{m/Y})(\mathcal{U}) = H^\bullet_{\mathrm{fppf}}(\mathcal{U}, O_{\mathcal{U}}^\times)$ for a Y-scheme $\mathcal{U}$ as a presheaf (here $\mathcal{U}$ varies in the small fppf site over Y).

To study control of the Picard groups under Galois action, assuming that f, g and π are all faithfully flat of finite presentation, we use the spectral sequence of Čech cohomology of the flat covering $\pi : X \twoheadrightarrow Y$ in the fppf site over Y [ECH, III.2.7]:

$$\check{H}^p(X_T/Y_T, \underline{H}^q(\mathbb{G}_{m/Y})) \Rightarrow H^n_{\mathrm{fppf}}(Y_T, O_{Y_T}^\times) \underset{\iota}{\xrightarrow{\sim}} H^n(Y_T, O_{Y_T}^\times) \tag{2.1}$$

for each S-scheme T. Here $F \mapsto H^n_{\mathrm{fppf}}(Y_T, F)$ (resp. $F \mapsto H^n(Y_T, F)$) is the right derived functor of the global section functor: $F \mapsto F(Y_T)$ from the category of fppf sheaves (resp. Zariski sheaves) over Y_T to the category of abelian groups. The isomorphism ι is the one given in [ECH, III.4.9].

Write $\underline{H}^\bullet_{Y_T}$ for $\underline{H}^\bullet(\mathbb{G}_{m/Y_T})$ and $\check{H}^\bullet(\underline{H}^0_{Y_T})$ for $\check{H}^\bullet(X_T/Y_T, \underline{H}^0_{Y_T})$. From this spectral sequence, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\check{H}^1(\underline{H}^0_{Y_T}) & \xrightarrow{\hookrightarrow} & \mathrm{Pic}_{Y/S}(T) & \xrightarrow{a} & \check{H}^0(\frac{X_T}{Y_T}, \underline{H}^1_{Y_T}) & \longrightarrow & \check{H}^2(\underline{H}^0_{Y_T}) \\
\| \uparrow & & \wr \uparrow & & \uparrow \| & & \uparrow \| \\
\check{H}^1(\underline{H}^0_{Y_T}) & \longrightarrow & \mathrm{Pic}_{Y/S}(T) & \xrightarrow{b} & \check{H}^0(\frac{X_T}{Y_T}, \mathrm{Pic}_X(T)) & \longrightarrow & \check{H}^2(\underline{H}^0_{Y_T}) \\
\uparrow & & \cup \uparrow & & \uparrow \cup & & \uparrow \\
?_1 & \longrightarrow & J_Y(T) & \xrightarrow[c]{} & \check{H}^0(\frac{X_T}{Y_T}, J_X(T)) & \longrightarrow & ?_2.
\end{array} \tag{2.2}$$

Here the horizontal exactness at the top two rows follows from the spectral sequence (2.1).

Take a correspondence $U \subset Y \times_S Y$ given by two finite flat projections $\pi_1, \pi_2 : U \to Y$ of constant degree (i.e., $\pi_{j,*}\mathcal{O}_U$ is locally free of finite rank $\deg(\pi_j)$ over $\mathcal{O}_Y$). Consider the pullback $U_X \subset X \times_S X$ given by the Cartesian diagram:

$$\begin{array}{ccc}
U_X = U \times_{Y \times_S Y} (X \times_S X) & \longrightarrow & X \times_S X \\
\downarrow & & \downarrow \\
U & \xrightarrow{\hookrightarrow} & Y \times_S Y
\end{array}$$

Let $\pi_{j,X} = \pi_j \times_S \pi : U_X \twoheadrightarrow X$ $(j = 1, 2)$ be the projections.

Consider a new correspondence $U_X^{(q)} = \overbrace{U_X \times_Y U_X \times_Y \cdots \times_Y U_X}^{q}$, whose projections are the iterated product

$$\pi_{j,X^{(q)}} = \pi_{j,X} \times_Y \cdots \times_Y \pi_{j,X} : U_X^{(q)} \to X^{(q)} \ (j = 1, 2).$$

Here is a first step to get a control result of Λ-TS groups:

Lemma 2.2. *Let the notation and the assumption be as above. In particular, $\pi : X \to Y$ is a finite flat morphism of geometrically reduced proper schemes over $S = \mathrm{Spec}(K)$ for a field K. Suppose that X and U_X are proper schemes over a field K satisfying one of the following conditions:*

1. *U_X is geometrically reduced, and for each geometrically connected component X° of X, its pull back to U_X by $\pi_{2,X}$ is also connected; i.e., $\pi^0(X) \xrightarrow[\sim]{\pi^*_{2,X}} \pi^0(U_X)$;*
2. *$(f \circ \pi_{2,X})_*\mathcal{O}_{U_X} = f_*\mathcal{O}_X$.*

If $\pi_2 : U \to Y$ has constant degree $\deg(\pi_2)$, then, for each $q > 0$, the action of $U^{(q)}$ on $H^0(X, \mathcal{O}^\times_{X^{(q)}})$ factors through the multiplication by $\deg(\pi_2) = \deg(\pi_{2,X})$.

This result is given as [H15, Lemma 3.1, Corollary 3.2]. Though in [H15, §3], an extra assumption of requiring the existence of compatible sections to $X \to Y \to S$, this assumption is nothing to do with the proof of the above lemma, and hence the proof there is valid without any modification.

To describe the correspondence action of U on $H^0(X, \mathcal{O}_X^\times)$ in down-to-earth terms, let us first recall that the Čech cohomology $\check{H}^q(\frac{X_T}{Y_T}, \underline{H}^0(\mathbb{G}_{m/Y}))$ for a general S-scheme T is given by

$$\frac{\{(c_{i_0,\dots,i_q}) | c_{i_0,\dots,i_q} \in H^0(X_T^{(q+1)}, O^\times_{X_T^{(q+1)}}) \text{ and } \prod_j (c_{i_0\dots\check{i}_j\dots i_{q+1}} \circ p_{i_0\dots\check{i}_j\dots i_{q+1}})^{(-1)^j} = 1\}}{\{db_{i_0\dots i_q} = \prod_j (b_{i_0\dots\check{i}_j\dots i_q} \circ p_{i_0\dots\check{i}_j\dots i_q})^{(-1)^j} | b_{i_0\dots\check{i}_j\dots i_q} \in H^0(X_T^{(q)}, O^\times_{X_T^{(q)}})\}} \tag{2.3}$$

where we agree to put $H^0(X_T^{(0)}, O^{(0)}_{X_T}) = 0$ as a convention,

$$X_T^{(q)} = \overbrace{X \times_Y X \times_Y \cdots \times_Y X}^{q} \times_S T, O_{X_T^{(q)}} = \overbrace{O_X \times_{O_Y} O_X \times_{O_Y} \cdots \times_{O_Y} O_X}^{q} \times_{O_S} O_T,$$

the identity $\prod_j (c \circ p_{i_0 \dots \check{i}_j \dots i_{q+1}})^{(-1)^j} = 1$ takes place in $O_{X_T^{(q+2)}}$ and $p_{i_0 \dots \check{i}_j \dots i_{q+1}} : X_T^{(q+2)} \to X_T^{(q+1)}$ is the projection to the product of X the j-th factor removed. Since $T \times_T T \cong T$ canonically, we have $X_T^{(q)} \cong \overbrace{X_T \times_T \cdots \times_T X_T}^{q}$ by transitivity of fiber product.

Consider $\alpha \in H^0(X, \mathcal{O}_X)$. Then we lift $\pi_{1,X}^* \alpha = \alpha \circ \pi_{1,X} \in H^0(U_X, \mathcal{O}_{U_X})$. Put $\alpha_U := \pi_{1,X}^* \alpha$. Note that $\pi_{2,X,*}\mathcal{O}_{U_X}$ is locally free of rank $d = \deg(\pi_2)$ over $\mathcal{O}_X$, the multiplication by α_U has its characteristic polynomial $P(T)$ of degree d with coefficients in $\mathcal{O}_X$. We define the norm $N_U(\alpha_U)$ to be the constant term $P(0)$. Since α is a global section, $N_U(\alpha_U)$ is a global section, as it is defined everywhere locally. If $\alpha \in H^0(X, \mathcal{O}_X^\times)$, $N_U(\alpha_U) \in H^0(X, \mathcal{O}_X^\times)$. Then define $U(\alpha) = N_U(\alpha_U)$, and in this way, U acts on $H^0(X, \mathcal{O}_X^\times)$.

For a degree q Čech cohomology class $[c] \in \check{H}^q(X_{/Y}, \underline{H}^0(\mathbb{G}_{m/Y}))$ with a Čech q-cocycle $c = (c_{i_0,\dots,i_q})$, $U([c])$ is given by the cohomology class of the Čech cocycle $U(c) = (U(c_{i_0,\dots,i_q}))$, where $U(c_{i_0,\dots,i_q})$ is the image of the global section $c_{i_0,\dots,i_q}$ under U. Indeed, $(\pi_{1,X}^* c_{i_0,\dots,i_q})$ plainly satisfies the cocycle condition, and $(N_U(\pi_{1,X}^* c_{i_0,\dots,i_q}))$ is again a Čech cocycle as N_U is a multiplicative homomorphism. By the same token, this operation sends coboundaries to coboundaries, and define the action of U on the cohomology group. Thus we get the following vanishing result:

Proposition 2.3. *Suppose that $S = \mathrm{Spec}(K)$ for a field K. Let $\pi : X \to Y$ be a finite flat covering of (constant) degree d of geometrically reduced proper varieties over K, and let $Y \xleftarrow{\pi_1} U \xrightarrow{\pi_2} Y$ be two finite flat coverings (of constant degree) identifying the correspondence U with a closed subscheme $U \overset{\pi_1 \times \pi_2}{\hookrightarrow} Y \times_S Y$. Write $\pi_{j,X} : U_X = U \times_Y X \to X$ for the base-change to X. Suppose one of the conditions* (1) *and* (2) *of* Lemma 2.2 *for (X, U). Then*

1. *The correspondence $U \subset Y \times_S Y$ sends $\check{H}^q(\underline{H}^0_Y)$ into $\deg(\pi_2)(\check{H}^q(\underline{H}^0_Y))$ for all $q > 0$.*
2. *If d is a p-power and $\deg(\pi_2)$ is divisible by p, $\check{H}^q(\underline{H}^0_Y)$ for $q > 0$ is killed by U^M if $p^M \geq d$.*
3. *The cohomology $\check{H}^q(\underline{H}^0_Y)$ with $q > 0$ is killed by d.*

This follows from Lemma 2.2, because on each Čech q-cocycle (whose value is a global section of iterated product $X_T^{(q+1)}$), the action of U is given by $U^{(q+1)}$ by (2.3). See [H15, Proposition 3.3] for a detailed proof.

Assume that a finite group G acts on $X_{/Y}$ faithfully. Then we have a natural morphism $\phi : X \times G \to X \times_Y X$ given by $\phi(x, \sigma) = (x, \sigma(x))$. Suppose that ϕ is surjective; for example, if Y is a geometric quotient of X by G; see [GME, §1.8.3]). Under this map, for any fppf abelian sheaf F, we have a natural map $\check{H}^0(X/Y, F) \to H^0(G, F(X))$ sending a Čech 0-cocycle $c \in H^0(X, F) = F(X)$ (with $p_1^* c = p_2^* c$) to $c \in H^0(G, F(X))$. Obviously, by the surjectivity of ϕ, the map $\check{H}^0(X/Y, F) \to H^0(G, F(X))$ is an isomorphism (e.g., [ECH, Example III.2.6, page 100]). Thus we get

Lemma 2.4. *Let the notation be as above, and suppose that ϕ is surjective. For any scheme T fppf over S, we have a canonical isomorphism: $\check{H}^0(X_T/Y_T, F) \cong H^0(G, F(X_T))$.*

We now assume $S = \mathrm{Spec}(K)$ for a field K and that X and Y are proper reduced connected curves. Then we have from the diagram (2.2) with the exact middle two columns and exact horizontal rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 \\
\uparrow & & {\scriptstyle \deg}\uparrow{\scriptstyle \text{onto}} & & {\scriptstyle \deg}\uparrow{\scriptstyle \text{onto}} & & \uparrow \\
\check{H}^1(\underline{H}^0_Y) & \longrightarrow & \mathrm{Pic}_{Y/S}(T) & \xrightarrow{b} & \check{H}^0(\frac{X_T}{Y_T}, \mathrm{Pic}_{X/S}(T)) & \longrightarrow & \check{H}^2(\underline{H}^0_Y) \\
\uparrow & & \cup\uparrow & & \uparrow\cup & & \uparrow \\
?_1 & \longrightarrow & J_Y(T) & \xrightarrow[c]{} & \check{H}^0(\frac{X_T}{Y_T}, J_X(T)) & \longrightarrow & ?_2,
\end{array}$$

Thus we have $?_j = \check{H}^j(\underline{H}^0_Y)$ $(j = 1, 2)$.

By Proposition 2.3, if $q > 0$ and X/Y is of degree p-power and $p | \deg(\pi_2)$, $\check{H}^q(\underline{H}^0_Y)$ is a p-group, killed by U^M for $M \gg 0$.

3. Exotic modular curves

We study a more general tower $\{X_r\}_r$ different from the standard one $\{X_1(Np^r)\}_r$. We introduce open compact subgroups of $\mathrm{GL}_2(\mathbb{A}^{(\infty)})$ giving rise to the general tower $\{X_r\}_r$.

Let $\Gamma := 1 + p^\epsilon\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, where $\epsilon = 2$ if $p = 2$ and $\epsilon = 1$ otherwise. Let $\gamma = 1 + p^\epsilon$, which is a topological generator of $\Gamma = \gamma^{\mathbb{Z}_p}$. We define the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\Gamma]] = \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$ and identify it with the power series ring $\mathbb{Z}_p[[T]]$ sending γ to $t = 1+T$. The group Γ is a maximal torsion-free subgroup of $\mathbb{Z}_p^\times$. Fix an exact sequence of profinite groups $1 \to H_p \to \Gamma \times \Gamma \xrightarrow{\pi_\Gamma} \Gamma \to 1$, and regard H_p as a subgroup of $\Gamma \times \Gamma$. This implies

$$\pi_\Gamma(a, d) = a^\alpha d^{-\delta} \tag{3.1}$$

for a pair $(\alpha, \delta) \in \mathbb{Z}_p^2$ with $\alpha\mathbb{Z}_p + \delta\mathbb{Z}_p = \mathbb{Z}_p$ and hence $H_p = \{(a, d) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times | a^\alpha d^{-\delta} = 1\}$. Thus H_p only depends on $\mathbf{P}^1(\mathbb{Z}_p)$; so, we freely identify (α, δ) with $(z\alpha, z\delta)$ for any $z \in \mathbb{Z}_p^\times$. Writing μ for the maximal torsion subgroup of $\mathbb{Z}_p^\times$, we pick a character $\xi : \mu \times \mu \to \mathbb{Z}_p^\times$ and define $H = H_\xi = H_{\alpha,\delta,\xi} := H_p \times \mathrm{Ker}(\xi)$ in $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times = \Gamma \times \Gamma \times \mu \times \mu$. We can take $\xi(\zeta, \zeta') = \zeta^{\alpha'} \zeta'^{-\delta'}$ for $(\alpha', \delta') \in \mathbb{Z}^2$. Write $\pi := \pi_\Gamma \times \xi : \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ and the image of H in $(\mathbb{Z}_p^\times)^2/(\Gamma^{p^{r-\epsilon}})^2$ as H_r. Then define, for $\widehat{\mathbb{Z}} = \prod_{l:\text{primes}} \mathbb{Z}_l$,

$$\begin{aligned}
&\widehat{\Gamma}_0(M) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \middle| c \in M\widehat{\mathbb{Z}} \right\},\ \widehat{\Gamma}_1(M) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M) \middle| d - 1 \in M\widehat{\mathbb{Z}} \right\}, \\
&\widehat{\Gamma}_1^1(M) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(M) \middle| a - 1 \in M\widehat{\mathbb{Z}} \right\}, \\
&\widehat{\Gamma}_s = \widehat{\Gamma}_{H,s} := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}_0(p^s) \cap \widehat{\Gamma}_1(N) \middle| (a_p, d_p) \in H_s \right\},\ \widehat{\Gamma}_s^r = \widehat{\Gamma}_{H,s}^r := \widehat{\Gamma}_0(p^s) \cap \widehat{\Gamma}_r \quad (s \geq r).
\end{aligned} \tag{3.2}$$

By definition, $\widehat{\Gamma}_r \cap \mathrm{SL}_2(\mathbb{Q}) = \Gamma_1(Np^r)$ if $H_p = \Gamma \times \{1\}$ (i.e., $(\alpha, \delta) = (0, 1)$) and $\xi(a, d) = \omega(d)$ for $\omega(a) = \lim_{n\to\infty} a^{p^n}$ if p is odd and otherwise $\omega(a) = \left(\frac{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}{a}\right)$ (the quadratic residue symbol)). We write this ξ as ω_d.

Consider the moduli problem over $\mathbb{Q}$ of classifying the following triples

$$(E, \mu_N \underset{\phi_N}{\overset{\hookrightarrow}{\longrightarrow}} E, \mu_{p^r} \underset{\phi_{p^r}}{\overset{\hookrightarrow}{\longrightarrow}} E[p^r] \underset{\varphi_{p^r}}{\overset{\twoheadrightarrow}{\longrightarrow}} \mathbb{Z}/p^r\mathbb{Z})_{/R},$$

where E is an elliptic curve defined over a $\mathbb{Q}$-algebra R and the sequence $\mu_{p^n} \hookrightarrow E[p^r] \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z}$ is meant to be exact in the category of finite flat group schemes. As is well known (e.g., [AME]), the triples are classified by a modular curve $U_{r/\mathbb{Q}}$, and we write Z_r for the compactification of U_r smooth at cusps. In Shimura's terminology, writing Z'_r for the canonical model attached to $U_r := \widehat{\Gamma}_1^1(p^r) \cap \widehat{\Gamma}_1(N)$, the curve Z'_r is defined over $\mathbb{Q}(\mu_{p^r})$ and is geometrically irreducible, while we have $Z_r = \mathrm{Res}_{\mathbb{Q}(\mu^{p^r})/\mathbb{Q}} Z'_r$ (when $N \geq 4$) which is not geometrically irreducible. We have the identity of the complex points $Z_r(\mathbb{C}) - \{\text{cusps}\} = \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/U_r\mathbb{R}^\times\mathrm{SO}_2(\mathbb{R})$.

Each element (u, a, d) of the group $G := (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ acts on Z_r by sending $(E, \mu_N \xrightarrow{\phi_N} E, \mu_{p^r} \underset{\hookrightarrow}{\xrightarrow{\phi_{p^r}}} E[p^r] \underset{\twoheadrightarrow}{\xrightarrow{\varphi_{p^r}}} \mathbb{Z}/p^r\mathbb{Z})$ to

$$(E, \phi_N \circ u : \mu_N \xrightarrow{\phi_N \circ u} E, \mu_{p^r} \underset{\hookrightarrow}{\xrightarrow{\phi_{p^r} \circ d}} E[p^r] \underset{\twoheadrightarrow}{\xrightarrow{a \circ \varphi_{p^r}}} \mathbb{Z}/p^r\mathbb{Z}), \tag{3.3}$$

where $a \circ \varphi_{p^r}(x) = a\varphi_{p^r}(x)$ and the action on ϕ_{p^r} and ϕ_N. For $z = (z_N, z_p) \in (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times$, we write the action of $(u, a, d) = (z_N, z_p, z_p)$ as $\langle z \rangle$. Via the inclusion $\Gamma \times \Gamma \subset G$, the two variable Iwasawa algebra $\mathbf{\Lambda} := \mathbb{Z}_p[[\Gamma \times \Gamma]]$ is embedded into $\mathbb{Z}_p[[G]] = \mathbf{\Lambda}[(\mathbb{Z}/N\mathbb{Z})^\times \times \mu \times \mu]$ for the maximal torsion subgroup μ of $\mathbb{Z}_p^\times$.

We consider the quotient curves $X_r := Z_r/H$. The complex points of X_r removed cusps is given by $Y_r(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\widehat{\Gamma}_r\mathbb{R}_+\mathrm{SO}_2(\mathbb{R})$. Indeed, the action of $(a_p, d_p) \in H$ regarded as an element $\left(\begin{smallmatrix} a_p & 0 \\ 0 & d_p \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_p) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is given by $(\phi_{p^r}, \varphi_{p^r}) \mapsto (\phi_{p^r} \circ d_p, \varphi_{p^r} \circ a_p)$. If $\det(\widehat{\Gamma}_r) = \widehat{\mathbb{Z}}^\times$, by [IAT, Chapter 6], X_r is a geometrically connected curve canonically defined over $\mathbb{Q}$. We have an adelic expression of their complex points.

$$X_s^r(\mathbb{C}) - \{\text{cusps}\} = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\widehat{\Gamma}_s^r \mathbb{R}_+^\times \mathrm{SO}_2(\mathbb{R}) = \Gamma_s^r\backslash\mathfrak{H} \text{ and } X_r(\mathbb{C}) = \Gamma_r\backslash\mathfrak{H},$$

where $\Gamma_s^r = \widehat{\Gamma}_s^r \cap \mathrm{SL}_2(\mathbb{Q})$ and $\Gamma_r = \widehat{\Gamma}_r \cap \mathrm{SL}_2(\mathbb{Q})$. If $\det(\widehat{\Gamma}_r) \subsetneq \widehat{\mathbb{Z}}^\times$, our curve $X_r^s = \mathrm{Res}_{F_\xi/\mathbb{Q}} V_{\widehat{\Gamma}_s^r}$ and $X_r = \mathrm{Res}_{F_\xi/\mathbb{Q}} V_{\widehat{\Gamma}_r}$ for Shimura's geometrically irreducible canonical model V_S defined over F_ξ for $S = \widehat{\Gamma}_s^r$ and $\widehat{\Gamma}_r$ (see [IAT, Chapter 6]). In any case, these curves are geometrically reduced curves defined over $\mathbb{Q}$ with equal number of geometrically connected components (i.e., it is $[F_\xi : \mathbb{Q}]$ for Shimura's field of definition $F_\xi \subset \mathbb{Q}^{ab}$ fixed by $\det(\widehat{\Gamma}_s^r) \subset \widehat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$).

We fix a $\mathbb{Z}_p$ basis $(\zeta_{Np^r} = \exp(\frac{2\pi i}{Np^r}))_r \in \mathbb{Z}_p(1) \times (\mathbb{Z}/N\mathbb{Z})(1)$. Then we identify μ_{Np^r} with $(\mathbb{Z}/Np^r\mathbb{Z})$ by $\zeta_{Np^r}^m \mapsto (m \bmod Np^r)$. For a triple $(E, \mu_N \xrightarrow{\phi_N} E, \mu_{p^r} \overset{\phi_{p^r}}{\hookrightarrow} E[p^r] \overset{\varphi_{p^r}}{\twoheadrightarrow} \mathbb{Z}/p^r\mathbb{Z})$, by the canonical duality $\langle\cdot,\cdot\rangle$ on $E[Np^r]$, we have a unique generator $v \in E[Np^r]/\mathrm{Im}(\phi_{Np^r})$ for $\phi_{Np^r} = \phi_N \times \phi_{p^r}$ such that $\langle v, \phi_{Np^r}(\zeta_{Np^r})\rangle = \zeta_{Np^r}$. Then the quotient $E' := E/(\mathrm{Im}(\phi_{Np^r}))$ has an inclusion $\mu_{Np^r} \xrightarrow{\phi'_{Np^r}} E'$ given by sending $\zeta_{Np^r}^a$ to $(av \bmod \mathrm{Im}(\phi_{Np^r})) \in E'$. This gives a new triple $(E', \phi'_N, \phi'_{p^r}, \varphi'_{p^r})$, where φ'_{p^r} is determined by $\langle x, \phi'_{p^r}(\zeta_{p^r})\rangle = \zeta_{p^r}^{\varphi'_{p^r}(x)}$ for $x \in E'[p^r]$. We define an operator $w_r = w_{\zeta_{Np^r}}$ acting on Z_r by sending $(E, \mu_N \xrightarrow{\phi_N} E, \mu_{p^r} \overset{\phi_{p^r}}{\hookrightarrow} E[p^r] \overset{\varphi_{p^r}}{\twoheadrightarrow} \mathbb{Z}/p^r\mathbb{Z})$ to the above $(E', \phi'_N, \phi'_{p^r}, \varphi'_{p^r})$. We have the following fact from the definition:

Lemma 3.1. *The tower $\{X_{r/\mathbb{Q}}\}_r$ with respect to (α, δ, ξ) is isomorphically sent by w_r defined over $\mathbb{Q}$ to the tower over $\mathbb{Q}$ with respect to (δ, α, ξ') for $\xi'(a, d) = \xi(d, a)$. In other words, H defining the tower $\{X_r\}_r$ is send to H' defining the other by the involution $(a, d) \mapsto (d, a)$. Regarding w_r as an involution of X_r defined over $\mathbb{Q}(\mu_{Np^r})$, if $\sigma_z \in \mathrm{Gal}(\mathbb{Q}(\mu_{Np^r})/\mathbb{Q})$ for $z \in (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times$ is given by $\sigma_z(\zeta_{Np^r}) = \zeta_{Np^r}^z$, we have $w_r^{\sigma_z} = \langle z\rangle \circ w_r = w_r \circ \langle z\rangle^{-1}$.*

The last assertion of the lemma follows from $w_{\zeta_{Np^r}}^{\sigma_z} = w_{\sigma_z(\zeta_{Np^r})} = w_{\zeta_{Np^r}^z} = \langle z\rangle \circ w_{\zeta_{Np^r}}$ and $w_r^2 = \mathrm{id}$.

The group $\widehat{\Gamma}_s^r$ $(s > r)$ normalizes $\widehat{\Gamma}_s$, and we have $\widehat{\Gamma}_s^0/\widehat{\Gamma}_s = \Gamma_s^0/\Gamma_s$ is canonically isomorphic to $(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)/H \bmod p^s$ by sending coset $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\widehat{\Gamma}_s$ to $(a_p, d_p) \bmod p^s \in (H \bmod p^s)$, and the moduli theoretic action of H coincides with the action of $\mathrm{Gal}(X_s/X_s^0) = ((\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)/H \bmod p^s)$. Through $\Gamma \cong (\Gamma \times \Gamma)/H_p$ (resp. $(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)/H \cong \mathbb{Z}_p^\times$), the one variable Iwasawa algebra Λ (resp. $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] = \Lambda[\mu]$) acts on the tower $\{X_r\}_r$ as correspondences.

If $\det(\widehat{\Gamma}_{H,r}) = \widehat{\mathbb{Z}}^\times$, as explained in [IAT, Chapter 6], $X_{r/\mathbb{Q}}$ and $X_{s/\mathbb{Q}}^r$ is geometrically irreducible. Though we do not need geometric irreducibility, we indicate here an easy criterion when geometric irreducibility holds. We note that $\det(\widehat{\Gamma}_r) \supset (\widehat{\mathbb{Z}}^{(p)})^\times$, where $\widehat{\mathbb{Z}}^{(p)} = \prod_{l \neq p} \mathbb{Z}_l \cong \widehat{\mathbb{Z}}/\mathbb{Z}_p$. Thus the problem is reduced to the study of the determinant map at p. By $\alpha\mathbb{Z}_p + \delta\mathbb{Z}_p = \mathbb{Z}_p$, it is easy to see by definition, embedding diagonally H into $\mathrm{GL}_2(\mathbb{Z}_p)$, that

(3.4) $\det : H_p \to \Gamma$ is an isomorphism if and only if $p \nmid (\alpha + \delta)$ or $\alpha \cdot \delta = 0$.

If $(\alpha', \delta') \in \mathbb{Z}^2$ with $\alpha'\mathbb{Z} + \delta'\mathbb{Z} = \mathbb{Z}$ and $\xi(a, d) = \omega(a)^{\alpha'}\omega(d)^{-\delta'}$,

(3.5) $\det : (H \cap \mu \times \mu) \to \mu$ is an isomorphism if $\alpha' + \delta'$ is prime to $2 \cdot (p - 1)$ or $\alpha' \cdot \delta' = 0$.

The second condition becomes also a necessary condition if we replace $\alpha' \cdot \delta' = 0$ by $\alpha' \cdot \delta' \equiv 0 \bmod p - 1$ if p is odd and by $\alpha' \cdot \delta' \equiv 0 \bmod 2$ if $p = 2$. If $\alpha' = \delta' = i$, then $\mathrm{Ker}(\xi) \supset (\zeta, \zeta)$, and hence $\det(H) \supset \mu^2$. To have a non-trivial element in $\mathrm{Ker}(\omega^i)$ in $\mu \setminus \mu^2$, ω^i has to have odd order.

(3.6) $\det : (H \cap \mu \times \mu) \cong \mu$ if $\alpha' = \delta' = i$ and ω^i has odd order.

The image $\det(H)$ can be a proper subgroup in $\mathbb{Z}_p^\times$, and the curves X_r and X_s^r become reducible over the subfield $F = F_{\xi/\mathbb{Q}}$ fixed by $\det(H)$ identifying $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ with $\mathbb{Z}_p^\times$.

As discussed in the introduction, an interesting case is when $\xi(a, d) = \omega^i(a)\omega^{-i}(d)$ $(i = 0, 1, \ldots, p-2)$ and $\alpha = \delta = 1$. Suppose $\alpha' = \delta' = i$ for $0 \leq i < p$ (so, $\alpha'\mathbb{Z} + \delta'\mathbb{Z} = i\mathbb{Z}$). In this case, the L-function $L(s, f_P)$ can have root number ± 1. By (3.6), $\det : \mathrm{Ker}(\xi) \to \mu$ is onto if and only if ω^i has odd order (including the case where $i = 0$), and hence $\det(\widehat{\Gamma}_{H,r}) = \widehat{\mathbb{Z}}^\times$ if $p > 2$ and ω^i has odd order. Otherwise, if $p > 2$, F_ξ is a unique quadratic extension of $\mathbb{Q}$ inside $\mathbb{Q}(\mu_p)$. If $p = 2$, if $\alpha = \delta = 1$ and $\alpha' = \delta' = 0$, $F_\xi = \mathbb{Q}[\sqrt{2}]$, and if $\alpha = \delta = 1$ and $\alpha' = \delta' = 1$, then $F_\xi = \mathbb{Q}(\sqrt{-1}, \sqrt{2})$.

Taking $(X, Y, U)_{/S}$ to be $(X_{s/\mathbb{Q}}, X^r_{s/\mathbb{Q}}, U(p))_{/\mathbb{Q}}$ for $s > r \geq 1$, to the projection $\pi : X_s \to X^r_s$, the result of the previous section is plainly applicable if X^r_s is geometrically irreducible, since $U(p)$ is also geometrically irreducible as it is the image of $X^r_{s+1} := \mathfrak{H}/(\Gamma^r_s \cap \Gamma_0(p^{s+1}))$ by the diagonal product of two degeneration maps from X^r_{s+1} in $X^r_s \times X^r_s$. If not, writing $X^r_{s/F_\xi} = \bigcup_i X^r_{s,i}$ for geometrically irreducible components $X^r_{s,i}$, then $U(p)$ restricted in each $X^r_{s,i} \times X^r_{s,i}$ is geometrically irreducible by the same argument above and its degree is a p-power independent of the components; so, we can apply the argument in Section 2 in these geometrically reducible cases.

Corollary 3.2. *Let F be a number field or a finite extension of $\mathbb{Q}_l$ for a prime l. Then we have, for integers r, s with $s \geq r \geq \epsilon$,*

(u) $\pi^* : J^r_{s/\mathbb{Q}}(F) \to \check{H}^0(X_s/X^r_s, J_{s/\mathbb{Q}}(F)) \overset{(*)}{=} J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-\epsilon}} - 1]$ *is a $U(p)$-isomorphism,*

where $J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-\epsilon}} - 1] = \mathrm{Ker}(\gamma^{p^{r-\epsilon}} - 1 : J_s(F) \to J_s(F))$ and $\epsilon = 1$ if $p > 2$ and $\epsilon = 2$ if $p = 2$.

Here the identity at $(*)$ follows from Lemma 2.4. The kernel $A \mapsto \mathrm{Ker}(\gamma^{p^{r-\epsilon}} - 1 : J_s(A) \to J_s(A))$ is an abelian fppf sheaf (as the category of abelian fppf sheaves is abelian and regarding a sheaf as a presheaf is a left exact functor), and it is represented by the scheme theoretic kernel $J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}} - 1]$ of the endomorphism $\gamma^{p^{r-\epsilon}} - 1$ of $J_{s/\mathbb{Q}}$. From the exact sequence $0 \to J_s[\gamma^{p^{r-\epsilon}} - 1] \to J_s \xrightarrow{\gamma^{p^{r-\epsilon}} - 1} J_s$, we get another exact sequence: $0 \to J_s[\gamma^{p^{r-\epsilon}} - 1](F) \to J_s(F) \xrightarrow{\gamma^{p^{r-\epsilon}} - 1} J_s(F)$. Thus

$$J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-\epsilon}} - 1] = J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}} - 1](F). \tag{3.7}$$

By a simple Hecke operator identity (e.g., [H15, (3.1)]), we have the following commutative diagram (i.e., the contraction property of the $U(p)$-operator):

$$\begin{array}{ccc} J_{r/R} & \xrightarrow{\pi^*} & J^r_{s/R} \\ \downarrow u & \swarrow u' & \downarrow u'' \\ J_{r/R} & \xrightarrow{\pi^*} & J^r_{s/R}, \end{array} \tag{3.8}$$

where the middle u' is given by $U^s_r(p^{s-r}) = [\Gamma^r_s \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r} \end{pmatrix} \Gamma_r]$ and u and u'' are $U(p^{s-r})$. Thus

(u1) $\pi^* : J_{r/R} \to J^r_{s/R}$ is a $U(p)$-isomorphism (for the projection $\pi : X^r_s \to X_r$).

The above (u) combined with (u1) and (3.7) implies the sheaf identity (u2) below for integers r, s with $s \geq r \geq \epsilon$:

(u2) $\pi^* : J^r_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}} - 1] = \mathrm{Ker}(\gamma^{p^{r-\epsilon}} - 1 : J_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}})$ is a $U(p)$-isomorphism.

We reformulate the above statement (u2) as follows:

Lemma 3.3. *For integers r, s with $s \geq r \geq \epsilon$, we have morphisms*

$$\iota^r_s : J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}} - 1] \to J^r_{s/\mathbb{Q}} \quad \text{and} \quad \iota^{r,*}_s : J^r_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}} - 1)(J_{s/\mathbb{Q}})$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc} J^r_{s/\mathbb{Q}} & \xrightarrow{\pi^*} & J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}} - 1] \\ \downarrow u & \swarrow \iota^r_s & \downarrow u'' \\ J^r_{s/\mathbb{Q}} & \xrightarrow{\pi^*} & J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}} - 1], \end{array} \tag{3.9}$$

$$\begin{array}{ccc} J^r_{s/\mathbb{Q}} & \xleftarrow{\pi_*} & J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}} - 1)(J_{s/\mathbb{Q}}) \\ \uparrow u^* & \nearrow \iota^{r,*}_s & \uparrow u''^* \\ J^r_{s/\mathbb{Q}} & \xleftarrow{\pi_*} & J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}} - 1)(J_{s/\mathbb{Q}}), \end{array} \tag{3.10}$$

where u and u'' are $U(p^{s-r}) = U(p)^{s-r}$ and u^* and u''^* are $U^*(p^{s-r}) = U^*(p)^{s-r}$. In particular, for an fppf extension $T_{/\mathbb{Q}}$, the evaluated map at T: $(J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}}-1)(J_{s/\mathbb{Q}}))(T) \xrightarrow{\pi_*} J_s^r(T)$ (resp. $J_s^r(T) \xrightarrow{\pi^*} J_s[\gamma^{p^{r-\epsilon}}-1](T)$) is a $U^*(p)$-isomorphism (resp. a $U(p)$-isomorphism).

Proof. We first prove the assertion for π^*. We note that the category of groups schemes fppf over a base S is a full subcategory of the category of abelian fppf sheaves. We may regard $J^r_{s/\mathbb{Q}}$ and $J_s[\gamma^{p^{r-\epsilon}}-1]_{/\mathbb{Q}}$ as abelian fppf sheaves over $\mathbb{Q}$ in this proof. Since these sheaves are represented by (reduced) algebraic groups over $\mathbb{Q}$, we can check being $U(p)$-isomorphism by evaluating the sheaf at a field K of characteristic 0 (e.g., [EAI, Lemma 4.18]). Since the degree of X_s over X_s^r ($r \geq \epsilon$) is a p-power, the kernel $\mathcal{K} := \mathrm{Ker}(J^r_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}}-1])$ is a p-abelian group scheme. By Proposition 2.3 (2) applied to $X = X_{s/\mathbb{Q}}$ and $Y = X^r_{s/\mathbb{Q}}$ (with $s \geq r$), $\mathcal{K} := \mathrm{Ker}(J^r_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}}-1])$ is killed by $U(p)^{s-r}$ as $d = p^{s-r} = \deg(X_s/X_s^r)$. Thus we get $\mathcal{K} \subset \mathrm{Ker}(U(p)^{s-r} : J^r_{s/\mathbb{Q}} \to J^r_{s/\mathbb{Q}})$. Since the category of fppf abelian sheaves is an abelian category (because of the existence of the sheafication functor from presheaves to sheaves under fppf topology described in [ECH, §II.2]), the above inclusion implies the existence of ι_s^r with $\pi^* \circ \iota_s^r = U(p)^{s-r}$ as a morphism of abelian fppf sheaves. Since the category of group schemes fppf over a base S is a full subcategory of the category of abelian fppf sheaves, all morphisms appearing in the identity $\pi^* \circ \iota_s^r = U(p)^{s-r}$ are morphism of group schemes. This proves the assertion for π^*.

Take a number field so that $X_s(K) \neq \emptyset$ (for example, the infinity cusp of X_s is rational over $\mathbb{Q}(\mu_{p^s})$). Then $\mathrm{Pic}^0_{J_s^r/K} \cong J_s^r$ for any $s \geq r \geq 0$ by the self-duality of the jacobian variety. Note that the second assertion is the dual of the first under this self-duality; so, over K, it can be proven reversing all the arrows and replacing $J_s[\gamma^{p^{r-\epsilon}}-1]_{/K}$ (resp. $\pi^*, U(p)$) by the quotient $J_s/(\gamma^{p^{r-\epsilon}}-1)J_s$ as fppf abelian sheaves (resp. $\pi_*, U^*(p)$). By Lemma 2.1, every morphism and every abelian variety of the diagram in question are all well defined over $\mathbb{Q}$. In particular $J_s/(\gamma^{p^{r-\epsilon}}-1)(J_s)$ is an abelian variety quotient over $\mathbb{Q}$ (cf., [NMD, Theorem 8.2.12] combined with [ARG, §V.7]). Then by Galois descent for projective varieties (e.g., [GME, §1.11]), the diagram descends to $\mathbb{Q}$. Since being $U^*(p)$-isomorphism or $U(p)$-isomorphism is insensitive to the descent process, we get the final assertion. □

Remark 3.4. For a finite extension k of $\mathbb{Q}$ or $\mathbb{Q}_l$ and an abelian variety $A_{/k}$, recall $\widehat{A}(k) := \varprojlim_n A(k)/p^n A(k)$ and for an infinite Galois extension κ/k, $\widehat{A}(\kappa) = \varinjlim_F \widehat{A}(F)$ with F running over all finite Galois extensions k inside κ (here note that $\widehat{A}(k)$ is not equal to $A(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ if k is a finite extension of $\mathbb{Q}_l$; see (S) in the introduction). Thus this process of taking projective limit and then possibly an inductive limit with respect to F preserves the commutative diagrams (3.8) and (3.9), and the statements Corollary 3.2, (u), (u1), (u2) and Lemma 3.3 are also valid replacing the abelian varieties A in each statement by $\widehat{A}$.

4. Hecke algebras for exotic towers

Hereafter, we fix the data (α, δ, ξ) which defines the exotic tower $\{X_r\}_r$. We introduce the Hecke algebra $\mathbf{h}_{\alpha,\delta,\xi}$ for the tower $\{X_r\}_r$. We assume in the rest of the paper the following condition:

(F) *The Hecke algebra $\mathbf{h}_{\alpha,\delta,\xi}$ is Λ-free.*

In practice, if the local ring $\mathbb{T}$ of $\mathbf{h}_{\alpha,\delta,\xi}$ we are dealing with is Λ-free, our argument works. However there is not a good way to confirm directly Λ-freeness of $\mathbb{T}$; so, we assume (F). If $(\alpha, \delta) = (0, 1)$ and $\xi(a, d)$ only depends on d, this is always true, and as we see in this section, the Λ-freeness of $\mathbf{h}_{\alpha,\delta,\xi}$ holds for $p \geq 5$ without any other assumptions, and even for $p = 3$, for most of (α, δ) including the self-dual case of $\alpha = \delta = 1$, the Λ-freeness of $\mathbf{h}_{\alpha,\delta,\xi}$ holds (see Proposition 18.2).

As described in (3.3), $z \in \mathbb{Z}_p^\times$ acts on X_r. Recall that $J_{r/\mathbb{Q}}$ (resp. $J^r_{s/\mathbb{Q}}$) is the Jacobian of X_r (resp. X_s^r). We regard J_r as the degree 0 component of the Picard scheme of X_r. For an extension $K_{/\mathbb{Q}}$, we consider the group of K-rational points $J_r(K)$.

For each prime l, we consider $\varpi_l := \left(\begin{smallmatrix} 1 & 0 \\ 0 & l \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_l)$, and regard $\varpi_l \in \mathrm{GL}_2(\mathbb{A})$ so that its component at each place $v \nmid l$ is trivial. Then $\Delta := \varpi_l^{-1}\widehat{\Gamma}^r_s\varpi_l \cap \widehat{\Gamma}^r_s$ gives rise to a modular curve

$X(\Delta)$ whose $\mathbb{C}$-points (outside cusps) is given by $\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}^{(\infty)}) \times (\mathfrak{H} \sqcup \overline{\mathfrak{H}}))/\Delta$. We have a projection $\pi'_l : X(\Delta) \to X^r_s$ given by $\mathfrak{H} \ni z \mapsto z/l \in \mathfrak{H}$ in addition to the natural one $\pi_l : X(\Delta) \to X^r_s$ coming from the inclusion $\Delta \subset \Gamma^r_s$. Then embedding $X(\Delta)$ into $X^r_s \times X^r_s$ by these two projections, we get the modular correspondence written by $T(l)$ if $l \nmid Np$ and $U(l)$ if $l|Np$. We can extend this definition to $T(n)$ for all $n > 0$ prime to Np via Picard/Albanese functoriality (see Lemma 2.1). We use the same symbol $T(n)$ and $U(l)$ to indicates the endomorphism (called the Hecke operator) given by the corresponding correspondence $T(n)$ and $U(l)$. The Hecke operator $U(p)$ acts on $J_r(K)$ and the p-adic limit $e = \lim_{n\to\infty} U(p)^{n!}$ is well defined on the Barsotti–Tate group $J_r[p^\infty]$ and the completed Mordell–Weil group $\widehat{J}_r(K)$ as defined in (S) above.

Let Γ be the maximal torsion-free subgroup of $\mathbb{Z}_p^\times$ given by $1 + p^\epsilon\mathbb{Z}_p$ for $\epsilon = 1$ if $p > 2$ and $\epsilon = 2$ if $p = 2$. Writing $\gamma = 1 + p^\epsilon \in \Gamma$, γ is a topological generator of the multiplicative group $\Gamma = \gamma^{\mathbb{Z}_p}$. As described in (3.3), $(u, a, d) \in G = (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ acts on J_r through the quotient $G/H \cong (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times$. This action of $(u, a, d) \in G$, we write as $\langle u, a, d\rangle$; so, for a prime $l \nmid Np$, $\langle l\rangle = \langle u, a, d\rangle$ for $u = (l \bmod N)$ and $a = d = l$ in $\mathbb{Z}_p^\times$.

Define $h_r(\mathbb{Z})$ by the subalgebra of $\mathrm{End}(J_r)$ generated by $T(n)$ with n prime to Np, $U(l)$ with $l|Np$ and the action of $\langle z\rangle$ coming from $z = (u, a, d) \in G$. Put $h_r(R) = h_r(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ for a ring R. Then we define $\mathbf{h}_r = \mathbf{h}_{r,\alpha,\delta,\xi} := e(h_r(\mathbb{Z}_p))$. The restriction morphism $h_s(\mathbb{Z}) \ni h \mapsto h|_{J_r} \in h_r(\mathbb{Z})$ for $s > r$ induces a projective system $\{\mathbf{h}_r\}_r$ whose limit gives rise to the big ordinary Hecke algebra

$$\mathbf{h} = \mathbf{h}_{\alpha,\delta,\xi}(N) := \varprojlim_r \mathbf{h}_r.$$

Identify $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ with $\mathrm{Gal}(X_r/X_0(Np^r))$. Writing $\langle l\rangle$ (the diamond operator) for the action of $l \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$ on X_r, we have an identity $l\langle l\rangle = T(l)^2 - T(l^2) \in h_r(\mathbb{Z}_p)$ for all primes $l \nmid Np$.

Since $\Gamma \subset \mathbb{Z}_p^\times \subset G/H$, we have a canonical Λ-algebra structure $\Lambda = \mathbb{Z}_p[[\Gamma]] \hookrightarrow \mathbf{h}$ sending γ to $\langle 1, a, d\rangle$ for $a, d \in \Gamma$ such that $\pi_\Gamma(a, d) = \gamma$ as in (3.1). If $(\alpha, \delta) = (0, 1) = (\alpha', \delta')$, it is now well known that $\mathbf{h}$ is a free of finite rank over Λ and $\mathbf{h}_r = \mathbf{h} \otimes_\Lambda \Lambda/(\gamma^{p^{r-\epsilon}} - 1)$ (cf. [H86a], [GK13] or [GME, §3.2.6]). More generally, by [PAF, Corollary 4.31], assuming $p \geq 5$, the same facts hold (and we expect this to be true without any assumption on primes). Anyway, if $p = 2, 3$, the specialization map $\mathbf{h} \otimes_\Lambda \Lambda/(\gamma^{p^{r-\epsilon}} - 1) \to \mathbf{h}_r$ is onto with finite kernel, and $\mathbf{h}$ is a torsion-free Λ-module of finite type. We will prove the Λ–freeness of $\mathbf{h}_{\alpha,\delta,\xi}(N)$ and isomorphisms $\mathbf{h} \otimes_\Lambda \Lambda/(\gamma^{p^{r-\epsilon}} - 1) \cong \mathbf{h}_r$ for most cases of $p = 3$ in Section 18 for the sake of completeness.

A prime P in $Ar_{\mathbf{h}} := \bigcup_{r>0} \mathrm{Spec}(\mathbf{h}_r)(\overline{\mathbb{Q}}_p) \subset \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ is called an *arithmetic point* of weight 2 in $\mathrm{Spec}(\mathbf{h})$. For a closed subscheme $\mathrm{Spec}(R)$ of $\mathrm{Spec}(\mathbf{h})$, we put

$$Ar_R := Ar_{\mathbf{h}} \cap \mathrm{Spec}(R). \tag{4.1}$$

In this paper, we only deal with arithmetic point of weight 2; so, we often omit the word "weight 2" and just call them arithmetic points/primes. Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism $\lambda : \mathbf{h} \to \overline{\mathbb{Q}}_p$ killing $\gamma^{p^r} - 1$ for $r \geq 0$ to a classical Hecke eigenform, we need to fix (once and for all) an embedding $\overline{\mathbb{Q}} \xrightarrow{i_p} \overline{\mathbb{Q}}_p$ of the algebraic closure $\overline{\mathbb{Q}}$ in $\mathbb{C}$ into a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We write i_∞ for the inclusion $\overline{\mathbb{Q}} \subset \mathbb{C}$.

More generally, for the jacobian variety $J(Z_r)$ of the curve Z_r defined above (3.3), we define $\mathbf{h}_r^{\mathrm{n.ord}}$ to be the maximal $\boldsymbol{\Lambda}$-algebra direct summand of $\mathrm{End}(J(Z_r)) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in which $U(p)$ is invertible. Then as before we put $\mathbf{h}^{\mathrm{n.ord}} = \mathbf{h}(N)^{\mathrm{n.ord}} := \varprojlim_r \mathbf{h}_r^{\mathrm{n.ord}}$, which is a $\boldsymbol{\Lambda}$-algebra. We consider

$$\mathbf{h}^{\mathrm{n.ord},\varphi} := \mathbf{h}^{\mathrm{n.ord}} \otimes_{\mathbb{Z}_p} W/\mathfrak{a}_\varphi \mathbf{h}^{\mathrm{n.ord}} \otimes_{\mathbb{Z}_p} W$$

where $\mathfrak{a}_\varphi$ is the kernel of the algebra homomorphism $W[[\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times]] \to W[[\mathbb{Z}_p^\times]]^\times$ induced by the character $(a, d) \mapsto \varphi(a, d)\xi(a, d) : \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$. If we take $\varphi(a, d) = a^\alpha d^{-\delta}$ for $(a, d) \in \Gamma \times \Gamma$ and $W = \mathbb{Z}_p$ with $\xi : \mu \times \mu \to \mathbb{Z}_p^\times$, we have $\mathbf{h}^{\mathrm{n.ord},\varphi} = \mathbf{h}_{\alpha,\delta,\xi}(N)$ under present notation. Then by [PAF, Corollary 4.31], $\mathbf{h}^{\mathrm{n.ord},\varphi}$ is Λ–free of finite rank for $\Lambda_\varphi = \mathbb{Z}_p[[\mathrm{Im}(\varphi)]]$. In particular, we have

Proposition 4.1. *Assume $p \geq 5$ or $(\alpha, \delta, \xi) = (0, 1, \omega_d)$, where $\omega_d(a, d) = \omega(d)$. Then $\mathbf{h}_{\alpha,\delta,\xi}(N)$ is Λ-free of finite rank for $\Lambda = \mathbb{Z}_p[[\Gamma^2/H_p]]$.*

Remark 4.2. For $p \le 3$, we will prove in Proposition 18.2 Λ-freeness of $\mathbf{h}_{\alpha,\delta,\xi}(N)$ if it is obtained by systematic twists of $\mathbf{h}_{0,1,\omega_d}(N)$. This covers the interesting cases of analytic families of abelian varieties, including some corresponding to the p-adic L-function $k \mapsto L(2k+2, k+1)$ as in the introduction.

We have injective limits $J_\infty(K) = \varinjlim_r \widehat{J}_r(K)$ and $J_\infty[p^\infty](K) = \varinjlim_r J_r[p^\infty](K)$ via Picard functoriality, on which e acts. Write $\mathcal{G} = \mathcal{G}_{\alpha,\delta,\xi} := e(J_\infty[p^\infty])$, which is called the Λ-adic Barsotti–Tate group in [H14] and whose arithmetic properties are scrutinized there. Adding superscript or subscript "ord", we indicate the image of e.

The compact cyclic group Γ acts on these modules by the diamond operators. Thus $J_\infty(K)^{\mathrm{ord}}$ is a module over $\Lambda := \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ by $\gamma \leftrightarrow t = 1+T$ for a fixed topological generator γ of $\Gamma = \gamma^{\mathbb{Z}_p}$. The big ordinary Hecke algebra $\mathbf{h}$ acts on J_∞^{ord} as endomorphisms of functors.

Let $\mathrm{Spec}(\mathbb{T})$ be a connected component of $\mathrm{Spec}(\mathbf{h})$ and $\mathrm{Spec}(\mathbb{I})$ be a primitive irreducible component of $\mathrm{Spec}(\mathbb{T})$. For each $\mathbf{h}$-module M, we put $M_{\mathbb{T}} := M \otimes_{\mathbf{h}} \mathbb{T}$; in particular, $J^{\mathrm{ord}}_{\infty,\mathbb{T}} := J^{\mathrm{ord}}_\infty \otimes_{\mathbf{h}} \mathbb{T}$ as an fppf sheaf. For $P \in Ar_{\mathbb{I}}$ with $P \in \mathrm{Spec}(\mathbb{I}/(\gamma^{p^{r-\epsilon}} - 1)\mathbb{I})(\overline{\mathbb{Q}}_p)$, we write $r(P)$ for the minimal r with this property. Then the corresponding Hecke eigenform f_P belongs to $S_2(\Gamma_1(Np^r))$. See Section 18 for an explicit description of the "Neben" character of f_P in terms of (α, δ, ξ).

5. Abelian factors of J_r.

We give a description of abelian factors A_s and B_s of the modular jacobian varieties $\{J_s\}_s$ of the exotic modular tower which behave coherently in the limit process under the Hecke operator action. Let $\pi_{s,r,*} : J_s \to J_r$ for $s > r$ be the morphism induced by the covering map $\pi_{s,r} : X_s \twoheadrightarrow X_r$ through Albanese functoriality. Then we define $\pi_s^r = w_r \circ \pi_{s,r,*} \circ w_s$. Note that π_s^r is well defined over $\mathbb{Q}$ (cf. Lemma 3.1), and satisfies $T(n) \circ \pi_s^r = \pi_s^r \circ T(n)$ for all n prime to Np and $U(q) \circ \pi_s^r = \pi_s^r \circ U(q)$ for all $q|Np$ (as $w_? \circ h \circ w_? = h^*$ for $h \in h_?(\mathbb{Z})$ $(? = s, r)$ by [MFM, Theorem 4.5.5].

Let $\mathrm{Spec}(\mathbb{T})$ be a connected component of $\mathrm{Spec}(\mathbf{h}(N))$. Write $\mathfrak{m}_{\mathbb{T}}$ for the maximal ideal of $\mathbb{T}$ and $1_{\mathbb{T}}$ for the idempotent of $\mathbb{T}$ in $\mathbf{h}(N)$. We assume the following condition

(A) We have $\varpi \in \mathfrak{m}_{\mathbb{T}}$ such that $(\varpi) \cap \Lambda$ is a factor of $(\gamma^{p^r} - 1)$ in Λ and that $\mathbb{T}/(\varpi)$ is free of finite rank over $\mathbb{Z}_p$.

We call a prime ideal P satisfying the above condition (A) a *principal arithmetic* point of $\mathrm{Spec}(\mathbb{T})$. Write ϖ_s for the image of $\varpi \oplus (1 - 1_{\mathbb{T}})$ in $\mathbf{h}_s$ $(s \ge r)$ and define an $h_s(\mathbb{Z})$-ideal by

$$\mathfrak{a}_s = (\varpi_s \mathbf{h}_s \oplus (1-e) h_s(\mathbb{Z}_p)) \cap h_s(\mathbb{Z}).$$

Write A_s for the identity connected component of $J_s[\mathfrak{a}_s] = \bigcap_{\alpha \in \mathfrak{a}_s} J_s[\alpha]$, and put $B_s = J_s/\mathfrak{a}_s J_s$, where $\mathfrak{a}_s J_s$ is a rational abelian subvariety of J_s given by $\mathfrak{a}_s J_s(\overline{\mathbb{Q}}) = \sum_{a \in \mathfrak{a}_s} a(J_s(\overline{\mathbb{Q}})) \subset J_s(\overline{\mathbb{Q}})$.

Taking a finite set G of generators of $\mathfrak{a}_s$, $\mathfrak{a}_s J_s$ is the image of $a : \bigoplus_{g \in G} J_s \xrightarrow{x \mapsto \sum_g g(x)} J_s$. The kernel $J_s[\mathfrak{a}_s] = \mathrm{Ker}(a)$ is a well defined fppf sheaves, which is represented by an extension of the abelian variety A_s by a finite étale group scheme both over $\mathbb{Q}$. Then by [NMD, Theorem 8.2.12], the quotient $\mathfrak{a} J_s = (\bigoplus_{g \in G} J_s)/\mathrm{Ker}(a)$ is well defined as an abelian scheme and is the sheaf fppf quotient. Then again $B_s := J_s/\mathfrak{a}_s J_s$ is the fppf sheaf quotient and also abelian variety quotient again by [NMD, Theorem 8.2.12]. By definition, A_s is stable under $h_s(\mathbb{Z})$ and $h_s(\mathbb{Z})/\mathfrak{a}_s \hookrightarrow \mathrm{End}(A_s)$.

Lemma 5.1. *Assume* (F) *and* (A). *Then we have* $\widehat{A}_s^{\mathrm{ord}} = \widehat{J}_s^{\mathrm{ord}}[\varpi_s]$ *and* $\widehat{J}_s[\mathfrak{a}_s] = \widehat{A}_s$. *The abelian variety* A_s $(s > r)$ *is the image of* A_r *in* J_s *under the morphism* $\pi^* = \pi_{s,r}^* : J_r \to J_s$ *induced by Picard functoriality from the projection* $\pi = \pi_{s,r} : X_s \to X_r$ *and is* $\mathbb{Q}$*-isogenous to* B_s. *The morphism* $J_s \to B_s$ *factors through* $J_s \xrightarrow{\pi_s^r} J_r \to B_r$. *In addition, the sequence of fppf sheaves*

$$0 \to \widehat{A}_s^{\mathrm{ord}} \to \widehat{J}_s^{\mathrm{ord}} \xrightarrow{\varpi} \widehat{J}_s^{\mathrm{ord}} \to \widehat{B}_s^{\mathrm{ord}} \to 0$$

is an exact sequence.

Passing to the limit, we get the following exact sequence of fppf sheaves:

$$0 \to \widehat{A}_\infty^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \xrightarrow{\varpi} J_\infty^{\mathrm{ord}} \to \widehat{B}_\infty^{\mathrm{ord}} \to 0, \tag{5.1}$$

where $J^{\mathrm{ord}}_\infty = \varinjlim_s \widehat{J}^{\mathrm{ord}}_s$ and $\widehat{X}^{\mathrm{ord}}_\infty = \varinjlim_s \widehat{X}^{\mathrm{ord}}_s$ for $X = A, B$.

Proof. Taking a finite set G of generators of $\mathfrak{a}_s$ containing ϖ_s, we get an exact sequence $0 \to J_s[\mathfrak{a}_s] \to J_s \xrightarrow{x\mapsto(g(x))_{g\in G}} \bigoplus_{g\in G} J_s$. Since $X \mapsto \widehat{X}$ as in (S) is left exact, we have $\widehat{A}_s \subset \bigcap_{a\in\mathfrak{a}_s} \widehat{J}_s[a]$ with finite quotient. Applying further the idempotent, since $\mathfrak{a}_s = ((\varpi_s) \oplus (1-e)h_s(\mathbb{Z}_p)) \cap h_s(\mathbb{Z})$, we find

$$\widehat{J}_s[\mathfrak{a}_s]^{\mathrm{ord}} = \bigcap_{a\in\mathfrak{a}_s} \widehat{J}_s[a]^{\mathrm{ord}} = \widehat{J}^{\mathrm{ord}}_s[\varpi_s].$$

We have an exact sequence

$$0 \to J_s[\mathfrak{a}_s][p^\infty]^{\mathrm{ord}} \to J_s[p^\infty]^{\mathrm{ord}} \xrightarrow{\varpi_s} J_s[p^\infty]^{\mathrm{ord}} \to \mathrm{Coker}(\varpi_s) \to 0,$$

and $\mathrm{Coker}(\varpi_s)$ is p-divisible and is dual to $J_s[\mathfrak{a}_s][p^\infty]^{\mathrm{ord}}$ under the w_s-twisted self Cartier duality of $J_s[p^\infty]^{\mathrm{ord}}$ (over $\overline{\mathbb{Q}}$; see [H14, §4]). This shows $\widehat{J}_s[\mathfrak{a}_s][p^\infty]^{\mathrm{ord}}$ is p-divisible (so, $(J_s[\mathfrak{a}_s]/A_s)^{\mathrm{ord}}$ has order prime to p), and hence $\widehat{A}^{\mathrm{ord}}_s = \widehat{J}_s[\mathfrak{a}_s]^{\mathrm{ord}}$.

Plainly by definition, $\pi^*(J_r[\mathfrak{a}_r]) \subset J_s[\mathfrak{a}_s]$. Since we have the following commutative diagram:

$$\begin{array}{ccc} h_s(\mathbb{Z}) & \longrightarrow & h_r(\mathbb{Z}) \\ \downarrow & & \downarrow \\ h_s(\mathbb{Z}_p)/(\varpi_s\mathbf{h}_s \oplus (1-e)h_s(\mathbb{Z}_p)) & = & h_r(\mathbb{Z}_p)/(\varpi_r\mathbf{h}_r \oplus (1-e)h_r(\mathbb{Z}_p)), \end{array}$$

we have $\dim A_s = \mathrm{rank}_{\mathbb{Z}}\, h_s(\mathbb{Z})/\mathfrak{a}_s = \mathrm{rank}_{\mathbb{Z}}\, h_r(\mathbb{Z})/\mathfrak{a}_r = \dim A_r$; so, $A_s = \pi^*(A_r)$.

The above commutative diagram also tells us that $\mathfrak{a}_s \supset \mathfrak{b}_s := \mathrm{Ker}(h_s(\mathbb{Z}) \twoheadrightarrow h_r(\mathbb{Z}))$ in $h_s(\mathbb{Z}_p)$. Thus the projection $J_s \twoheadrightarrow J_s/\mathfrak{a}_sJ_s = B_s$ factors through $J_r = J_s/\mathfrak{b}_rJ_s$. Indeed, the natural projection: $J_s/\mathfrak{b}_sJ_{s/\mathbb{Q}} \twoheadrightarrow J_{r/\mathbb{Q}}$ has to be a finite morphism (as the tangent space at the origin of the two are isomorphic), and we conclude $J_s/\mathfrak{b}_sJ_s = J_r$ by the universality of the categorical quotient $J_s/\mathfrak{a}_sJ_s$ (cf., [NMD, page 219]).

Assuming $X_s(K) \neq \emptyset$, we have $J_s \cong \mathrm{Pic}^0_{J_s/K}$ by the polarization of the canonical divisor (e.g., [ARG, VII.6]). The dual sequence (over K) of the exact sequence of abelian varieties: $0 \to J_s[\mathfrak{a}_s] \to J_s \xrightarrow{x\mapsto(g(x))_{g\in G}} \bigoplus_{g\in G} J_s$ is $\bigoplus_{g\in G} J_s \xrightarrow{x\mapsto\sum_g g(x)} J_s \to B_s \to 0$. Thus A_s is isogenous to B_s over K, and by Galois descent, A_s is $\mathbb{Q}$-isogenous to B_s. Indeed, for the complementary abelian subvariety $A_s^\perp$ in J_s of A_s, we have $J_s/A_s^\perp = B_s$, and the $\mathbb{Q}$-isogeny follows without taking duality. Here note that the quotient $J_s/A_s^\perp$ exists as an abelian variety and also as an fppf sheaves by [NMD, Theorem 8.2.12] (and [ARG, V.7]).

As explained just below (A), we have $\mathrm{Im}(\bigoplus_{g\in G} J_s \xrightarrow{x\mapsto\sum_g g(x)} J_s) = \mathfrak{a}_sJ_s$ as fppf sheaves. Then applying the argument of [H15, Section 1] to the exact sequence of fppf sheaves

$$0 \to J_s[\mathfrak{a}_s] \to \bigoplus_{g\in G} J_s \to \mathfrak{a}J_s \to 0,$$

we confirm the exactness of $0 \to \widehat{J}_s[\mathfrak{a}_s] \to \bigoplus_{g\in G} \widehat{J}_s \to \widehat{\mathfrak{a}J_s} \to 0$ as fppf sheaves. Thus applying the idempotent e, we see $\mathrm{Im}(\bigoplus_{g\in G} \widehat{J}^{\mathrm{ord}}_s \xrightarrow{x\mapsto\sum_g g(x)} \widehat{J}^{\mathrm{ord}}_s) = \widehat{\mathfrak{a}_sJ_s}^{\mathrm{ord}}$. Since the morphism $\bigoplus_{g\in G} \widehat{J}^{\mathrm{ord}}_s \xrightarrow{x\mapsto\sum_g g(x)} \widehat{J}^{\mathrm{ord}}_s$ factors through $\varpi_s(\widehat{J}^{\mathrm{ord}}_s)$ as all $g = \varpi_s x$ with $x \in \mathbf{h}_s$, noting $\varpi \in G$, $\varpi_s(\widehat{J}^{\mathrm{ord}}_s) \hookrightarrow \mathfrak{a}_s(\widehat{J}^{\mathrm{ord}}_s)$. Thus $\widehat{\mathfrak{a}_sJ_s}^{\mathrm{ord}} = \varpi_s(\widehat{J}^{\mathrm{ord}}_s)$ as fppf sheaves, and the sequence $0 \to \widehat{A}^{\mathrm{ord}}_s \to \widehat{J}^{\mathrm{ord}}_s \xrightarrow{\varpi_s} \widehat{\mathfrak{a}_sJ_s} = \varpi_s(\widehat{J}^{\mathrm{ord}}_s) \to 0$ is sheaf exact. Since $B_s = J_s/\mathfrak{a}_sJ_s$ as fppf sheaves, $0 \to \varpi_s(\widehat{J}^{\mathrm{ord}}_s) \to \widehat{J}^{\mathrm{ord}}_s \to \widehat{B}^{\mathrm{ord}}_s \to 0$ is exact as fppf sheaves. Combining the two exact sequence, we obtain the exactness of the last sequence in the lemma. □

Assuming $X_s(K) \neq \emptyset$, we have $J_s \cong \mathrm{Pic}^0_{J_s/K}$ and the Rosati involution $h \mapsto h^*$ and $T(n) \mapsto T^*(n)$ which brings $h_r(\mathbb{Z})$ to $h^*_r(\mathbb{Z}) \subset \mathrm{End}(J_{r/K})$. At the level of double coset operator $[\Gamma\alpha\Gamma']$, the involution has the effect $[\Gamma\alpha\Gamma']^* = [\Gamma'\alpha^\iota\Gamma]$. Thus the involution $h \mapsto h^*$ gives rise to an isomorphism

$h_r(\mathbb{Z}) \cong h_r(\mathbb{Z})^*$ in $\mathrm{End}(J_{r/\mathbb{Q}})$ (even if $X_r(\mathbb{Q}) = \emptyset$). Note that $X_1(Np^r)(\mathbb{Q})$ contains the infinity cusp; so, for the standard tower, we have $X_r(\mathbb{Q}) \neq \emptyset$.

The Weil involution $w_s = [\Gamma_s \begin{pmatrix} 0 & -1 \\ Np^s & 0 \end{pmatrix} \Gamma_s]$ satisfies $w_s[\Gamma_s^\iota \alpha^\iota \Gamma_s^\iota] = [\Gamma_s \alpha \Gamma_s] w_s$. Thus $w_s \circ T^*(n) = T(n) \circ w_s$ for all n including $U(l)$. We write $\{X'^r_s\}_{s>r}$ for the dual tower defined for (δ, α, ξ') with $\xi'(a,d) = \xi(d,a)$ which corresponds to $\{(\widehat{\Gamma}^r_s)^\iota = w_s \widehat{\Gamma}^r_s w_s^{-1}\}_{s>r}$. Thus the l-component of $(\widehat{\Gamma}^r_s)^\iota$ for $l|N$ is given by $\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_l) | c \in N\mathbb{Z}_l, a-1 \in N\mathbb{Z}_l \right\}$. Then w_s gives an isomorphism $w_r : X^r_s \to X'^r_s$ defined over $\mathbb{Q}$. Note that the fixed isomorphism $\mu_{p^s} \cong \mathbb{Z}/p^s\mathbb{Z}$ ($\zeta_{p^s} \mapsto 1$) induces an isomorphism $X^r_s \cong X'^r_s$ over $\mathbb{Q}(\mu_{Np^r})$. As an automorphism of $X^r_{s/\mathbb{Q}[\mu_{p^s}]}$, w_s satisfies $w_s^{[z,\mathbb{Q}]} = \langle z \rangle \circ w_s = w_s \circ \langle z \rangle^{-1}$ for the Artin symbol $[z, \mathbb{Q}]$ with $z \in \widehat{\mathbb{Z}}^\times$ (see Lemma 3.1).

Take a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ and an irreducible component $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(\mathbb{T})$. Assume that $\mathbb{I}$ is primitive in the sense of [H86a, Section 3]. For each arithmetic $P \in Ar_{\mathbb{I}}$, the corresponding cusp form f_P is new at each prime $l|N$ if and only if $\mathbb{I}$ is primitive.

We get directly from Lemma 5.1 the following proposition giving sufficient conditions for the validity of (A) for $A_{f,s}$ when $f = f_P$ is in a p-adic analytic family indexed by $P \in \mathrm{Spec}(\mathbb{I})$.

Proposition 5.2. *Let* $\mathrm{Spec}(\mathbb{T})$ *be a connected component of* $\mathrm{Spec}(\mathbf{h})$ *and* $\mathrm{Spec}(\mathbb{I})$ *be a primitive irreducible component of* $\mathrm{Spec}(\mathbb{T})$*. Assume* Λ*-freeness of* $\mathbb{T}$ *(i.e.,* (F)*). Then the condition* (A) *holds for the following choices of* (ϖ, A_s, B_s)*:*

(1) *Suppose that an eigen cusp form* $f = f_P$ *new at each prime* $l|N$ *belongs to* $\mathrm{Spec}(\mathbb{T})$ *and that* $\mathbb{T} = \mathbb{I}$ *is regular (or more generally a unique factorization domain). Then writing the level of* f_P *as* Np^r*, the algebra homomorphism* $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ *given by* $f|T(l) = \lambda(T(l))f$ *gives rise to the prime ideal* $P = \mathrm{Ker}(\lambda)$*. Since* P *is of height 1, it is principal generated by* $\varpi \in \mathbb{T}$*. This* ϖ *has its image* $a_s \in \mathbb{T}_s = \mathbb{T} \otimes_\Lambda \Lambda_s$ *for* $\Lambda_s = \Lambda/(\gamma^{p^{s-\epsilon}} - 1)$*. Write* $\mathbf{h}_s = \mathbf{h} \otimes_\Lambda \Lambda_s = \mathbb{T}_s \oplus 1_s \mathbf{h}_s$ *as an algebra direct sum for an idempotent* 1_s*. Then, the element* $\varpi_s = a_s \oplus 1_s \in \mathbf{h}_s$ *for the identity* 1_s *of* X_s *satisfies* (A)*.*

(2) *Fix* $r > 0$*. Then* $\varpi \in \mathfrak{m}_{\mathbb{T}}$ *for a factor* $\varpi | (\gamma^{p^{r-\epsilon}} - 1)$ *in* Λ*, satisfies* (A)*.*

Definition 5.3. *If* ϖ *generates a principal arithmetic point* $P \in Ar_{\mathbf{h}}$ *with associated Hecke eigenform* f_P *having minimal level* $Np^{r(P)}$*, we denote by* A_P *(resp.* B_P*) the abelian variety* A_r *(resp.* B_r*) for* $r = r(P)$*.*

We record two lemmas giving Hecke module structure of $\widehat{A}^{\mathrm{ord}}_P(K)$.

Lemma 5.4. *Let* K *be a number field. Write* $r_P := \mathrm{rank}_{\mathbb{Z}} A_P(K)$*. Then we have* $[H_P : \mathbb{Q}] | r_P$*, and writing* $r_P = h_P[H_P : \mathbb{Q}]$*, for an arithmetic point* $P \in \mathrm{Spec}(\mathbb{T})$*, we have* $\mathrm{rank}_{\mathbb{Z}_p} \widehat{A}^{\mathrm{ord}}_P(K) = \mathrm{rank}_{\mathbb{Z}_p} \widehat{B}^{\mathrm{ord}}_P(K) = h_P \cdot \mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}/P$ *and* $h_P = \dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}$*.*

Proof. Note that $A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a vector space over the field H_P; so, $\mathrm{rank}_{\mathbb{Z}} A_P(K) = h_P[H_P : \mathbb{Q}]$ for $h_P = \dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore $\widehat{A}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p \cong (H_P \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{h_P}$ as $h_r(\mathbb{Z}_p)$-modules (for the level Np^r of f_P. Here $h_r(\mathbb{Z}_p)$ is the Hecke algebra of level Np^r of weight 2 as defined at the beginning of Section 4. For the idempotent $1_{\mathbb{T}}$ in $\mathbf{h}$ of $\mathbb{T}$, we denote by the same symbol $1_{\mathbb{T}}$ the idempotent in $h_r(\mathbb{Z}_p)$ which is the image of the original $1_{\mathbb{T}}$ as $\mathbf{h}_r$ is a direct factor of $h_r(\mathbb{Z}_p)$. Then we have $1_{\mathbb{T}}(H_P \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cong \mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by definition. Thus we have

$$\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong 1_{\mathbb{T}}(A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p) \cong 1_{\mathbb{T}}(H_P \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{h_P} \cong (\mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{h_P}.$$

This shows the last identity for $\widehat{A}^{\mathrm{ord}}_P(K)$. The identity for $\widehat{B}^{\mathrm{ord}}_P(K)$ also follows since A_P is isogenous to B_P over $\mathbb{Q}$. □

Lemma 5.5. *Let* K *be a finite extension of* $\mathbb{Q}_p$*. Then we have* $\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong (\mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{[K:\mathbb{Q}_p]}$ *as* $\mathbb{T}/P$*-modules, and* $\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{T}/P \otimes_{\mathbb{Z}_p} K$ *as* $\mathbb{T}/P \otimes_{\mathbb{Z}_p} K$*-modules.*

Proof. By [Ma55], The group $A_P(K)$ contains a subgroup $W^{\dim A}$ for the p-adic integer ring W of K. Thus we need to show that $A_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H_P \otimes_{\mathbb{Q}} K$ as H_P-module. Since K is complete, X_r

has K-rational point. Thus we can take the identity $\mathbf{0}$ of J_r belongs to the image of X_r under the canonical Albanese embedding $X_r \hookrightarrow J_r$.

Take a Néron model $A_{P/W}$ of A_P over W, and write $\mathcal{A}_{/W}$ its formal completion along the identity of the special fiber of the Néron model. Then $Lie(\mathcal{A})$ is isomorphic to the tangent space of $\mathcal{A}_{/W}$ at the identity. Since $\mathcal{A}$ is a smooth formal group over W, $Lie(\mathcal{A}) \cong W^{\dim \mathcal{A}} = W^{\dim A_P}$. Note that $Lie(\mathcal{A}) \otimes_W K \cong Lie(\mathcal{A} \times_W K)$ which is isomorphic to the tangent space at the identity of $A_{P/K}$. Since $A_P \subset J_r$ and $Lie(J_{r/K})$ is isomorphic to the tangent space $T_{r/K}$ of X_r at the base point $\mathbf{0} \in X_r(K)$. Since $\Omega_{X_r/K}$ is isomorphic to the space of K-rational cusp forms of weight 2 on X_r (which is dual $\mathrm{Hom}(h_r(K), K)$ as $h_r(K)$-module), $T_{r/K} \cong h_r(K)$ as $h_r(K)$-modules. Thus by definition, $Lie(\mathcal{A}) \otimes_W K \cong Lie(A_{P/\mathbb{Q}}) \otimes_{\mathbb{Q}} K \cong H_P \otimes_{\mathbb{Q}} K$,

The abelian variety has semi-stable reduction over $\mathbb{Z}_p[\mu_{p^\alpha}]$, and we now take $K = \mathbb{Q}_p[\mu_{p^\alpha}]$. Then the formal group $\mathcal{A}^{\mathrm{ord}}_{/W} \subset \mathcal{A}_{/W}$ gives rise to the connected component of the Barsotti–Tate group $\widehat{A}^{\mathrm{ord}}_P[p^\infty]_{/W}$. Note that $\mathcal{A}^{\mathrm{ord}}(W)$ contains $W^{\mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}/P}$ as a subgroup of finite index. By logarithm map, for the Lie algebra $Lie(\mathcal{A}^{\mathrm{ord}})$ of $\mathcal{A}^{\mathrm{ord}}$, we find $Lie(\mathcal{A}^{\mathrm{ord}}) \otimes_W K \cong \mathbb{T}/P \otimes_{\mathbb{Z}_p} K$ as $\mathbb{T}/P$-modules. Since $\mathcal{A}^{\mathrm{ord}}$ is defined over $\mathbb{Q}_p$ as a p-adic Lie group, the $\mathbb{T}/P$-module structure descends to $\mathbb{Q}_p$-points (as described above), and we find that $\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as $\mathbb{T}/P$-modules. By extending scalar to an arbitrary K, we find $\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{T}/P \otimes_{\mathbb{Z}_p} K \cong (\mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{[K:\mathbb{Q}_p]}$ as $\mathbb{T}/P$-modules. □

For a given Hecke eigenform $f \in S_2(\Gamma_1(N))$, we now show that for almost all primes ordinary for f, the local ring $\mathbb{T}$ given by f is regular. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a Dirichlet character, and consider the space $S_2(\Gamma_0(N), \chi)$ of cusp forms of weight 2 with Neben character χ. Write $\mathbb{Z}[\chi]$ for the subalgebra of $\overline{\mathbb{Q}}$ generated by the values of χ. Then we can consider the Hecke algebra $h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi])$ inside $\mathrm{End}_{\mathbb{C}}(S_2(\Gamma_0(N), \chi))$ generated over $\mathbb{Z}[\chi]$ by all Hecke operators $T(n)$ and $U(l)$. Then this Hecke algebra is free of finite rank over $\mathbb{Z}$, and hence its reduced part (modulo the nilradical) has a well defined discriminant D_χ over $\mathbb{Z}$. Here is a criterion from [F02, Theorem 3.1] for regularity of $\mathbb{T}$:

Theorem 5.6. *Assume Λ-freeness of $\mathbf{h}_{\alpha,\delta,\xi}$. Let f be a Hecke eigenform of conductor N with $f|T(p) = a_p f$ for $a_p \in \overline{\mathbb{Q}}$, of weight 2 and with Neben character χ. Let p be a prime outside $6D_\chi N\varphi(N)$ (for $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$). Suppose that for the prime ideal $\mathfrak{p}$ of $\mathbb{Z}[a_p]$ induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, ($a_p \bmod \mathfrak{p}$) is different from 0 and $\pm\sqrt{\chi(p)}$. Then for the connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h}_{\alpha,\delta,\xi})$ acting non-trivially on the p-stabilized Hecke eigenform corresponding to f in $S_2(\Gamma_0(Np), \chi)$, $\mathbb{T}$ is a regular integral domain isomorphic to $W \otimes_{\mathbb{Z}_p} \Lambda = W[[T]]$ for a complete discrete valuation ring W unramified at p.*

The result is valid always for $p \geq 5$ and for $p = 3$ under (F) (see Propositions 4.1 and 18.2). Here is a proof of this fact since [F02, Theorem 3.1] is slightly different from the above theorem.

Proof. Let $e^\circ := \lim_{n\to\infty} T(p)^{n!} \in h_2(\Gamma_0(N), \chi; A)$ for $\mathbb{Z}_p[\chi]$-algebra A. Put $h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; A) := e^\circ h_2(\Gamma_0(N), \chi; A)$. Since $U(p) \equiv T(p) \mod p$ on $A[[q]]$, the natural algebra homomorphism:

$$h_2^{\mathrm{ord}}(\Gamma_0(Np), \chi; A) \to h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; A)$$

sending $U(p)$ to the unit root of $X^2 - T(p)X + \chi(p)p \in h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; A)[X]$ and $T(l)$ to $T(l)$ for all primes $l \neq p$ is a well defined surjective A-algebra homomorphism.

Since $p \nmid 6D_\chi N\varphi(N)$, we have $p > 3$ and $p \nmid \varphi(Np)$. Write $\mathbf{h}$ for $\mathbf{h}_{\alpha,\delta,\xi}(N)$. Then $\mathbf{h}$ is Λ-free by (F) and an exact control is valid (see Propositions 4.1 and 18.2). By the diamond operators $\langle z\rangle$ for $z \in (\mathbb{Z}/Np\mathbb{Z})^\times$, $\mathbf{h}$ is an algebra over $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times]$. We can decompose $\mathbf{h} = \oplus_\psi \mathbf{h}(\psi)$ so that the diamond operator $\langle z\rangle$ for $z \in (\mathbb{Z}/N\mathbb{Z})^\times$ acts by $\psi(z)$ on $\mathbf{h}(\psi)$, where ψ runs over all even characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$. From the exact control $\mathbf{h}/T\mathbf{h} \cong \mathbf{h}_1$ ($T = \gamma - 1 \in \Lambda$), we thus get

$$\mathbf{h}(\chi)/T\mathbf{h}(\chi) \cong h_2^{\mathrm{ord}}(\Gamma_0(Np), \chi; \mathbb{Z}_p[\chi]) =: h$$

for the character χ of $(\mathbb{Z}/Np\mathbb{Z})^\times$, where

$$h_2(\Gamma_0(Np), \chi; \mathbb{Z}_p[\chi]) = h_2(\Gamma_1(Np), \chi; \mathbb{Z}) \otimes_{\mathbb{Z}[(\mathbb{Z}/Np\mathbb{Z})^\times],\chi} \mathbb{Z}_p[\chi]$$

and $\mathbb{Z}_p[\chi]$ is the $\mathbb{Z}_p$-subalgebra of $\overline{\mathbb{Q}}_p$ generated by the values of χ. Here the tensor product is with respect to the algebra homomorphism $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \to \mathbb{Z}_p[\chi]$ induced by χ. Writing $\Sigma = \mathrm{Hom}_{\mathrm{alg}}(\mathbf{h}(\chi), \overline{\mathbb{F}}_p)$, for each $\lambda \in \Sigma$, $\overline{\Sigma} := \{\mathfrak{m}_\lambda = \mathrm{Ker}(\lambda) | \lambda \in \Sigma\}$ is the set of all maximal ideals of $\mathbf{h}(\chi)$. Thus we have compatible decompositions $\mathbf{h}(\chi) = \bigoplus_{\mathfrak{m}\in\overline{\Sigma}} \mathbf{h}(\chi)_\mathfrak{m}$ and $h = \bigoplus_{\mathfrak{m}\in\overline{\Sigma}} h_\mathfrak{m}$ (see [BCM, III.4.6]). Here the subscript "$\mathfrak{m}$" indicates the localizations at the maximal ideal $\mathfrak{m}$.

Identify Σ with $\mathrm{Hom}_{\mathrm{alg}}(h, \overline{\mathbb{F}}_p)$. Write Σ° for the subset of $\Sigma = \mathrm{Hom}_{\mathrm{alg}}(h, \overline{\mathbb{F}}_p)$ made of λ's such that there exists

$$\lambda^\circ \in \mathrm{Hom}_{\mathrm{alg}}(h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{F}_p[\chi]), \overline{\mathbb{F}}_p)$$

with $\lambda(T(l)) = \lambda^\circ(T(l))$ for all primes $l \nmid pN$. Here we put

$$h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{F}_p[\chi]) := h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{Z}_p[\chi]) \otimes_\mathbb{Z} \mathbb{F}_p.$$

Accordingly let $\overline{\Sigma}^\circ$ denote the set of maximal ideals corresponding to $\lambda \in \Sigma^\circ$. Since p-new forms in $S_2(\Gamma_0(Np), \chi)$ have $U(p)$-eigenvalues $\pm\sqrt{\chi(p)}$ (see [MFM, Theorem 4.6.17]), by $a_p \not\equiv \pm\sqrt{\chi(p)}$ mod $\mathfrak{p}$, we have further decomposition $h = h_N \oplus h'$ so that h_N is the direct sum of $h_\mathfrak{m}$ for $\mathfrak{m}$ running over $\overline{\Sigma}^\circ$. Since $\mathbf{h}(\chi)/T\mathbf{h}(\chi) \cong h$, by Hensel's lemma (e.g., [BCM, III.4.6]), we have a unique algebra decomposition $\mathbf{h}(\chi) = \mathbf{h}_N \oplus \mathbf{h}'$ so that $\mathbf{h}_N/T\mathbf{h}_N = h_N$ and $\mathbf{h}'/T\mathbf{h}' = h'$.

Since $T(p) \equiv U(p) \mod (p)$ in h_N, we get $h_N \cong h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{Z}_p[\chi])$. Since $p \nmid D_\chi$, the reduction map modulo p: $\mathrm{Hom}_{\mathrm{alg}}(h, \overline{\mathbb{Q}}_p) \to \Sigma$ is a bijection. In particular, we have $h = h^{new} \oplus h^{old}$ where h^{new} is the direct sum of $h_{\mathfrak{m}_\lambda}$ for λ coming from the eigenvalues of N-primitive forms. Again by Hensel's lemma, we have the algebra decomposition $\mathbf{h}_N = \mathbf{h}^{new} \oplus \mathbf{h}^{old}$ with $\mathbf{h}^?/T\mathbf{h}^? = h^?$ for $? = new, old$. Since h^{new} is reduced by the theory of new forms ([H86a, §3] and [MFM, §4.6]) and unramified over $\mathbb{Z}_p$ by $p \nmid D_\chi\varphi(N)$, we conclude $h^{new} \cong \bigoplus_W W$ for discrete valuation rings W finite unramified over $\mathbb{Z}_p$ (one of the direct summand W acts on f non-trivially; i.e., W given by $\mathbb{Z}_p[f] = \mathbb{Z}_p[a_n | n = 1, 2, \dots] \subset \overline{\mathbb{Q}}_p$ for $T(n)$-eigenvalues a_n of f). Thus again by Hensel's lemma, we have a unique algebra direct factor $\mathbb{T}$ of $\mathbf{h}^{new}$ such that $\mathbb{T}/T\mathbb{T} = \mathbb{Z}_p[f] = W$. Since W is unramified over $\mathbb{Z}_p$, by the theory of Witt vectors [BCM, IX.1], we have a unique section $W \hookrightarrow \mathbb{T}$ of $\mathbb{T} \twoheadrightarrow \mathbb{Z}_p[f] = W$. Then $W[[T]] \subset \mathbb{T}$ which induces a surjection after reducing modulo T. Then by Nakayama's lemma, we have $\mathbb{T} = W[[T]] = W \otimes_{\mathbb{Z}_p} \Lambda$ as desired. □

6. Limit abelian factors

We recall some elementary facts (e.g. [H15, §6, after (6.6)]) with proof to good extent. Let $\iota : C_{r/\mathbb{Q}} \subset J_{r/\mathbb{Q}}$ (resp. $\pi : J_{r/\mathbb{Q}} \twoheadrightarrow D_{r/\mathbb{Q}}$) be an abelian subvariety (resp. an abelian variety quotient) stable under Hecke operators (including $U(l)$ for $l|Np$) and w_r. Here the stability means that $\mathrm{Im}(\iota)$ and $\mathrm{Ker}(\pi)$ are stable under Hecke operators. Then ι and π are Hecke equivariant. Let $\iota_s : C_s := \pi_{s,r}^*(C) \subset J_s$ for $s > r$ and D_s is the quotient abelian variety of $\pi_s : J_s \xrightarrow{\pi_s^r} J_r \twoheadrightarrow D_r$, where $\pi_s^r = w_r \circ \pi_{s,r,*} \circ w_s$. The twisted projection π_s^r is rational over $\mathbb{Q}$ as $w_s^{[z,\mathbb{Q}]} = \langle z\rangle \circ w_s = w_s \circ \langle z\rangle^{-1}$ for $z \in \widehat{\mathbb{Z}}^\times$.

Since the two morphisms $J_r \to J_s^r$ and $J_s^r \to J_s[\gamma^{p^{r-\epsilon}} - 1]$ (Picard functoriality) are $U(p)$-isomorphism of fppf abelian sheaves by (u1) and Corollary 3.2, we get the following two isomorphisms of fppf abelian sheaves for $s > r > 0$:

$$C_r[p^\infty]^{\mathrm{ord}} \xrightarrow[\pi_{r,s}^*]{\sim} C_s[p^\infty]^{\mathrm{ord}} \text{ and } \widehat{C}_r^{\mathrm{ord}} \xrightarrow[\pi_{r,s}^*]{\sim} \widehat{C}_s^{\mathrm{ord}}, \tag{6.1}$$

since $\widehat{C}_s^{\mathrm{ord}}$ is the isomorphic image of $\widehat{C}_r^{\mathrm{ord}} \subset \widehat{J}_r$ in $\widehat{J}_s[\gamma^{p^{r-\epsilon}} - 1]$. By the w-twisted Cartier duality [H14, §4], we have

$$D_s[p^\infty]^{\mathrm{ord}}_{/\mathbb{Q}} \xrightarrow[\pi_s^r]{\sim} D_r[p^\infty]^{\mathrm{ord}}_{/\mathbb{Q}}. \tag{6.2}$$

This isomorphism (6.2) is over $\mathbb{Q}$ not over the discrete valuation ring $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$ as explained in [H16, §11] (after the proof of Proposition 11.3 in [H16]), but the isomorphism (6.1) is usually valid over $\mathbb{Z}_{(p)}$ (see Section 17). By Kummer sequence, we have the following commutative diagram

$$\begin{array}{ccc} \widehat{D}_s^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} = (D_s(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z})^{\mathrm{ord}} & \underset{\hookrightarrow}{\longrightarrow} & H^1(D_s[p^m]^{\mathrm{ord}}) \\ {\scriptstyle \pi_s^r}\downarrow & & \wr\downarrow{\scriptstyle (6.2)} \\ \widehat{D}_r^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} = (D_r(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z})^{\mathrm{ord}} & \underset{\hookrightarrow}{\longrightarrow} & H^1(D_r[p^m]^{\mathrm{ord}}) \end{array}$$

This shows the injectivity of the following map

$$\widehat{D}_s^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} \to \widehat{D}_r^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z}.$$

Taking the w-twisted dual C_s of D_s (which interchanges (α,δ) to (δ,α)), from $\widehat{C}_s^{\mathrm{ord}}(\kappa)\cong\widehat{C}_r^{\mathrm{ord}}(\kappa)$, the source and the target of the above map has the same order; so, it is an isomorphism. Passing to the projective/injective limit, we get

$$\widehat{D}_s^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} \widehat{D}_r^{\mathrm{ord}} \text{ and } (D_s\otimes_{\mathbb{Z}}\mathbb{T}_p)^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} (D_r\otimes_{\mathbb{Z}}\mathbb{T}_p)^{\mathrm{ord}} \tag{6.3}$$

as fppf abelian sheaves. In short, we get

Lemma 6.1. *Suppose that κ is a field extension of finite type of either a number field or a finite extension of $\mathbb{Q}_l$. Then we have the following isomorphism*

$$\widehat{C}_r(\kappa)^{\mathrm{ord}} \xrightarrow[\pi_{s,r}^*]{\sim} \widehat{C}_s(\kappa)^{\mathrm{ord}} \quad \textit{and} \quad \widehat{D}_s(\kappa)^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} \widehat{D}_r(\kappa)^{\mathrm{ord}}$$

for all $s>r$ including $s=\infty$.

Taking C_s to be A_s (and hence $D_s=B_s$ by Lemma 5.1) and applying this lemma to the exact sequence (5.1), we get a new exact sequence (for ϖ in (A)):

$$0\to\widehat{A}_r^{\mathrm{ord}}\to J_\infty^{\mathrm{ord}}\xrightarrow{\varpi}J_\infty^{\mathrm{ord}}\to\widehat{B}_\infty^{\mathrm{ord}}\to 0, \tag{6.4}$$

since $\widehat{A}_\infty^{\mathrm{ord}}=\varinjlim_s\widehat{A}_s^{\mathrm{ord}}\cong\widehat{A}_r^{\mathrm{ord}}$ by the lemma.

We make $\widehat{B}_\infty^{\mathrm{ord}}$ explicit. By computation, $\pi_s^r\circ\pi_{r,s}^*=p^{s-r}U(p^{s-r})$. To see this, as Hecke operators from Γ_s-coset operations, we have $\pi_{r,s}^*=[\Gamma_s]$ (restriction with respect to Γ_r/Γ_s) and $\pi_{r,s,*}=[\Gamma_r]$ (trace map with respect to Γ_r/Γ_s). Thus we have

$$\pi_s^r\circ\pi_{r,s}^*=[\Gamma_s]\cdot w_s\cdot[\Gamma_r^\iota]\cdot w_r=[\Gamma_s]\cdot[w_sw_r]\cdot[\Gamma_r]=[\Gamma_s^r:\Gamma_s][\Gamma_s^r\left(\begin{smallmatrix}1&0\\0&p^{s-r}\end{smallmatrix}\right)\Gamma_r]=p^{s-r}U(p^{s-r}). \tag{6.5}$$

Lemma 6.2. *We have the following two commutative diagrams for $s'>s$*

$$\begin{array}{ccc} \widehat{C}_{s'}^{\mathrm{ord}} & \xleftarrow[\pi_{s,s'}^*]{\sim} & \widehat{C}_s^{\mathrm{ord}} \\ {\scriptstyle \pi_{s'}^s}\downarrow & & \downarrow{\scriptstyle p^{s'-s}U(p)^{s'-s}} \\ \widehat{C}_s^{\mathrm{ord}} & = & \widehat{C}_s^{\mathrm{ord}}, \end{array}$$

and

$$\begin{array}{ccc} \widehat{D}_{s'}^{\mathrm{ord}} & \xrightarrow[\pi_{s'}^s]{\sim} & \widehat{D}_s^{\mathrm{ord}} \\ {\scriptstyle \pi_{s,s'}^*}\uparrow & & \uparrow{\scriptstyle p^{s'-s}U(p)^{s'-s}} \\ \widehat{D}_s^{\mathrm{ord}} & = & \widehat{D}_s^{\mathrm{ord}}. \end{array}$$

In particular, we get $\widehat{D}_\infty^{\mathrm{ord}}:=\varinjlim_s\widehat{D}_r^{\mathrm{ord}}=\widehat{D}_r^{\mathrm{ord}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

Proof. By $\pi_{r,s}^*$ (resp. π_s^r), we identify $\widehat{C}_s^{\mathrm{ord}}$ with $\widehat{C}_r^{\mathrm{ord}}$ (resp. $\widehat{D}_s^{\mathrm{ord}}$ with $\widehat{D}_r^{\mathrm{ord}}$) as in Lemma 6.1. Then the above two diagrams follow from (6.5).

For a free $\mathbb{Z}_p$-module F of finite rank, we suppose to have a commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{p^n} & F \\ \| \downarrow & & \downarrow p^{-n} \\ F & \underset{\hookrightarrow}{\longrightarrow} & p^{-n}F. \end{array}$$

Thus we have $\varinjlim_{n, x\mapsto p^n x} F = \varinjlim_{n,x\mapsto p^{-n}x} p^{-n}F \cong F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If T is a torsion $\mathbb{Z}_p$-module with $p^B T = 0$ for $B \gg 0$, we have $\varinjlim_{n,x\mapsto p^n x} T = 0$. Thus for general $M = F \oplus T$, we have $\varinjlim_{n,x\mapsto p^n x} M \cong M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Identifying $\widehat{D}_s^{\mathrm{ord}}$ with $\widehat{D}_r^{\mathrm{ord}}$ by π_s^r for all $s \geq r$, the transition map of the inductive limit $\varinjlim_s \widehat{D}_s^{\mathrm{ord}}$ is given by the following commutative diagram

$$\begin{array}{ccc} \widehat{D}_r^{\mathrm{ord}} & \longrightarrow & \widehat{D}_s^{\mathrm{ord}} \\ \| \downarrow & & \wr \downarrow \pi_s^r \\ \widehat{D}_r^{\mathrm{ord}} & \xrightarrow[x\mapsto p^{s-r}U(p)^{s-r}]{} & \widehat{D}_r^{\mathrm{ord}}, \end{array}$$

where the top arrow is induced by $\pi_{r,s}^*$. Thus applying the above result for $M = \widehat{D}_s^{\mathrm{ord}}(K)$, we find $\varinjlim_s \widehat{D}_s^{\mathrm{ord}}(K) = \widehat{D}_r^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. □

Applying this lemma to $D_s = B_s$, we get from (6.4), the following exact sequence:

Corollary 6.3. *Assume* (F). *Let K be either a number field or a finite extension of $\mathbb{Q}_l$ for a prime l. For (ϖ, A_r, B_r) satisfying* (A), *we get the following exact sequence of étale/fppf sheaves over K:*

$$0 \to \widehat{A}_r^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \xrightarrow{\varpi} J_\infty^{\mathrm{ord}} \xrightarrow{\rho_\infty} \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$

In particular, for $K' = K^S$ if K is a number field and $K' = \overline{K}$ if K is local, we have the following exact sequence of Galois modules: $0 \to \widehat{A}_r^{\mathrm{ord}}(K') \to J_\infty^{\mathrm{ord}}(K') \xrightarrow{\varpi} J_\infty^{\mathrm{ord}}(K') \to \widehat{B}_r^{\mathrm{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$.

Proof. Since a finite étale extension R of K is a product of finite field extensions of K, we may assume that R is a field extension of K. Then by (S), $\widehat{B}_s(R)^{\mathrm{ord}} \cong \widehat{B}_r(R)^{\mathrm{ord}}$ is a $\mathbb{Z}_p$-module of finite type. Then by the above lemma Lemma 6.2, taking D_s to be B_s, we find that $\varinjlim_s \widehat{B}_s(R)^{\mathrm{ord}} = \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Since passing to injective limit is an exact functor, this proves the first exact sequence:

$$0 \to \widehat{A}_r^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \xrightarrow{\varpi} J_\infty^{\mathrm{ord}} \to \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$

Since $\widehat{X}^{\mathrm{ord}}(K') = \varinjlim_{K'/F/K} \widehat{X}^{\mathrm{ord}}(F)$ for F running a finite extension of K, we get the exactness of $0 \to \widehat{A}_r^{\mathrm{ord}}(K') \to J_\infty^{\mathrm{ord}}(K') \xrightarrow{\varpi} J_\infty^{\mathrm{ord}}(K')$.

Since $0 \to \widehat{A}_r^{\mathrm{ord}}(K') \to J_s^{\mathrm{ord}}(K') \xrightarrow{\varpi} J_s^{\mathrm{ord}}(K') \to \widehat{B}_s^{\mathrm{ord}}(K') \to 0$ is an exact sequence of Galois modules, passing to the limit, we still have the exactness of

$$0 \to \widehat{A}_r^{\mathrm{ord}}(K') \to J_\infty^{\mathrm{ord}}(K') \xrightarrow{\varpi} J_\infty^{\mathrm{ord}}(K') \to \widehat{B}_s^{\mathrm{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$$

proving the last assertion. □

Let $\mathcal{G}_s = J_s[p^\infty]^{\mathrm{ord}}$. Then we have an exact sequence $A_r[p^\infty]^{\mathrm{ord}} \hookrightarrow \mathcal{G}_s \xrightarrow{\varpi} \mathcal{G}_s \twoheadrightarrow B_s[p^\infty]^{\mathrm{ord}}$ of fppf sheaves. In this case, $\varinjlim_{x\mapsto p^{s-r}U(p)^{s-r}} B_s[p^\infty]^{\mathrm{ord}} = 0$ as $B_s[p^\infty]^{\mathrm{ord}}$ is p-torsion. Passing to the limit, we recover the following fact proven in [H14, §3, (DV) and §5]:

Corollary 6.4. *We have an exact sequence of fppf sheaves over $\mathbb{Q}$:*

$$0 \to A_r[p^\infty]^{\mathrm{ord}} \to \mathcal{G}_\infty \xrightarrow{\varpi} \mathcal{G}_\infty \to 0$$

which extends canonically to an exact sequence of fppf sheaves over $\mathbb{Z}_{(p)}[\mu_{p^\infty}]$.

7. Generality of Galois cohomology

We prove some general result on Galois cohomology for our later use. Let S be a set of places of a number field K. Suppose that S contains all archimedean places and p-adic places of K (and primes for bad reduction of the abelian varieties when we deal with abelian varieties). Let K^S be the maximal extension unramified outside S.

Lemma 7.1. *Let $\{M_n\}_n$ be a projective system of finite $\mathbb{Z}_p[\mathrm{Gal}(K^S/K)]$-modules M_n. Write $M_\infty := \varprojlim_n M_n$ and $M_\infty^\vee := \varinjlim_n M_n^\vee$ for the Pontryagin dual $M_n^\vee$ of M_n. Write G (resp. G_v for a place v of K) for the (point by point) stabilizer of $M_\infty^\vee$ in $\mathrm{Gal}(K^S/K)$ (resp. $\mathrm{Gal}(\overline{K}_v/K_v)$) and $\overline{G} = \mathrm{Gal}(K^S/K)/G$ (resp. $\overline{G}_v := \mathrm{Gal}(\overline{K}_S/K_v)/G_v)$). Then, we have*

1. *$Ш^1(K^S/K, M_\infty) = \varprojlim_n Ш^1(K^S/K, M_n)$, and if S is a finite set, we have $Ш^1(K^S/K, M_\infty^\vee) = \varinjlim_n Ш^1(K^S/K, M_n^\vee)$.*
2. *$Ш^2(K^S/K, M_\infty^\vee) = \varinjlim_n Ш^2(K^S/K, M_n^\vee)$, and if S is a finite set, we have*

$$H^2(K^S/K, M_\infty) = \varprojlim_n H^2(K^S/K, M_n) \quad \text{and} \quad Ш^2(K^S/K, M_\infty) = \varprojlim_n Ш^2(K^S/K, M_n).$$

Proof. We first prove the assertion in (1) for projective limit. Since $H^0(?, M_n)$ ($? = K^S/K$ and K_v) is finite for all n, we have $\varprojlim_n H^1(?, M_n) = H^1(?, M_\infty)$ for $? = K^S/K$ and K_v by [CNF, Corollary 2.7.6]. By definition, we have an exact sequence:

$$0 \to Ш^1(K^S/K, M_n) \to H^1(K^S/K, M_n) \to \prod_{v\in S} H^1(K_v, M_n).$$

Since any continuous cochain with values in $\varprojlim_n M_n$ is a projective limit of continuous cochains with values in M_n, we have a natural map $H^1(?, \varprojlim_n M_n) \to \varprojlim_n H^1(?, M_n)$ for $? = K^S/K$ and K_v. Passing to the limit, we get the following commutative diagram with exact rows

$$\begin{array}{ccccc} Ш^1(K^S/K, \varprojlim_n M_n) & \hookrightarrow & H^1(K^S/K, \varprojlim_n M_n) & \longrightarrow & \prod_{v\in S} H^1(K_v, \varprojlim_n M_n) \\ \downarrow & & \wr\downarrow & & \wr\downarrow \\ \varprojlim_n Ш^1(K^S/K, M_n) & \hookrightarrow & \varprojlim_n H^1(K^S/K, M_n) & \longrightarrow & \prod_{v\in S} \varprojlim_n H^1(K_v, M_n). \end{array}$$

This shows $Ш^1(K^S/K, \varprojlim_n M_n) = \varprojlim_n Ш^1(K^S/K, M_n)$ as desired. As for the injective limit, we first note that the cohomology functor commutes with the limit. However it may not commute with infinite product; so, we need to assume that S is finite (this fact is pointed out by D. Harari).

As for (2), since cohomology functor commutes with injective limit, the assertion (2) for injective limits follows from the same argument as in the case of (1), noting that by local Tate duality, the direct product $\prod_{v\in S} H^2(K_v, M_n)$ in the definition of $Ш^2$ can be replaced by the direct sum; so, we have the assertion for the injective limit. If S is finite, $H^1(K^S/K, M_n)$ is finite (e.g., [ADT, I.5.1]). Thus by [CNF, Corollary 2.7.6], we have $\varprojlim_n H^2(?, M_n) = H^2(?, \varprojlim_n M_n)$ for $? = K^S/K$ and K_v, and hence once again the same argument works (replacing H^1 by H^2). □

Let A be an abelian variety over a field K. Since the Galois group $\mathrm{Gal}(\overline{K}/K)$ and $\mathrm{Gal}(K^S/K)$ is profinite and $A(\overline{K})$ and $A(K^S)$ are discrete modules, for $q > 0$, the continuous cohomology group $H^q(K^S/K, A)$ for a number field K and $H^q(K, A)$ for a local field K are torsion discrete modules (see [MFG, Corollary 4.26]).

Lemma 7.2. *If K is either a number field or a local field of characteristic 0, we have a canonical isomorphism for $0 < q \in \mathbb{Z}$:*

$$H^q(\widehat{A}) \cong H^q(A)_p = H^q(A)[p^\infty], \tag{7.1}$$

where $H^q(?) = H^q(K^S/K, ?)$ if K is a number field, and $H^q(?) = H^q(K, ?)$ if K is local.

Proof. By (S), if K is a number field, we have

$$H^q(K^S/K,\widehat{A}) \cong H^q(K^S/K, A\otimes_{\mathbb{Z}}\mathbb{Z}_p) \overset{(*)}{\cong} H^q(K^S/K,A)\otimes_{\mathbb{Z}}\mathbb{Z}_p = H^q(K^S/K,A)_p,$$

as $H^q(K^S/K,A)$ is a torsion module. Here the identity $(*)$ follows from the universal coefficient theorem (e.g., [CNF, 2.3.4] or [CGP, (0.8)]).

Now suppose that K is an l-adic with $l\neq p$ or archimedean local field. Then $\widehat{A}=A[p^\infty]$, and we have a natural inclusion $0\to\widehat{A}(\overline{K})\to A(\overline{K})\to Q\to 0$ for the quotient Galois module Q. Thus Q is p-torsion-free and p-divisible; i.e., the multiplication by p is invertible on Q. Therefore $H^q(K,Q)_p := H^q(K,Q)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is a $\mathbb{Q}_p$-vector space for $q\geq 0$ (so, $H^q(K,Q)_p=0$ though we do not need this vanishing). By the exact sequence $H^{q-1}(K,Q)_p\to H^q(K,\widehat{A})_p\to H^q(K,A)_p\to H^q(K,Q)_p$, we conclude $H^q(K,\widehat{A})_p\cong H^q(K,A)_p$ as the two modules are p-torsion.

If $l=p$, we have $A(\overline{K})=\widehat{A}(\overline{K})\oplus A^{(p)}(\overline{K})$ under the notation of (S), and hence $H^q(K,A)_p = H^q(\widehat{A})_p\oplus H^q(K,A^{(p)})_p$. Since $\widehat{A}(\overline{K})$ is a union of p-profinite group, we have $H^q(\widehat{A})_p=H^q(\widehat{A})$. Since $A^{(p)}(\overline{K})$ is prime-to-p torsion, we have $H^q(K,A^{(p)})_p=0$. Thus $H^q(K,A)_p=H^q(K,A)\otimes_{\mathbb{Z}}\mathbb{Z}_p = H^q(K,\widehat{A})\oplus H^q(K,A^{(p)})_p = H^q(K,\widehat{A})$. □

8. Diagrams of Selmer groups and Tate–Shafarevich groups

We describe commutative diagrams involving different Selmer groups and Tate–Shafarevich groups, which are the base of the proof of the control result in later sections. We assume $p>2$ for simplicity.

Recall the definition of the p-part of the Selmer group and the Tate–Shafarevich group for an abelian variety A defined over a number field K:

$$Ш_K(A)_p = \mathrm{Ker}(H^1(K^S/K,A)_p \xrightarrow{\mathrm{Res}} \prod_{v\in S} H^1(K_v,A)_p),$$
$$\mathrm{Sel}_K(A)_p = \mathrm{Ker}(H^1(K^S/K,A[p^\infty]) \xrightarrow{\mathrm{Res}} \prod_{v\in S} H^1(K_v,A)_p).$$

As long as S is sufficiently large containing all bad places for A in addition to all archimedean and p-adic places, these groups are independent of S (see [ADT, I.6.6]) and are p-torsion modules.

Lemma 8.1. *We can replace A in the above definition by $\widehat{A}$, and we get*

$$\begin{aligned} Ш_K(A)_p = Ш_K(\widehat{A}) &= \mathrm{Ker}(H^1(K^S/K,\widehat{A}) \xrightarrow{\mathrm{Res}} \bigoplus_{v\in S} H^1(K_v,\widehat{A})),\\ \mathrm{Sel}_K(A)_p = \mathrm{Sel}_K(\widehat{A}) &= \mathrm{Ker}(H^1(K^S/K,A[p^\infty]) \xrightarrow{\mathrm{Res}} \bigoplus_{v\in S} H^1(K_v,\widehat{A})).\end{aligned} \tag{8.1}$$

Proof. It is known that image of global cohomology classes lands in the direct sum $\bigoplus_{v\in S} H^1(K_v,\widehat{A})$ in the product $\prod_{v\in S} H^1(K_v,\widehat{A})$ (see [ADT, I.6.3]).

By Lemma 7.2, we have $Ш_K(\widehat{A}) = Ш_K(A)_p = Ш_K(A)\otimes_{\mathbb{Z}}\mathbb{Z}_p$. Thus we may replace the p-primary part of the traditional Ш-functor $A\mapsto Ш_K(A)_p$ by the completed one $A\mapsto Ш_K(\widehat{A})$. □

Since $A\mapsto Ш_K(\widehat{A})$ is covariant, from Lemma 3.3 (and Remark 3.4), we get the commutative diagram for $X=Ш$ and Sel:

$$\begin{array}{ccc} X_K(\widehat{J}_r) & \xrightarrow{\pi^*} & X_K(\widehat{J}^r_s) \\ \downarrow u & \swarrow u' & \downarrow u'' \\ X_K(\widehat{J}_r) & \xrightarrow{\pi^*} & X_K(\widehat{J}^r_s), \end{array} \tag{8.2}$$

Similarly to the diagram as above, from Corollary 3.2, we get the following commutative diagram:

$$\begin{array}{ccc} X_K(\widehat{J}^r_s) & \xrightarrow{\pi^*} & X_K(\widehat{J}_s[\gamma^{p^{r-\epsilon}}-1]) \\ \downarrow u & \swarrow \iota^r_s & \downarrow u'' \\ X_K(\widehat{J}^r_s) & \xrightarrow{\pi^*} & X_K(\widehat{J}_s[\gamma^{p^{r-\epsilon}}-1]). \end{array} \tag{8.3}$$

These diagrams provide us the following canonical isomorphisms

$$X_K(\widehat{J}_r)^{\mathrm{ord}} \cong X_K(\widehat{J}_s[\gamma^{p^{r-\epsilon}}-1])^{\mathrm{ord}} \quad \text{for } X = Ш \text{ and Sel.} \tag{8.4}$$

For any group subvariety $A_{/\mathbb{Q}}$ of J_s proper over $\mathbb{Q}$ stable under $U(p)$ or any abelian variety quotient $A_{/\mathbb{Q}}$ of J_s stable under $U(p)$, we have $\widehat{A} = \widehat{A}^{\mathrm{ord}} \oplus (1-e)\widehat{A}$, and hence $H^q(?,\widehat{A}) = H^q(?,\widehat{A}^{\mathrm{ord}}) \oplus H^q(?,(1-e)\widehat{A})$ for $? = \overline{K}$ and K^S. This shows $H^q(?,\widehat{A})^{\mathrm{ord}} = H^q(?,\widehat{A}^{\mathrm{ord}})$, and hence $X^q_K(\widehat{A}^{\mathrm{ord}}) = X^q_K(\widehat{A})^{\mathrm{ord}} = X^q_K(A)^{\mathrm{ord}}_p$. Thus hereafter, we attach the superscript "ord" inside the cohomology/Tate–Shafarevich group if the coefficient is p-adically completed in the sense of (S).

We define the ind Λ-TS group and the ind Λ-Selmer group by

$$\begin{aligned} Ш_K(J_\infty)^{\mathrm{ord}} &:= Ш_K(J_\infty^{\mathrm{ord}}) = \varinjlim_r Ш_K(\widehat{J}_r^{\mathrm{ord}}) = \varinjlim_r Ш_K(J_r)^{\mathrm{ord}}_p, \\ \mathrm{Sel}_K(J_\infty)^{\mathrm{ord}} &:= \mathrm{Sel}_K(J_\infty^{\mathrm{ord}}) = \varinjlim_r \mathrm{Sel}_K(\widehat{J}_r^{\mathrm{ord}}) = \varinjlim_r \mathrm{Sel}_K(J_r)^{\mathrm{ord}}_p \end{aligned} \tag{8.5}$$

which are naturally $\mathbf{h}$-modules.

Write $H^q_S(M) = \bigoplus_{v\in S} H^q(K_v,M)$ and $H^q(M) = H^q(K^S/K,M)$ for a $\mathrm{Gal}(K^S/K)$-module M. By [ADT, I.6.6], $Ш(K^S/K,A)_p = Ш_K(A)_p$ for an abelian variety $A_{/K}$ as long as S contains all bad places of A and all archimedean and p-adic places. Consider a triple $(\varpi, A_s = J_s[\mathfrak{a}_s], B_s = J_s/\mathfrak{a}_sJ_s)$ satisfying the condition (A) of Section 5 and (F) in Section 4. Note that $\widehat{J}^{\mathrm{ord}}_s[\varpi_s] = \widehat{A}^{\mathrm{ord}}_s$ (see Lemma 5.1), we have $H^q(?,\widehat{J}^{\mathrm{ord}}_s[\varpi_s]) = H^q(?,\widehat{A}^{\mathrm{ord}}_s)$. This implies $Ш_S(\widehat{J}^{\mathrm{ord}}_s[\varpi_s]) = Ш_S(\widehat{A}^{\mathrm{ord}}_s) \cong Ш_K(\widehat{A}^{\mathrm{ord}}_r)$, where the last identity follows from [ADT, I.6.6]. Recall the following exact sequence from Corollary 6.3:

$$0 \to \widehat{A}^{\mathrm{ord}}_r(K') \to J^{\mathrm{ord}}_\infty(K') \xrightarrow{\varpi} J^{\mathrm{ord}}_\infty(K') \to \widehat{B}^{\mathrm{ord}}_r(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0, \tag{8.6}$$

where $J^{\mathrm{ord}}_\infty = \varinjlim_s \widehat{J}^{\mathrm{ord}}_s$ and $K' = K^S$ and $\overline{K}_v$. We separate it into two short exact sequences:

$$\begin{aligned} 0 \to & \widehat{A}^{\mathrm{ord}}_r(K') \to J^{\mathrm{ord}}_\infty(K') \xrightarrow{\varpi} \varpi(J^{\mathrm{ord}}_\infty)(K') \to 0, \\ 0 \to & \varpi(J^{\mathrm{ord}}_\infty)(K') \to J^{\mathrm{ord}}_\infty(K') \to \widehat{B}^{\mathrm{ord}}_r(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0. \end{aligned} \tag{8.7}$$

Define

$$\begin{aligned} \mathrm{Sel}_K(\varpi(J^{\mathrm{ord}}_\infty)) &:= \mathrm{Ker}(i : H^1(K^S/K, \varpi(J^{\mathrm{ord}}_\infty)[p^\infty]) \to H^1_S(\varpi(J^{\mathrm{ord}}_\infty))), \\ Ш_K(\varpi(J^{\mathrm{ord}}_\infty)) &:= \mathrm{Ker}(i : H^1(K^S/K, \varpi(J^{\mathrm{ord}}_\infty)) \to H^1_S(\varpi(J^{\mathrm{ord}}_\infty))), \\ C_{MW}(K_v) &:= \mathrm{Coker}(J^{\mathrm{ord}}_\infty(K_v) \to \widehat{B}^{\mathrm{ord}}_r(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p). \end{aligned} \tag{8.8}$$

Here we have written $H^1_S(X) := \prod_{v\in S} H^1(K_v,X)$.

Look into the following commutative diagram of sheaves with exact rows:

$$\begin{array}{ccccccc} A_r[p^\infty]^{\mathrm{ord}} & \xrightarrow{\hookrightarrow} & J^{\mathrm{ord}}_\infty[p^\infty] & \xrightarrow{\varpi[p^\infty]} & J^{\mathrm{ord}}_\infty[p^\infty] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\ \widehat{A}^{\mathrm{ord}}_r & \xrightarrow{\hookrightarrow} & J^{\mathrm{ord}}_\infty & \xrightarrow{\varpi} & J^{\mathrm{ord}}_\infty & \longrightarrow & \widehat{B}^{\mathrm{ord}}_r \otimes \mathbb{Q}_p. \end{array} \tag{8.9}$$

Since $\widehat{B}^{\mathrm{ord}}_r \otimes \mathbb{Q}_p$ is a sheaf of $\mathbb{Q}_p$-vector spaces and $J^{\mathrm{ord}}_\infty[p^\infty]$ is p-torsion, the inclusion map i factors through the image $\mathrm{Im}(\varpi) = \varpi(J^{\mathrm{ord}}_\infty)$; so, for a finite extension K of $\mathbb{Q}$ or $\mathbb{Q}_l$,

$$\varpi(J^{\mathrm{ord}}_\infty)[p^\infty] = J^{\mathrm{ord}}_\infty[p^\infty]. \tag{8.10}$$

From the exact sequence, $\varpi(J^{\mathrm{ord}}) \hookrightarrow J^{\mathrm{ord}} \twoheadrightarrow \widehat{B}^{\mathrm{ord}}_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, taking its cohomology sequence, we get the bottom sequence of the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(\varpi(J^{\mathrm{ord}}_\infty)[p^\infty]) & = & H^1(J^{\mathrm{ord}}_\infty[p^\infty]) \\ \downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\ \prod_{v\in S} C_{MV}(K_v) & \xrightarrow{\hookrightarrow} & H^1_S(\varpi(J^{\mathrm{ord}}_\infty)) & \xrightarrow{\twoheadrightarrow} & H^1_S(J^{\mathrm{ord}}_\infty). \end{array}$$

By the snake lemma, we get an exact sequence

$$0 \to \mathrm{Sel}_K(\varpi(J_\infty^{\mathrm{ord}})) \to \mathrm{Sel}_K(J_\infty^{\mathrm{ord}}) \to \prod_{v|p} C_{MW}(K_v), \tag{8.11}$$

since $\widehat{B}_r^{\mathrm{ord}}(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = C_{MW}(K_v) = 0$ if $v \nmid p$ by (S) in the introduction.

Define error terms

$$\begin{aligned} E^s_{\mathrm{Sel}}(F) &:= \frac{\varpi(J_s^{\mathrm{ord}})[p^\infty](F)}{\varpi(J_s^{\mathrm{ord}}[p^\infty](F))},\ E_{\mathrm{Sel}}(F) := \frac{J_\infty^{\mathrm{ord}}[p^\infty](F)}{\varpi(J_\infty^{\mathrm{ord}}[p^\infty](F))} = \varinjlim_s \frac{\varpi(\widehat{J}_s^{\mathrm{ord}})[p^\infty](F)}{\varpi(\widehat{J}_s^{\mathrm{ord}}[p^\infty](F))}, \\ E^s(F) &:= \frac{\varpi(J_s^{\mathrm{ord}})(F)}{\varpi(J_s^{\mathrm{ord}}(F))},\ E^\infty(F) := \frac{\varpi(J_\infty^{\mathrm{ord}})(F)}{\varpi(J_\infty^{\mathrm{ord}}(F))} = \varinjlim_s \frac{\varpi(\widehat{J}_s^{\mathrm{ord}})(F)}{\varpi(\widehat{J}_s^{\mathrm{ord}}(F))} \end{aligned} \tag{8.12}$$

for $F = K, K_v$, and put $E_S^\infty(K) = \prod_{v \in S} E^\infty(K_v)$. If F is a finite extension of $\mathbb{Q}_l$ for $l \neq p$, by (S) in the introduction, we have $E_{\mathrm{Sel}}(F) = E^\infty(F)$. If $A_r = A_P$ for an arithmetic point P, we often write $E_P^\infty(F)$ for $E^\infty(F)$ as it depends on P. Noting $\mathcal{G} \xrightarrow{\varpi} \mathcal{G}$ is an epimorphism of sheaves for $\mathcal{G} = J_\infty^{\mathrm{ord}}[p^\infty]$ by Corollary 6.4, we then get the following commutative diagram with two bottom exact rows and columns:

$$\begin{array}{ccccccc} \mathrm{Ker}(\iota_{\mathrm{Sel},*}) & \longrightarrow & \mathrm{Sel}_K(\widehat{A}_r^{\mathrm{ord}}) & \xrightarrow{\iota_{\mathrm{Sel},*}} & \mathrm{Sel}_K(J_\infty^{\mathrm{ord}}) & \xrightarrow{\varpi_{\mathrm{Sel},*}} & \mathrm{Sel}_K(\varpi(J_\infty^{\mathrm{ord}})) \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ E_{\mathrm{Sel}}(K) & \hookrightarrow & H^1(\widehat{A}_r^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\iota_*} & H^1(J_\infty^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\varpi_*} & H^1(J_\infty^{\mathrm{ord}}[p^\infty]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_S^\infty(K) & \hookrightarrow & H^1_S(\widehat{A}_r^{\mathrm{ord}}) & \xrightarrow{\iota_{S,*}} & H^1_S(J_\infty^{\mathrm{ord}}) & \xrightarrow[\twoheadrightarrow]{\varpi_{S,*}} & H^1_S(\varpi(J_\infty^{\mathrm{ord}})). \end{array} \tag{8.13}$$

Here the last map $\varpi_{S,*}$ could have 2-torsion finite cokernel if $p = 2$.

We look into Λ-TS groups. Let $\varpi \in \mathbf{h}$ coming from $\varpi_r \in \mathrm{End}(J_{r/\mathbb{Q}})$ and suppose that $(\varpi) = \varpi\mathbf{h} \supset (\gamma^{p^{r-\epsilon}} - 1)$. The long exact sequence obtained from (8.7) produces the following commutative diagram with exact columns and bottom two exact rows:

$$\begin{array}{ccccccc} \mathrm{Ker}(\iota_{Ш,*}) & \longrightarrow & Ш_K(\widehat{A}_r^{\mathrm{ord}}) & \xrightarrow{\iota_{Ш,*}} & Ш_K(J_\infty^{\mathrm{ord}}) & \xrightarrow{\varpi_{Ш,*}} & Ш_K(\varpi(J_\infty^{\mathrm{ord}})) \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ E^\infty(K) & \hookrightarrow & H^1(\widehat{A}_r^{\mathrm{ord}}) & \xrightarrow{\iota_*} & H^1(J_\infty^{\mathrm{ord}}) & \xrightarrow{\varpi_*} & H^1(\varpi(J_\infty^{\mathrm{ord}})) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_S^\infty(K) & \hookrightarrow & H^1_S(\widehat{A}_r^{\mathrm{ord}}) & \xrightarrow{\iota_{S,*}} & H^1_S(J_\infty^{\mathrm{ord}}) & \xrightarrow[\twoheadrightarrow]{\varpi_{S,*}} & H^1_S(\varpi(J_\infty^{\mathrm{ord}})). \end{array} \tag{8.14}$$

By the vanishing of $H^2_S(\widehat{A}_r)$ ([ADT, Theorem I.3.2] and Lemma 7.2), $\varpi_{S,*}$ are surjective. In each term of the diagram (8.14), we can bring the superscript "ord" inside the functor Ш and H^1 to outside the functor as the ordinary projector acts on $\widehat{J}_s$, J_∞ and $\widehat{A}_r$ and gives direct factor of the sheaf. The diagram "ord" inside is the one obtained directly from the short exact sequence of Corollary 6.3. Thus we get from [BCM, Proposition I.1.4.2] the following fact:

Lemma 8.2. *Suppose* (A) *and* (F). *If $E_S^\infty(K)$ is finite, the sequence*

$$0 \to E^\infty(K) \to Ш_K(\widehat{A}_r^{\mathrm{ord}}) \to Ш_K(J_\infty^{\mathrm{ord}}) \xrightarrow{\varpi} Ш_K(\varpi(J_\infty^{\mathrm{ord}}))$$

is exact up to finite error. Moreover, if in addition $C_{MW}(K) = 0$, the sequence extends to the following exact sequence up to finite error:

$$0 \to E^\infty(K) \to Ш_K(\widehat{A}_r^{\mathrm{ord}}) \to Ш_K(J_\infty^{\mathrm{ord}}) \xrightarrow{\varpi} Ш_K(J_\infty^{\mathrm{ord}}).$$

Proof. Applying [BCM, I.1.4.2 (1)] to the third column of (8.14), the first column is exact. Thus $\mathrm{Ker}(i_{\text{Ш},*})$ is isomorphic to $E^\infty(K)$ up to finite error. Applying [BCM, I.1.4.2 (1)] again to the first three terms of the bottom row, we get exactness of $0 \to \mathrm{Ker}(\iota_{\text{Ш},*}) \to \text{Ш}_K(\widehat{A}_r^{\mathrm{ord}}) \to \text{Ш}_K(J_\infty^{\mathrm{ord}})$. Replacing $\mathrm{Ker}(\iota_{\text{Ш},*})$ by $E^\infty(K)$, we get the exactness of $0 \to E^\infty(K) \to \text{Ш}_K(\widehat{A}_r^{\mathrm{ord}}) \to \text{Ш}_K(J_\infty^{\mathrm{ord}})$ up to finite error. Since $E_S^\infty(K)$ is finite, again by [BCM, Proposition I.1.4.2 (1)], we get the exactness of $\text{Ш}_K(\widehat{A}_r^{\mathrm{ord}}) \to \text{Ш}_K(J_\infty^{\mathrm{ord}}) \xrightarrow{\varpi} \text{Ш}_K(\varpi(J_\infty^{\mathrm{ord}}))$ up to finite error. This finishes the proof for the first sequence.

To extend the exact sequence, we look into the sheaf exact sequence: $0 \to \varpi(J_\infty^{\mathrm{ord}}) \xrightarrow{i} J_\infty^{\mathrm{ord}} \to \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$, which produces the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Ker}(i_{\text{Ш},*}) & \longrightarrow & \text{Ш}_K(\varpi(J_\infty^{\mathrm{ord}})) & \xrightarrow{i_{\text{Ш},*}} & \text{Ш}_K(J_\infty^{\mathrm{ord}}) \\
\cap\downarrow & & \cap\downarrow & & \cap\downarrow \\
C_{MW}(K) & \hookrightarrow & H^1(\widehat{A}_r^{\mathrm{ord}}) & \overset{i_*}{\twoheadrightarrow} & H^1(J_\infty^{\mathrm{ord}}) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{v|p} C_{MW}(K_v) & \hookrightarrow & H^1_S(\widehat{A}_r^{\mathrm{ord}}) & \overset{i_{S,*}}{\twoheadrightarrow} & H^1_S(J_\infty^{\mathrm{ord}}).
\end{array} \tag{8.15}$$

Again the left column is exact as the right column is exact. Then by the assumption $C_{MW}(K) = 0$, we get a canonical injection:

$$i_{\text{Ш},*} : \text{Ш}_K(\varpi(J_\infty^{\mathrm{ord}})) \hookrightarrow \text{Ш}_K(J_\infty^{\mathrm{ord}}). \tag{8.16}$$

Thus we can replace the end term $\text{Ш}_K(\varpi(J_\infty^{\mathrm{ord}}))$ of the first exact sequence by $\text{Ш}_K(J_\infty^{\mathrm{ord}})$, getting the extended exact sequence up to finite error. □

9. Vanishing of the first error term E^∞ for l-adic fields with $l \neq p$

In this section, we prove vanishing of the error term $E^\infty(K)$ for local fields of residual characteristic $l \neq p$, which combined with a similar (but more difficult) result for p-adic fields given in Section 17 will be used in the following sections to prove the control result up to finite error of the limit Selmer group, the limite Mordell–Weil group and the limit Tate–Shafarevich group.

More generally, for the moment, we denote by K either a number field or an l-adic field (the prime l can be p unless we mention that $l \neq p$).

Lemma 9.1. *Let K either a number field or an l-adic field. Then the Pontryagin dual $E^\infty(K)^\vee$ of $E^\infty(K)$ is a $\mathbb{Z}_p$-module of finite type (i.e., $E^\infty(K)$ is p-torsion of finite corank).*

Proof. Let $K' = \overline{K}$ if K is local and $K = K^S$ if K is global. We have an exact sequence

$$0 \to E^\infty(K) \to H^1(K'/K, \widehat{A}_r^{\mathrm{ord}}) \to H^1(K'/K, \widehat{J}_\infty^{\mathrm{ord}}).$$

By [ADT, I.3.4], if K is local, $H^1(K'/K, \widehat{A}_r^{\mathrm{ord}}) \cong \mathrm{Pic}^0_{A/K}(K)^\vee$; so, we get the desired result. If K is global, $\widehat{A}_r^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}} \mathbb{F} \to H^1(K'/K, \widehat{A}_r^{\mathrm{ord}}[p]) \twoheadrightarrow H^1(K'/K, \widehat{A}_r^{\mathrm{ord}})[p]$ is exact, and the middle term is finite by Tate's computation of the global cohomology (taking S to be finite); so, $H^1(K'/K, \widehat{A}_r^{\mathrm{ord}})$ has Pontryagin dual finite type over $\mathbb{Z}_p$. This finishes the proof. □

Proposition 9.2. *Assume* (A) *and* (F). *Let K be a finite extension of $\mathbb{Q}_l$. Then we have*

1. $E^s(K) \cong \mathrm{Ker}(\widehat{A}_r^{t,\text{co-ord}}(K)^\vee \to \widehat{J}_s^{t,\text{co-ord}}(K)^\vee)$ *which is under $l \neq p$ in turn isomorphic to*
$$= \mathrm{Ker}(H_0(K, T_p\widehat{A}_r^{\mathrm{ord}}(-1)) \to H_0(K, T_p\widehat{J}_s^{\mathrm{ord}}(-1)))$$
for the negative Tate twist indicated by "(−1)",
2. *If $l \neq p$, the order $|E^s(K)|$ is finite and is bounded for all s.*
3. *If $l \neq p$ and $H_0(K, T_pA_r^{\mathrm{ord}}(-1)) = 0$ ($\Leftrightarrow A_r^{t,\text{co-ord}}(K) = 0$), we have $E^s(K) = 0$ for all s.*
4. *Suppose $l \neq p$. If $\mathbb{T}$ is an integral domain and A_r has good reduction over the l-adic integer ring W of K, then $E^s(K) = 0$ for all s.*

We will prove the finiteness and boundedness of $E^s(K)$ when $l = p$ later in Section 17 under some extra assumptions (see Theorem 17.2).

Proof. We have an exact sequence

$$0 \to E^s(K) := \mathrm{Coker}(\varpi_s : \widehat{J}_s^{\mathrm{ord}}(K) \to \varpi(\widehat{J}_s^{\mathrm{ord}})(K)) \xrightarrow{i_s} H^1(K, \widehat{A}_r^{\mathrm{ord}}) \xrightarrow{\iota_s^1} H^1(K, \widehat{J}_s^{\mathrm{ord}}).$$

By [ADT, I.3.4], if K is local, for an abelian variety A over K, we have $H^1(K, A) = A^t(K)^\vee$ for $A^t = \mathrm{Pic}^0_{A/K}$. Note that $\widehat{A}^t = A^t[p^\infty]$ if $l \neq p$ by (S) at the end of the introduction. From local Tate duality and Lemma 7.2 (combined with Weil pairing $A^t[p^\infty]$ and $T_pA(-1)$), we conclude

$$E^s(K) = \mathrm{Ker}(\widehat{A}_r^{t,\text{co-ord}}(K)^\vee \to \widehat{J}_s^{t,\text{co-ord}}(K)^\vee) = \mathrm{Ker}(H_0(K, T_p\widehat{A}_r^{\mathrm{ord}}(-1)) \to H_0(K, T_p\widehat{J}_s^{\mathrm{ord}}(-1)))$$

proving the first assertion.

We now claim that $\widehat{A}_r^{\text{co-ord}}(K) \subset A_r[p^\infty](K)$ is finite. If A_r has good reduction over W, we have $A_r[p^\infty](K) \cong A_r[p^\infty](\mathbb{F})$ which is finite. Take the Néron model $A_{r/W}$ of A_r. By the universal property of the Néron model, we find $A_r(W) = A_r(K)$; so, $A_{r/W}[p^\infty](W) = A_{r/W}[p^\infty](K)$. Since multiplication by p on $A_{r/W}$ is étale (as $l \neq p$; see [NMD, Lemma 7.3.2], we find $A_{r/W}[p^\infty](W) \cong A_{r/W}[p^\infty](\mathbb{F}) \subset A_{r/W}(\mathbb{F})$, which is finite (as $A_{r/W} \otimes \mathbb{F}$ is of finite type over $\mathbb{F}$). This shows the claim. The claim proves the assertion (2) as $E^s(K) \hookrightarrow A_r[p^\infty](K)$ by (1). The assertion (3) also follows from (1).

We now prove (4). If A_r has good reduction over W, by Lemma 9.3 following this proof, $\varpi(\widehat{J}_{s,\mathbb{T}}^{\mathrm{ord}})$ and $\widehat{J}_{s,\mathbb{T}}^{\mathrm{ord}}$ have good reduction over the l-adic integer ring W of K, we have $\varpi(\widehat{J}_{s,\mathbb{T}}^{\mathrm{ord}})(K) = \varpi(\widehat{J}_{s,\mathbb{T}}^{\mathrm{ord}})[p^\infty](K) \cong \varpi(\widehat{J}_{s,\mathbb{T}}^{\mathrm{ord}})[p^\infty](\mathbb{F})$ and similarly $\widehat{J}_{s,\mathbb{T}}^{t,\text{co-ord}}(K) = \widehat{J}_{s,\mathbb{T}}^{t,\text{co-ord}}[p^\infty](K) \cong J_{s,\mathbb{T}}^{t,\text{co-ord}}[p^\infty](\mathbb{F})$ for the residue field $\mathbb{F}$ of W. From the sheaf exact sequence $\widehat{A}_r \hookrightarrow \widehat{J}_{s,\mathbb{T}} \twoheadrightarrow \varpi(\widehat{J}_{s,\mathbb{T}})$, the corresponding sequence of their Néron models is exact by [NMD, Proposition 7.5.3 (a)]. Over finite field, by the vanishing of $H^1(\mathbb{F}, X)$ for an abelian variety $X_{/\mathbb{F}}$ (Lang's theorem [L56]), we find $\mathrm{Coker}(\widehat{J}_{s,\mathbb{T}}^{\mathrm{ord}}(\mathbb{F}) \to \varpi(\widehat{J}_{s,\mathbb{T}})^{\mathrm{ord}}(\mathbb{F})) = 0$; so, $E^s(K) = 0$. □

Lemma 9.3. *Suppose $l \neq p$, and let W be the l-adic integer ring of K. Then if A_r has good reduction (resp. additive, semi-stable) over W and $\mathbb{T}$ is integral, then $\widehat{J}_{s,\mathbb{T}}^{\mathrm{ord}} = J_{s,\mathbb{T}}[p^\infty]^{\mathrm{ord}}$ and $\varpi(\widehat{J}_{s,\mathbb{T}})^{\mathrm{ord}} = \varpi(J_{s,\mathbb{T}}[p^\infty]^{\mathrm{ord}})$ are contained in an abelian factor of J_s with good reduction (resp. additive, semi-stable) over W for all s.*

We indicated this fact in the proof of Proposition 9.2 saying that $\widehat{J}_{s,\mathbb{T}}^{\mathrm{ord}}$ and $\varpi(\widehat{J}_{s,\mathbb{T}})^{\mathrm{ord}}$ has good reduction over W.

Proof. As is well known (e.g., [H11, Remark after Conjecture 3.4], the l-type (i.e., the l-local representation π_l) of automorphic representation occurring in a given ordinary p-adic analytic family is independent of the member of the family. In other words, if l-type is a ramified principal series $\pi(\alpha, \beta)$ or a Steinberg representation $\sigma(|\cdot|_l\alpha, \alpha)$, $\alpha|_{\mathbb{Z}_l^\times}$ and $\beta|_{\mathbb{Z}_l}^\times$ are independent of members, and if π_l is super-cuspidal, the associate p-adic Galois representation restricted to the l-inertia group is independent of the members up to isomorphism. Then by the criterion of Néron-Ogg-Shafarevich, the reduction of any member A_P is independent of an arithmetic point P. □

10. Control of Λ-Selmer groups

We start with a lemma.

Lemma 10.1. *For a number field or an l-adic field K and $\mathcal{G} = J_\infty^{\mathrm{ord}}[p^\infty]$, the Pontryagin dual $\mathcal{G}(K)^\vee$ is a Λ-torsion module of finite type. For any arithmetic prime P, $\mathcal{G}(K)^\vee \otimes_{\mathbf{h}} \mathbf{h}/P^n$, $\mathcal{G}(K) \otimes_{\mathbf{h}} \mathbf{h}/P^n$ and $\mathcal{G}(K)[P^n]$ are all finite for any positive integer n.*

Proof. We give a detailed argument when K is a number field and briefly touch an l-adic field as the argument is essentially the same. Let $P \in Ar_{\mathbf{h}}$, and suppose K is a number field. Suppose that the Galois representation ρ_P associated with P contains an open subgroup G of $SL_2(\mathbb{Z}_p)$. Let L be the Pontryagin dual module of $\mathcal{G}(\overline{\mathbb{Q}})$. If the cusp form f_P associated to P has conductor divisible by

N, the localization L_P is free of rank 2 over the valuation ring $V = \mathbf{h}_P$ finite over Λ_P (e.g., [HMI, Proposition 3.78]). If not, by the theory of new form (e.g. [H86a, §3.3]), L_P is free of rank 2 over a local ring of the form $V[X_1, \dots, X_m]/(X_1^{e_1}, \dots, X_m^{e_m}) = \mathbf{h}_P$ with nilradical coming from old forms (e.g., [H13a, Corollary 1.2]). The contragredient $\widetilde{\rho}_P = {}^t\rho_P^{-1}$ of ρ_P is realized by L_P/PL_P. Then G is also contained in $\mathrm{Im}(\widetilde{\rho}_P)$, and $H_0(K, L_P/PL_P) \cong H(K, L_P)/PH_0(K, L_P)$ is a surjective image of $H_0(G, L_P/PL_P)$, which vanishes. Thus $H_0(K, L_P/PL_P) = 0$, which implies $H_0(K, L)_P = 0$ by Nakayama's lemma. In particular, $H_0(K, L)$ is a Λ-torsion module whose support is outside P.

If ρ_P does not contain an open subgroup of $SL_2(\mathbb{Z}_p)$, by Ribet [R85] (see also [GME, Theorem 4.3.18]), there exists an imaginary quadratic field M such that $\widetilde{\rho}_P = \mathrm{Ind}_M^{\mathbb{Q}} \varphi$ for an infinite order Hecke character φ of $\mathrm{Gal}(\overline{\mathbb{Q}}.M)$. Then it is easy to show that $H_0(K, L_P/PL_P) = 0$, and in the same way as above, we find $H_0(K, L)_P = 0$ and that $H_0(K, L) \otimes_{\mathbf{h}} \mathbf{h}/P^n$ is finite for all n. Thus for any arithmetic prime $P \in Ar_{\mathbf{h}}$, $H_0(K, L)_P = 0$ and hence $\mathcal{G}(K)^\vee = H_0(K, L)$ is Λ-torsion with support outside P. Thus in any case, $\mathcal{G}(K)^\vee \otimes_{\mathbf{h}} \mathbf{h}/P^n = H_0(K, L) \otimes_{\mathbf{h}} \mathbf{h}/P^n$ is finite for all n as $\mathbf{h}$ is a semi-local ring of dimension 2 finite torsion-free over Λ. The module $\mathcal{G}(K)[P^n]$ is just the dual of $H_0(K, L) \otimes_{\mathbf{h}} \mathbf{h}/P^n$ and hence is finite. Then $(\mathcal{G}(K) \otimes_{\mathbf{h}} \mathbf{h}/P^n)^\vee = H_0(K, L)[P^n]$, which is finite by the above fact that $H_0(K, L)$ is Λ-torsion with support outside P.

If K is l-adic, replacing K by its finite extension, we may assume that A_P has split semi-stable reduction. Write F for the residue field of P. Then either $\widetilde{\rho}_P(\mathrm{Frob})$ for a Frobenius element Frob of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/K)$ has infinite order without eigenvalue 1 or the space $V(\widetilde{\rho}_P)$ fits into a non-split extension $F \hookrightarrow V \twoheadrightarrow F(-1)$ for the Tate twist $F(-1)$ (by the degeneration theory of Mumford–Tate; cf., [DAV, Appendix]). Because of this description $H_0(K, L_P/PL_P) = 0$, and by the same argument above, the results follows. □

Since (ϖ) is supported by finitely many arithmetic primes, $E_{\mathrm{Sel}}(K)^\vee := (\mathcal{G}(K) \otimes_{\mathbf{h}} \mathbf{h}/(\varpi))^\vee \cong \mathcal{G}(K)^\vee[\varpi]$ is finite by the above lemma; so, we get

Corollary 10.2. *Assume* (A). *If K is a number field, then $E_{\mathrm{Sel}}(K)$ is finite.*

Let $\mathbb{T}$ be the local ring such that $\varpi \in \mathfrak{m}_{\mathbb{T}}$. We define $\Omega_{\mathbb{T}}$ to be the set of points $P \in \mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ such that

$$P \cap \Lambda \text{ contains } t^{p^s} - 1 \text{ for some } 0 \le s \in \mathbb{Z} \text{ and } P \text{ is principal as a prime ideal.} \tag{10.1}$$

So, $\Omega_{\mathbb{T}}$ is a subset of $\mathrm{Spec}(\mathbb{T}) \cap Ar_{\mathbf{h}}$ made of principal ideals. We see $E_{\mathrm{Sel}}(K)_{\mathbb{T}} = \mathrm{Coker}(\varpi : \mathcal{G}(K)_{\mathbb{T}} \to \mathcal{G}(K)_{\mathbb{T}})$, where $M_{\mathbb{T}} = M \otimes_{\mathbf{h}} \mathbb{T}$ for an $\mathbf{h}$-module M. Thus for the Galois representation $\rho_{\mathbb{T}}$ acting on $T\mathcal{G}_{\mathbb{T}} = \varprojlim_s T_p\mathcal{G}[\gamma^{p^s} - 1]$, if $\rho_{\mathbb{T}}$ modulo $\mathfrak{m}_{\mathbb{T}}$ is absolutely irreducible over $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, we conclude $E_{\mathrm{Sel}}(K) = 0$. Here $\overline{\rho}_{\mathbb{T}} = (\rho_{\mathbb{T}} \bmod \mathfrak{m}_{\mathbb{T}})$ is the semi-simple two dimensional representation whose trace is given by $\mathrm{Tr}(\rho_{\mathbb{T}}) \mod \mathfrak{m}_{\mathbb{T}}$. Indeed, the Galois module $\mathcal{G}[\mathfrak{m}_{\mathbb{T}}]$ has Jordan-Hölder sequence whose sub-quotients are all isomorphic to $\overline{\rho}_{\mathbb{T}}$; so, by Nakayama's lemma, $\mathcal{G}(K) = 0$. Write $\overline{\rho}_{\mathbb{T}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \begin{pmatrix} \overline{\nu}_p\overline{\psi} & * \\ 0 & \overline{\varphi} \end{pmatrix} \bmod \mathfrak{m}_{\mathbb{T}}$ with the nearly ordinary character $\overline{\varphi}$ (i.e., $\overline{\varphi}([p, \mathbb{Q}_p])$ is equal to the image modulo $\mathfrak{m}_{\mathbb{T}}$ of $U(p)$). Here $\overline{\nu}_p = \nu_p \mod p$. Then it is plain that $\mathcal{G}(K_v) = 0$ for all place $v|p$ of K if $\overline{\nu}_p\overline{\psi}$ and $\overline{\varphi}$ are both non-trivial over $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_v)$ for all $v|p$. We record this fact as

Corollary 10.3. *Let $p > 2$, K be a number field, and suppose one of the following two conditions:*

(1) *$\overline{\rho}_{\mathbb{T}}$ is irreducible over* $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$*;*
(2) *$\overline{\nu}_p\overline{\psi}$ and $\overline{\varphi}$ are both non-trivial over* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_v)$ *for all p-adic places $v|p$ of K.*

Then we have $E_{\mathrm{Sel}}(K) = 0$.

Recall $M_{\mathbb{T}} = M \otimes_{\mathbf{h}} \mathbb{T}$ for an $\mathbf{h}$-module M.

Theorem 10.4. *Let K be a number field and* $\mathrm{Spec}(\mathbb{T})$ *be the connected component such that $\varpi \in \mathfrak{m}_{\mathbb{T}}$. Suppose $p > 2$,* (A) *and* (F).

(1) *Assume one of the following two conditions*
  (e1) *$E^\infty(K_v)_{\mathbb{T}}$ is finite for all $v|p$,*
  (e2) *A_r does not have split multiplicative reduction modulo $\mathfrak{p}$ at all primes $\mathfrak{p}|p$ of K.*

Then the following sequence

$$0 \to \mathrm{Sel}_K(A_r^{\mathrm{ord}}) \to \mathrm{Sel}_K(J_{\infty,\mathbb{T}}^{\mathrm{ord}}) \xrightarrow{\varpi} \mathrm{Sel}_K(J_{\infty,\mathbb{T}}^{\mathrm{ord}}). \tag{10.2}$$

is exact up to finite error.

(2) *Assume one of the following two conditions:*

(E1) $E_{\mathrm{Sel}}(K)_{\mathbb{T}} = E^\infty(K_v)_{\mathbb{T}} = 0$ *for all* $v|Np$,

(E2) $\mathbb{T}$ *is an integral domain,* A_r *has good reduction at all* $v|Np$ *and* $|\varphi(\mathrm{Frob}_v) - 1|_p = 1$ *for all* $v|p$. *Here* Frob_v *is a Frobenius element in* $\mathrm{Gal}(\overline{K}_v/K_v)$ *acting trivially on* $K[\mu_{p^\infty}]$.

Then the sequence (10.2) *is exact, and if in addition* $C_{MW}(K_v)_{\mathbb{T}} = 0$ *for all* $v|p$, *we have* $\mathrm{Sel}_K(\varpi(J_{\infty,\mathbb{T}}^{\mathrm{ord}})) \cong \mathrm{Sel}_K(J_{\infty,\mathbb{T}}^{\mathrm{ord}})$.

By (8.11), we have an exact sequence: $0 \to \mathrm{Sel}_K(\varpi(J_\infty^{\mathrm{ord}})) \to \mathrm{Sel}_K(J_\infty^{\mathrm{ord}}) \to \prod_{v|p} C_{MW}(K_v)$, and if A_P has potentially good reduction at p for all $P \in \Omega_{\mathbb{T}}$, $\prod_{v|p} E^\infty(K_v)$ vanishes under (E2) (see Theorem 17.2). By our choice of $\mathbb{T}$, we have $\widehat{A}_{r,\mathbb{T}}^{\mathrm{ord}} = \widehat{A}_r^{\mathrm{ord}}$ and $\widehat{B}_{r,\mathbb{T}}^{\mathrm{ord}} = \widehat{B}_r^{\mathrm{ord}}$. By Corollary 10.3, (E2) implies $E_{\mathrm{Sel}}(K) = 0$. Indeed, $\overline{\nu}_p\overline{\psi}\overline{\varphi}([p, \mathbb{Q}_p]) = 1$ as A_r has good reduction modulo p. We prove the theorem under (E1) or (e1), since Theorem 17.2 combined with Proposition 9.2 (4) shows (e2)⇒(e1) and (E2)⇒(E1) for $v|p$.

Proof. Recall the following commutative diagram with two bottom exact rows and three right exact columns from (8.13) (tensored with $\mathbb{T}$ over $\mathbf{h}$):

$$\begin{array}{ccccccc} \mathrm{Ker}(\iota_{\mathrm{Sel},*}) & \xrightarrow{i_0} & \mathrm{Sel}_K(\widehat{A}_r^{\mathrm{ord}}) & \xrightarrow{\iota_{\mathrm{Sel},*}} & \mathrm{Sel}_K(J_{\infty,\mathbb{T}}^{\mathrm{ord}}) & \xrightarrow{\varpi_{\mathrm{Sel},*}} & \mathrm{Sel}_K(\varpi(J_{\infty,\mathbb{T}}^{\mathrm{ord}})) \\ {\scriptstyle i}\downarrow & & \cap\downarrow{\scriptstyle a} & & \cap\downarrow & & \cap\downarrow \\ E_{\mathrm{Sel}}(K) & \hookrightarrow & H^1(\widehat{A}_r^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\iota_*} & H^1(J_{\infty,\mathbb{T}}^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\varpi_*} & H^1(J_{\infty,\mathbb{T}}^{\mathrm{ord}}[p^\infty]) \\ {\scriptstyle e}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_S^\infty(K) & \underset{e_0}{\hookrightarrow} & H_S^1(\widehat{A}_r^{\mathrm{ord}}) & \xrightarrow{\iota_{S,*}} & H_S^1(J_{\infty,\mathbb{T}}^{\mathrm{ord}}) & \xrightarrow{\varpi_{S,*}} \!\!\!\! \twoheadrightarrow & H_S^1(\varpi(J_{\infty,\mathbb{T}}^{\mathrm{ord}})). \end{array} \tag{10.3}$$

By Proposition 9.2 and by the assumption (e1), $E_S^\infty(K)$ is finite. Since the middle two columns are exact, the left column is exact with injection i (e.g., [BCM, I.1.4.2 (1)]). Since the bottom row is exact with injection e_0, the map i_0 is injective and $\mathrm{Im}(i_0) = \mathrm{Ker}(\iota_{\mathrm{Sel},*})$. Suppose (E1). Then all the terms of the left column vanish. So $\mathrm{Ker}(\iota_{\mathrm{Sel},*}) = 0$ and the sequence:

$$0 \to \mathrm{Sel}_K(\widehat{A}_r^{\mathrm{ord}}) \to \mathrm{Sel}_K(J_\infty^{\mathrm{ord}}) \to \mathrm{Sel}_K(\varpi(J_\infty^{\mathrm{ord}})) \underset{\hookrightarrow}{\xrightarrow{(8.11)}} \mathrm{Sel}_K(J_\infty^{\mathrm{ord}}) \tag{10.4}$$

is exact. The cokernel $\mathrm{Coker}(\mathrm{Sel}_K(J_\infty^{\mathrm{ord}}) \xrightarrow{\varpi} \mathrm{Sel}_K(\varpi(J_\infty^{\mathrm{ord}}))$ is global in nature and seems difficult to determine, although $\mathrm{Coker}(\mathrm{Sel}_K(\varpi(J_\infty^{\mathrm{ord}})) \to \mathrm{Sel}_K(J_\infty^{\mathrm{ord}}))$ is local as in (8.11), and if $C_{MW}(K_v) = 0$ for all $v|p$, it vanishes.

Now we assume (e1). We need to prove the sequence (10.4) is exact up to finite error. By Corollary 10.2, $E_{\mathrm{Sel}}(K)$ is finite. Since we know $E^\infty(K_v) = 0$ for v prime to p by Proposition 9.2, we conclude from (e1) that $E_S^\infty(K)$ is finite. Then the diagram (8.13) has two bottom rows exact up to finite error. Since the Pontryagin dual of all the modules in the above diagram are Λ-modules of finite type, we can work with the category of Λ-modules of finite type up to finite error (e.g., [BCM, VII.4.5]). Then in this new category, the bottom two rows are exact and the extreme left terms a pseudo-null. Thus the dual sequence of the theorem is exact up to finite error, and by taking dual back, the sequence in the theorem is exact up to finite error. □

Corollary 10.5. *Assume* (F) *and* $p > 2$. *Then we have*

(1) *The Pontryagin dual* $\mathrm{Sel}_K(J_\infty^{\mathrm{ord}})^\vee$ *of* $\mathrm{Sel}_K(J_\infty^{\mathrm{ord}})$ *is a* Λ*-module of finite type.*

(2) *If further* $\mathrm{Sel}_K(\widehat{A}_r^{\mathrm{ord}}) = 0$ *for a single element* $\varpi \in \mathfrak{m}_{\mathbb{T}}$ *satisfying* (A) *and* (E1), *then* $\mathrm{Sel}_K(J_\infty^{\mathrm{ord}}) = 0$ *and* $\mathrm{Sel}_K(\widehat{A}_r^{\mathrm{ord}}) = 0$ *for every* $\varpi \in \mathfrak{m}_{\mathbb{T}}$ *satisfying* (A) *and* (E1).

(3) *Suppose that $\mathbb{T}$ is an integral domain. If $\mathrm{Sel}_K(\widehat{A}_r^{\mathrm{ord}})$ is finite for some ϖ satisfying* (A) *and* (e1)*, then $\mathrm{Sel}_K(J_\infty^{\mathrm{ord}})^\vee$ is a torsion $\mathbb{T}$-module of finite type. Thus if $\mathbb{T}$ is a unique factorization domain, for almost all $P \in \Omega_{\mathbb{T}}$, $\mathrm{Sel}_K(\widehat{A}_P^{\mathrm{ord}})$ is finite.*

Proof. The condition (A) and (e2) is satisfied by any non-trivial factor ϖ of $(\gamma^{p^r}-1)/(\gamma-1)$. Thus $\mathrm{Sel}_K(J_\infty^{\mathrm{ord}})^\vee/\varpi\cdot\mathrm{Sel}_K(J_\infty^{\mathrm{ord}})^\vee$ pseudo isomorphic to $\mathrm{Sel}_K(\widehat{A}_r^{\mathrm{ord}})^\vee$ which is $\mathbb{Z}_p$-module of finite type; so, by the topological Nakayama's lemma, we conclude that $\mathrm{Sel}_K(J_\infty^{\mathrm{ord}})^\vee$ is a Λ-module of finite type. The last two assertions can be proven similarly. □

11. Control of Λ-BT groups and its cohomology

Recall $\mathcal{G} := \mathcal{G}_{\alpha,\delta,\xi} = J_\infty^{\mathrm{ord}}[p^\infty]$ which is a Λ-BT group in the sense of [H14]. Here the set S is supposed to be finite. We study the control of the Tate–Shafarevich group of $\mathcal{G}$.

Theorem 11.1. *Let K be a number field. Suppose $|S|<\infty$,* (F) *and* (A) *for ϖ. Then the sequence $0 \to Ш(K^S/K,\widehat{A}_r^{\mathrm{ord}}[p^\infty]) \to Ш(K^S/K,\mathcal{G}) \xrightarrow{\varpi} Ш(K^S/K,\mathcal{G})$ is exact up to finite error.*

Proof. From the exact sequence $0\to\widehat{A}_r^{\mathrm{ord}}[p^\infty]\to\mathcal{G}\xrightarrow{\varpi}\mathcal{G}\to 0$ of Corollary 6.4, we get a commutative diagram with exact bottom two rows and exact columns:

$$\begin{array}{ccccccc}
\mathrm{Ker}(\iota_{Ш,*}) & \longrightarrow & Ш(K^S/K,\widehat{A}_r^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\iota_{Ш,*}} & Ш(K^S/K,\mathcal{G}) & \xrightarrow{\varpi_{Ш,*}} & Ш(K^S/K,\mathcal{G}) \\
\cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\
E_{BT}^\infty(K) & \xrightarrow{\hookrightarrow} & H^1(\widehat{A}_r^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\iota_*} & H^1(\mathcal{G}) & \xrightarrow{\varpi_*} & H^1(\mathcal{G}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_{v\in S}E_{BT}^\infty(K_v) & \xrightarrow{\hookrightarrow} & H^1_S(\widehat{A}_r^{\mathrm{ord}}[p^\infty]) & \xrightarrow{\iota_{S,*}} & H^1_S(\mathcal{G}) & \xrightarrow{\varpi_{S,*}} & H^1_S(\mathcal{G}),
\end{array} \tag{11.1}$$

where $E_{BT}^\infty(k)=E_{\mathrm{Sel}}(k)=\mathrm{Coker}(\varpi:\mathcal{G}(k)\to\mathcal{G}(k))$.

By Lemma 10.1, $E_{BT}^\infty(K)$ and $E_{BT}^\infty(K_v)$ are finite. Thus as long as S is finite, $\prod_{v\in S}E_{BT}^\infty(K_v)$ is finite. Then the above diagram proves the desired exactness. □

Corollary 11.2. *Let the notation and the assumption be as in the theorem. Assume that $\mathbb{T}$ is an integral domain and that $Ш(K^S/K,\widehat{A}_{P_0}^{\mathrm{ord}}[p^\infty])$ is finite for a principal arithmetic prime $P_0\in\Omega_{\mathbb{T}}$. Then $Ш(K^S/K,\mathcal{G}_{\mathbb{T}})^\vee$ is a torsion $\mathbb{T}$-module of finite type. In particular, $Z_g(\overline{\mathbb{Q}}_p)$ is finite for the support $Z_g\subset\mathrm{Spec}(\mathbb{T})$ of $Ш(K^S/K,\mathcal{G}_{\mathbb{T}})^\vee$, and for all principal arithmetic points $P\in\mathrm{Spec}(\mathbb{T})-Z_g$, $Ш(K^S/K,\widehat{A}_P^{\mathrm{ord}}[p^\infty])$ is finite.*

12. The second error term C_{MW}.

Put $E_{MW}(k) := \mathrm{Coker}(\varpi(J_{\infty,\mathbb{T}}^{\mathrm{ord}})(k)\hookrightarrow J_{\infty,\mathbb{T}}^{\mathrm{ord}}(k))$, and recall $C_{MW}(k)=\mathrm{Coker}(J_{\infty,\mathbb{T}}^{\mathrm{ord}}(k)\xrightarrow{\rho_\infty}\widehat{B}^{\mathrm{ord}}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$. If $P\in\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ is an arithmetic point generated by $\varpi\in\mathbb{T}$, we write $E_{MW}^P(k)$ (resp. $C_{MW}^P(k)$) for $E_{MW}(k)$ (resp. $C_{MW}(k)$).

Proposition 12.1. *Let k be a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_l$. Then $C_{MW}^P(k)\xrightarrow{\sim}E_{MW}^P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$ canonically. In particular, if $E_{MW}^P(k)$ is compact (e.g., $\dim_{\mathbb{T}/P\otimes_{\mathbb{Z}_p}\mathbb{Q}_p}\widehat{B}_P^{\mathrm{ord}}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p=1$ and $C_{MW}^P(k)\neq 0$), $C_{MW}^P(k)$ is isogenous to $\widehat{A}_P^{\mathrm{ord}}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$. If $\mathbb{T}$ is an integral domain, then the error term $C_{MW}^P(k)\cong E_{MW}^P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$ vanishes for almost all $P\in\Omega_{\mathbb{T}}$, and more precisely, $C_{MW}^P(k)\cong E_{MW}^P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p=0$ if and only if $P\in\Omega_{\mathbb{T}}$ is outside the support of the maximal $\mathbb{T}$-torsion submodule of J.*

Proof. Since $C_{MW}^P(k)$ is p-divisible torsion, we have $C_{MW}^P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p=0$ and $\mathrm{Tor}_1^{\mathbb{Z}_p}(C_{MW}^P(k),\mathbb{Q}_p/\mathbb{Z}_p)=C_{MW}^P(k)$ (with $\mathrm{Tor}_1^{\mathbb{Z}_p}(\widehat{B}_P^{\mathrm{ord}}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p,\mathbb{Q}_p/\mathbb{Z}_p)=0$). To see $E_{MW}^P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p\cong C_{MW}^P(k)$, we look into the exact sequence $E^P_{MW}(k) \hookrightarrow \widehat{B}^{\mathrm{ord}}_P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \twoheadrightarrow C^P_{MW}(k)$. Tensoring $\mathbb{Q}_p/\mathbb{Z}_p$ with the sequence, we get the following exact sequence:

$$0 \to C^P_{MW}(k) = \mathrm{Tor}_1^{\mathbb{Z}_p}(C^P_{MW}(k), \mathbb{Q}_p/\mathbb{Z}_p) \to E^P_{MW}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

This shows the first assertion. If $E^P_{MW}(k)$ is compact, by $E^P_{MW}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \cong C^P_{MW}(k)$, we have $\mathrm{rank}_{\mathbb{Z}_p} E^P_{MW}(k) = \mathrm{corank}\, C^P_{MW}(k) = \dim_{\mathbb{Q}_p} \widehat{B}^{\mathrm{ord}}_P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$; so, $C^P_{MW}(k)$ is isogenous to $\widehat{B}^{\mathrm{ord}}_P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$ (and hence to $\widehat{B}^{\mathrm{ord}}_P(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$).

We now assume that $E^P_{MW}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \cong C^P_{MW}(k) \neq 0$. Then we have the following short exact sequence:

$$0 \to \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}})(k) \to J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k) \xrightarrow{\rho_{\infty,\mathbb{T}}} \mathrm{Im}(\rho_{\infty,\mathbb{T}})(\cong E^P_{MW}(k)) \to 0$$

with $\mathbb{Z}_p$-free $\mathrm{Im}(\rho_{\infty,\mathbb{T}})$. Then by [BCM, I.2.5], we still have a short exact sequence:

$$0 \to \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}})(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \to J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\rho_{\infty,\mathbb{T}}} \mathrm{Im}(\rho_{\infty,\mathbb{T}})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Note that $(\mathrm{Im}(\rho_{\infty,\mathbb{T}})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a P-torsion $\mathbb{T}$-module of finite type. Thus P belongs to the support $Z_k \subset \mathrm{Spec}(\mathbb{T})$ of the maximal $\mathbb{T}$-torsion submodule of $(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)^\vee$. Since $(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a $\mathbb{T}$-module of finite type, $Z_k(\overline{\mathbb{Q}}_p)$ is a finite set. From our proof, we conclude $E^P_{MW}(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \neq 0 \Leftrightarrow P \in Z_k(\overline{\mathbb{Q}}_p)$. □

We state a proposition showing that the limit Mordel–Weil group is of co-finite type over Λ:

Proposition 12.2. *Assume* (F). *Let* $P = (\varpi) \in \Omega_{\mathbb{T}}$. *Let* $p > 2$ *and* k *be either a number field or an* l*-adic field. The following sequence*

$$0 \to \widehat{A}^{\mathrm{ord}}_P(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\iota} J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\pi} C^P_{MW}(k) \to 0 \tag{12.1}$$

is exact up to finite error except for $\mathrm{Ker}(\varpi)/\mathrm{Im}(\iota)$ *which is a image of* $E^\infty_P(k)$ *with finite kernel (so, it is at worst a* Λ*-torsion module of finite type). In particular, if* $\mathbb{T}$ *is an integral domain, the Pontryagin dual* $J := (J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p)^\vee$ *is a* $\mathbb{T}$*-module of finite type.*

Proof. Since $\varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)) = J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)/\widehat{A}^{\mathrm{ord}}_P(k)$, we have the following three exact sequences:

$$\begin{aligned}
&0 \to \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)) \to \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}})(k) \to E^\infty_P(k) \to 0,\\
&0 \to \widehat{A}^{\mathrm{ord}}_P(k) \to J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k) \to \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)) \to 0,\\
&0 \to \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}})(k) \to J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k) \to E^P_{MW}(k) \to 0.
\end{aligned} \tag{12.2}$$

Tensoring with $\mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbb{Z}_p$, by $\mathrm{Tor}_1^{\mathbb{Z}_p}(X, \mathbb{Q}_p/\mathbb{Z}_p) = X[p^\infty]$, we get the following exact sequences

$$\begin{aligned}
&\varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}})[p^\infty](k) \to E^\infty_P(k) \to \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k))\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}})(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \to 0,\\
&\varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k))[p^\infty] \xrightarrow{a} \widehat{A}^{\mathrm{ord}}_P(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \to J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k))\otimes\mathbb{Q}_p/\mathbb{Z}_p \to 0,\\
&0 \to \varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}})(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \to J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \to C^P_{MW}(k) \to 0,
\end{aligned} \tag{12.3}$$

where we replaced $E^P_{MW}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p$ by $C^P_{MW}(k)$ at the end of the last sequence (using Proposition 12.1). The last sequence is exact since, by Corollary 6.3, $E^P_{MW}(k)$ is a flat $\mathbb{Z}_p$-module [BCM, I.2.5]. The image $\mathrm{Im}((\varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)))[p^\infty] \xrightarrow{a} \widehat{A}^{\mathrm{ord}}_P(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p)$ is killed by ϖ. Since (ϖ) is arithmetic, by Lemma 10.1, this image is finite (i.e., factoring through the k-rational quotient of $\mathcal{G}_{\mathbb{T}}(k) = (\varpi(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)))[p^\infty]$ killed by ϖ). Thus the sequence:

$$0 \to \widehat{A}^{\mathrm{ord}}_P(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\iota} J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\pi} C^P_{MW}(k) \to 0$$

ι has finite kernel, $\mathrm{Ker}(\varpi)/\mathrm{Im}(\iota)$ is the image of $E^\infty_P(k)$ with finite kernel (by the first sequence combined with the second of (12.3)) and the last three right terms are exact.

Suppose that $\mathbb{T}$ is an integral domain. By Lemma 9.1, $E^\infty_P(k)$ has p-torsion with finite corank over $\mathbb{Z}_p$. Therefore $\mathrm{Ker}(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k) \xrightarrow{\varpi} J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k))^\vee$ is a $\mathbb{T}$-torsion module of finite type. Then by Nakayama's lemma, the Pontryagin dual $(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a $\mathbb{T}$-module of finite type. □

The exceptional finite subset in $P \in \Omega_{\mathbb{T}}$ for the control of the limit Mordell–Weil group $J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is contained in the union of the following three type of proper closed subschemes of $\mathrm{Spec}(\mathbb{T})$ whose $\overline{\mathbb{Q}}_p$-points are finite:

(a) the support $Z_k \subset \mathrm{Spec}(\mathbb{T})$ of the maximal $\mathbb{T}$-torsion submodule of $J^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)^\vee$ whose $\overline{\mathbb{Q}}_p$-points are finite by Proposition 12.1 (this set depends on the field k);
(b) the support $Z_g \subset \mathrm{Spec}(\mathbb{T})$ of the maximal $\mathbb{T}$-torsion submodule of $Ш(k^S/k, \mathcal{G}_{\mathbb{T}})^\vee$ whose $\overline{\mathbb{Q}}_p$-points are finite by Theorem 11.1 (this applies to a number field k);
(c) $Z_p \subset \mathrm{Spec}(\mathbb{T})$ made of $P \in \Omega_{\mathbb{T}}$ not satisfying (e2) (this applies to the case where k is either a number field or a p-adic field).

Corollary 12.3. *Suppose that $\mathbb{T}$ is an integral domain with infinite $\Omega_{\mathbb{T}}$. Let $P \in \omega_{\mathbb{T}} - Z_p(\overline{\mathbb{Q}}_p)$. Let K be a number field. Suppose either that $Ш_K(\widehat{A}^{\mathrm{ord}}_P)$ is finite or that the sequence*

$$0 \to \widehat{A}^{\mathrm{ord}}_P(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\iota} J^{\mathrm{ord}}_{\infty,\mathbb{T}}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi} J^{\mathrm{ord}}_{\infty,\mathbb{T}}(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

is exact up to finite error. Then $E^\infty_P(K)$ is finite.

Proof. First suppose $|Ш_K(\widehat{A}^{\mathrm{ord}}_P)| < \infty$. We look into the diagram (8.14) for $P = (\varpi)$. Since $P \notin Z_p$, by Proposition 9.2 and Theorem 17.2, $E^\infty_S(K)$ is finite. Since the left column and the first three term of the first row are exact in (8.14), from finiteness of $Ш_K(\widehat{A}^{\mathrm{ord}}_P)$, we conclude $E^\infty_P(K)$ is finite.

If the sequence in the corollary is exact up to finite error, since $E^\infty_P(K)$ is isogenous to $\mathrm{Ker}(\varpi)/\mathrm{Im}(\iota)$ by Proposition 12.2, it is finite. □

13. Control of limit Tate–Shafarevich groups and Mordell–Weil groups

We first study a relation of the Tate–Shafarevich group $Ш_K(\widehat{A}^{\mathrm{ord}}_P)$ and $Ш(K^S/K, \widehat{A}^{\mathrm{ord}}_P[p^\infty])$.

Proposition 13.1. *Suppose that $\mathbb{T}$ is an integral domain flat over Λ. Let K be a number field and pick an arithmetic point $P \in \mathrm{Spec}(\mathbb{T})$. Assume $|S| < \infty$ and that P is principal with $P = (\varpi)$. Let Ker^{MW}_P be the kernel of the natural diagonal map: $\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \prod_{v|p} \widehat{A}^{\mathrm{ord}}_P(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Then we have the following exact sequence*

$$0 \to \mathrm{Ker}^{MW}_P \to Ш(K^S/K, \widehat{A}^{\mathrm{ord}}_P[p^\infty]) \xrightarrow{\Pi_P} Ш_K(\widehat{A}^{\mathrm{ord}}_P).$$

Moreover Π_P is onto if $K = \mathbb{Q}$ and $\dim_{H_P} A_P(\mathbb{Q}) \geq 1$. Thus assuming $K = \mathbb{Q}$ and $\dim_{H_P} A_P(\mathbb{Q}) \geq 1$ and $|Ш(\mathbb{Q}^S/\mathbb{Q}, \widehat{A}^{\mathrm{ord}}_P[p^\infty])| < \infty$, we have $Ш_{\mathbb{Q}}(\widehat{A}^{\mathrm{ord}}_P)$ is finite and $\dim_{H_P} A_P(\mathbb{Q}) = 1$. For general K, if $Ш(K^S/K, \widehat{A}^{\mathrm{ord}}_P[p^\infty])$ vanishes (resp. is finite), the error term Ker^{MW}_P vanishes (resp. is finite). Similarly if $|Ш_K(\widehat{A}^{\mathrm{ord}}_P)| < \infty$ and $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$, the two modules Ker^{MW}_P and $Ш(K^S/K, \widehat{A}^{\mathrm{ord}}_P[p^\infty])$ are finite.

Proof. For $K' = K^S$ and $\overline{K}_v$, $\widehat{A}_r(K')$ (and hence $\widehat{A}^{\mathrm{ord}}_r(K')$) is p-divisible $\mathbb{Z}_p$-modules; so, the $\mathbb{Z}_p$-module $\widehat{A}^{\mathrm{ord}}_r(K')/\widehat{A}^{\mathrm{ord}}_r[p^\infty](K')$ is a $\mathbb{Q}_p$-vector space (i.e., it is isomorphic to $\widehat{A}^{\mathrm{ord}}_r(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$). From the short exact sequence $\widehat{A}^{\mathrm{ord}}_r[p^\infty](K') \hookrightarrow \widehat{A}^{\mathrm{ord}}_r(K') \twoheadrightarrow \widehat{A}^{\mathrm{ord}}_r(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ $(K' = K^S, \overline{K}_v)$ of Galois modules, we get the following commutative diagram with the bottom two exact rows:

$$\begin{array}{ccccccc}
\mathrm{Ker}^{MW}_P & \hookrightarrow & Ш(K^S/K, \widehat{A}^{\mathrm{ord}}_r[p^\infty]) & \xrightarrow{\iota_{Ш,*}} & Ш_K(\widehat{A}^{\mathrm{ord}}_r) & & \\
\downarrow & & \cap\downarrow & & \cap\downarrow & & \\
\widehat{A}^{\mathrm{ord}}_r(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow[I]{\hookrightarrow} & H^1(\widehat{A}^{\mathrm{ord}}_r[p^\infty]) & \xrightarrow{\iota_*} & H^1(\widehat{A}^{\mathrm{ord}}_r) & \longrightarrow & 0 \\
\delta\downarrow & & \mathrm{Res}[p^\infty]\downarrow & & \mathrm{Res}\downarrow & & \\
\prod_{v|p}(\widehat{A}^{\mathrm{ord}}_r(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) & \xrightarrow[I_S]{\hookrightarrow} & H^1_S(\widehat{A}^{\mathrm{ord}}_r[p^\infty]) & \xrightarrow{\iota_{S,*}} & H^1_S(\widehat{A}^{\mathrm{ord}}_r) & \longrightarrow & 0.
\end{array} \tag{13.1}$$

The injectivity of I_S and exactness of the bottom row prove the exact sequence in the proposition by [BCM, I.1.4.2 (1)].

Assume that $K = \mathbb{Q}$. By Lemma 5.5, $\widehat{A}_r^{\mathrm{ord}}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as $\mathbb{T}/P$-modules, and hence $\widehat{A}_r^{\mathrm{ord}}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ up to finite error. If $\dim_{H_P} A_r(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} = m \geq 1$, by Lemmas 5.4 and 5.5, we find that $\widehat{A}_r^{\mathrm{ord}}(\mathbb{Q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \cong (\mathbb{T}/P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^m$. Since the source and the target of the map δ are p-divisible, and from the finiteness of $Ш(\mathbb{Q}^S/\mathbb{Q}, \widehat{A}_r^{\mathrm{ord}}[p^\infty])$, δ is not a zero map, which implies surjectivity of δ as the corank of the target is equal to or less than m. Therefore by snake lemma, if $K = \mathbb{Q}$ and $\dim_{H_P} A_r(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} = m \geq 1$, we have an exact sequence:

$$0 \to \mathrm{Ker}_P^{MW} \to Ш(\mathbb{Q}^S/\mathbb{Q}, \widehat{A}_P^{\mathrm{ord}}[p^\infty]) \xrightarrow{\Pi_P} Ш_{\mathbb{Q}}(\widehat{A}_P^{\mathrm{ord}}) \to 0.$$

Thus we conclude $m = 1$ and $Ш_{\mathbb{Q}}(\widehat{A}_P^{\mathrm{ord}})$ is finite if $Ш(\mathbb{Q}^S/\mathbb{Q}, \widehat{A}_P^{\mathrm{ord}}[p^\infty])$ is finite.

If $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$, then $A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{i} \prod_{v|p} A_P(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has non-trivial image which is an H_P-vector space. This it has dimension 1 over H_P, and hence i is an injection. Thus $\delta_0 : \widehat{A}_r^{\mathrm{ord}}(K) \to \prod_{v|p} \widehat{A}_r^{\mathrm{ord}}(K_v)$ is a morphism of $\mathbb{Z}_p$-module of finite type with finite kernel. Then it is clear after tensoring $\mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbb{Z}_p$, δ has finite kernel; i.e., Ker_P^{MW} is finite, and hence finiteness of $Ш_K(\widehat{A}_P^{\mathrm{ord}})$ implies that of $Ш(K^S/K, \widehat{A}_P^{\mathrm{ord}}[p^\infty])$. □

Recall the troublesome exceptional subsets of $\mathrm{Spec}(\mathbb{T})$:

(a) the support $Z_k \subset \mathrm{Spec}(\mathbb{T})$ of the maximal $\mathbb{T}$-torsion submodule of $J_{\infty,\mathbb{T}}^{\mathrm{ord}}(k)^\vee$ whose $\overline{\mathbb{Q}}_p$-points are finite by Proposition 12.1 (this set depends on the field k);

(b) the support $Z_g \subset \mathrm{Spec}(\mathbb{T})$ of the maximal $\mathbb{T}$-torsion submodule of $Ш(k^S/k, \mathcal{G}_{\mathbb{T}})^\vee$ whose $\overline{\mathbb{Q}}_p$-points are finite by Theorem 11.1 (for a number field k);

(c) $Z_p \subset \mathrm{Spec}(\mathbb{T})$ made of $P \in \Omega_{\mathbb{T}}$ not satisfying (e2) (so, $r(P) \leq 1$ if $P \in Z_p(\overline{\mathbb{Q}}_p) \cap \Omega_{\mathbb{T}}$).

We now consider the following conditions:

(P0) There exists $P_0 \in \Omega_{\mathbb{T}} - (Z_g \cup Z_p)(\overline{\mathbb{Q}}_p)$ (so we have finite $E_S^\infty(K) := \prod_{v|p} E_{P_0}^\infty(K_v)$) such that $\dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$ and $Ш_K(\widehat{A}_{P_0}^{\mathrm{ord}})$ is finite,

(P1) There exists $P_0 \in \Omega_{\mathbb{T}} - (Z_g \cup Z_p)(\overline{\mathbb{Q}}_p)$ such that $\dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1$ and $Ш_K(\widehat{A}_{P_0}^{\mathrm{ord}})$ is finite.

We have the following implication (P0) $\Rightarrow P_0 \in \Omega_{\mathbb{T}} - (Z_p \cup Z_p)(\overline{\mathbb{Q}}_p)$ by Proposition 13.1. Note that finiteness of $E_S^\infty(K)$ follows if $A + P_0$ does not have split multiplicative reduction over $\mathbb{Z}_p[\mu_{p^\infty}]$ (so, in particular, A_{P_0} has potential good reduction at p; see Theorem 17.2).

Lemma 13.2. *Let K be a number field and $\mathbb{T}$ be a normal integral domain with infinite $\Omega_{\mathbb{T}}$. Suppose* (P1). *Then*

(1) $\mathrm{rank}_{\mathbb{T}}(J_\infty^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \leq 1$,

(2) *If* $\dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, *then* $\mathrm{rank}_{\mathbb{T}}(J_\infty^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee = 0$,

(3) *If* $\mathrm{rank}_{\mathbb{T}}(J_\infty^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee = \dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$, *then P_0 is outside $Z_K(\overline{\mathbb{Q}}_p)$.*

Write $\mathcal{J} := J_{\infty,\mathbb{T}}^{\mathrm{ord}}$ as a sheaf and put $J = \mathcal{J}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$.

Proof. Suppose $\dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ as in (2). Then the assertion (2) follows from Corollary 10.5 (3) as $|\mathrm{Sel}_K(\widehat{A}_{P_0}^{\mathrm{ord}})| < \infty$ under (P1).

By the finiteness of $Ш_K(\widehat{A}_{P_0}^{\mathrm{ord}})$ and $E_S^\infty(K)$, $E_P^\infty(K)$ is finite by Lemma 8.2. Then by Proposition 12.2, for a generator ϖ_0 of P_0,

$$J^\vee \xrightarrow{\varpi_0} J^\vee \to (\widehat{A}_{P_0}^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \to 0$$

is exact up to finite error. Localizing at P_0, we have the following exact sequence:

$$J_{P_0}^\vee \xrightarrow{\varpi} J_{P_0}^\vee \to (\widehat{A}_P^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$$

Since $\dim_{\mathbb{T}/P \otimes \mathbb{Q}_p} \widehat{A}_P^{\mathrm{ord}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq 1$ by Lemma 5.4, by Nakayama's lemma,

$$\mathrm{rank}_{\mathbb{T}} J^\vee = \mathrm{rank}_{\mathbb{T}_{P_0}} J_{P_0}^\vee \leq 1$$

proving (1).

We now prove the last assertion; so, we assume $K = \mathbb{Q}$. Since $Ш_K(A^{\mathrm{ord}}_{P_0})$ is finite, $E^{\infty}_{P_0}(K)$ is finite. Thus $J[\varpi_0]$ is isogenous to $\widehat{A}^{\mathrm{ord}}_{P_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ by Proposition 12.2. Pick a pseudo isomorphism $J^\vee \cong \mathbb{T} \oplus X$ for a torsion $\mathbb{T}$-module X of finite type. Then for a generator ϖ_0 of P_0, we have an exact sequence up to finite error:

$$0 \to X[P_0] \to J^\vee \xrightarrow{\varpi_0} J^\vee \to \mathbb{T}/P_0 \oplus X/P_0X \to 0.$$

Since $J^\vee/\varpi_0 J^\vee = J[\varpi_0]^\vee \cong (\widehat{A}^{\mathrm{ord}}_{P_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \mathbb{T}/P_0$ up to finite error by Lemma 5.4, X/P_0X has to be finite. Thus $P_0 \notin \mathrm{Supp}(X)(\overline{\mathbb{Q}}_p) = Z_K(\overline{\mathbb{Q}}_p)$. □

We now assume (P1) (as we know definitely that the finiteness of $Ш_K(\widehat{A}^{\mathrm{ord}}_P)$ for almost all $P \in \omega_{\mathbb{T}}$ if $\dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ by Corollary 10.5. Under (P1), we have two possibilities by Lemma 13.2; i.e.,

Case 0: $\mathrm{rank}_{\mathbb{T}}(J^{\mathrm{ord}}_\infty(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee = 0$ and
Case 1: $\mathrm{rank}_{\mathbb{T}}(J^{\mathrm{ord}}_\infty(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee = 1$.

Lemma 13.3. *Let $\mathcal{J} := J^{\mathrm{ord}}_{\infty,\mathbb{T}}$, and write $\omega_{\mathbb{T}} = \Omega_{\mathbb{T}}/\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subset \mathrm{Spec}(\mathbb{T})$ (the Galois conjugacy classes of $\Omega_{\mathbb{T}}$). Suppose that $\mathbb{T}$ is a unique factorization domain. Then we have*

$$\mathcal{J}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \bigoplus_{P \in \omega_{\mathbb{T}}} \widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

as $\mathbb{T}$-modules.

Proof. Then $\mathcal{J}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \varinjlim_s \mathcal{J}_s(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for $\mathcal{J}_s := \widehat{J}^{\mathrm{ord}}_{s,\mathbb{T}}$. Since for $P \in \Omega_{\mathbb{T}}$, $\widehat{A}^{\mathrm{ord}}_P$ only depends on $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-orbit of P, we have $\mathcal{J}_s(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \bigoplus_{P \in \omega_{\mathbb{T}} \cap \mathrm{Spec}(\mathbb{T}_s)} \widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, since $J^{\mathrm{ord}}_{s,\mathbb{T}}(K) \subset \sum_{P \in \mathrm{Spec}(\mathbb{T}_s) \cap \omega_{\mathbb{T}}} A_P$. Passing to the limit, we obtain the desired assertion. □

Theorem 13.4. *Let K be a number field, and put $Z_{\mathbb{T},K} := (Z_p \cup Z_g \cup Z_K)(\overline{\mathbb{Q}}_p)$. Suppose* (P1) *and that $\mathbb{T}$ is a unique factorization domain. Then we have*

(0) $\mathrm{rank}_{\mathbb{T}}(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \le \dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \le 1$.
(1) *If $A_{P_0}(K)$ is finite (i.e., $P_0 \notin Z_K(\overline{\mathbb{Q}}_p)$), then* $\mathrm{rank}_{\mathbb{T}}(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee = 0$ *and* $Ш_K(\widehat{A}^{\mathrm{ord}}_P)$ *is finite for almost all $P \in \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}$; in this case, we put $Ct_{\mathbb{T}} := \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}$.*
(2) *If $A_{P_0}(K)$ is infinite (i.e.,* (P0) *holds) and* $\mathrm{rank}_{\mathbb{T}}(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee = 0$*, we have $P_0 \in Z_{\mathbb{T},K}$, and assuming existence of a point $P_1 \in \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}$ with finite* $Ш_K(\widehat{A}^{\mathrm{ord}}_{P_1})$, $Ш_K(\widehat{A}^{\mathrm{ord}}_P)$ *is finite for almost all $P \in \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}$; again, in this case, we put $Ct_{\mathbb{T}} := \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}$.*
(3) *If $A_{P_0}(K)$ is infinite (i.e.,* (P0) *holds) and* $\mathrm{rank}_{\mathbb{T}}(J^{\mathrm{ord}}_{\infty,\mathbb{T}}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee = 1$*, then there exists an infinite subset $Ct_{\mathbb{T}} \subset \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}$ including P_0 and $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$ for all $P \in Ct_{\mathbb{T}}$ and $Ш_K(\widehat{A}^{\mathrm{ord}}_P)$ is finite for almost all $P \in Ct_{\mathbb{T}}$.*

Moreover if $P = (\varpi) \in Ct_{\mathbb{T}}$, we have the following two exact sequences up to finite error:

$$(13.2)\qquad \begin{aligned} &0 \to \widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J^{\mathrm{ord}}_\infty(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varpi_{MW}} J^{\mathrm{ord}}_\infty(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0 \\ &0 \to Ш_K(\widehat{A}^{\mathrm{ord}}_P) \to Ш_K(J^{\mathrm{ord}}_\infty) \xrightarrow{\varpi_{Ш}} Ш_K(J^{\mathrm{ord}}_\infty). \end{aligned}$$

In particular, for $P \in Ct_{\mathbb{T}}$, $E^\infty_P(K)$ is finite.

The definition of $Ct_{\mathbb{T}}$ in the assertion (3) will be given in the proof. The first inequality of the generic rank follows from Lemma 13.2; so, we prove the rest.

Proof. Write $J := \mathcal{J}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$, $\mathcal{J}_s := J^{\mathrm{ord}}_{s,\mathbb{T}}$ with $\mathcal{J} = \mathcal{J}_\infty$ and $\varpi(\mathcal{J}_s) := \varpi(J^{\mathrm{ord}}_{s,\mathbb{T}})$ as before. By Lemma 13.2 (1), we get the assertion (0): $\mathrm{rank}_{\mathbb{T}} J^\vee \le 1$, and by Lemma 13.2 (2), if $A_{P_0}(K)$ is finite, $\mathrm{rank}_{\mathbb{T}} J^\vee = 0$, proving the first part of (1). Therefore we may assume $\mathrm{rank}_{\mathbb{T}} J^\vee \le 1$.

We prove the two assertions (2–3) and the rest of (1). Tensoring $\mathcal{J}(K)$ and $\mathcal{J}_s(K)$ with the exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$, we get the tensored exact sequence with $\mathcal{G}_{s,\mathbb{T}}(K)$ finite if $s < \infty$ (Lemma 13.3):

$$0 \to \mathrm{Tor}^1_{\mathbb{Z}_p}(\mathcal{J}_s(K), \mathbb{Q}_p/\mathbb{Z}_p) = \mathcal{G}_{s,\mathbb{T}}(K) \to \mathcal{J}_s(K) \xrightarrow{j_s} \mathcal{J}_s(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathcal{J}_s(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0,$$

where $\mathrm{Ker}(j_s)$ is finite by Lemma 10.1. Passing to the limit, the following sequence is exact:

$$0 \to \mathcal{G}_{\mathbb{T}}(K) \to \mathcal{J}(K) \xrightarrow{j_\infty} \mathcal{J}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to J \to 0. \tag{13.3}$$

Suppose that $\mathrm{rank}_{\mathbb{T}} J^\vee = 0$ to prove (2). Then $J^\vee$ is supported by finite set $Z_K(\overline{\mathbb{Q}}_p) = \mathrm{Supp}(J^\vee)(\overline{\mathbb{Q}}_p) \subset \mathrm{Spec}(\mathbb{T})$. For $P \in \mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$, the following three conditions are equivalent:

(Z1) $J^\vee[P] = (J/PJ)^\vee$ is infinite,
(Z2) $P \in \mathrm{Supp}_{\mathbb{T}}(J^\vee)(\overline{\mathbb{Q}}_p)$,
(Z3) $J^\vee/PJ^\vee = J[P]^\vee$ is infinite.

By Lemma 8.2 combined with Proposition 12.2, this shows that $P_0 \notin Z_{\mathbb{T},K}$ under the assmption of (1) and $P_0 \in Z_{\mathbb{T},K}(\overline{\mathbb{Q}}_p)$ under the assumption of the assertion (2). Then the assertions (1) follows from Corollary 10.5 (3) applied to P_0, since the corresponding Selmer group is finite by the assumption.

Now we deal with the assertion (2) and the exactness of (13.2) in the cases of the assertions (1) and (2). If J/PJ is infinite, $(\mathcal{J}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\otimes_{\mathbb{T}}\mathbb{T}/P \neq 0$ which implies $P \in \Omega_{\mathbb{T}}$ by Lemma 13.3. Thus $Z_K(\overline{\mathbb{Q}}_p) \subset \Omega_{\mathbb{T}}$. Thus by Lemma 13.3, independently of the choice of P, there exists some finite $s_0 > 0$ such that if $s \geq s_0$, we have $\mathcal{J}_s(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = \mathcal{J}_{s_0}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and hence $\mathcal{J}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = \mathcal{J}_{s_0}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathcal{J}_{s_0}(K)/\mathcal{G}_{s_0,\mathbb{T}}(K) & \xrightarrow[\hookrightarrow]{j} & \mathcal{J}_{s_0}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p & \xrightarrow[\twoheadrightarrow]{} & \mathcal{J}_{s_0}(K)\otimes\mathbb{Q}_p/\mathbb{Z}_p \\
{\scriptstyle i}\downarrow & & \|\downarrow & & {\scriptstyle a}\downarrow \\
\mathcal{J}(K)/\mathcal{G}_{\mathbb{T}}(K) & \xrightarrow[\hookrightarrow]{j_\infty} & \mathcal{J}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p & \xrightarrow[\twoheadrightarrow]{} & J.
\end{array}$$

The map a is surjective, and $\mathrm{Ker}(a)$ is finite as $\mathrm{Supp}(\mathcal{J}_{s_0}(K)/\mathcal{G}_{s_0,\mathbb{T}}(K)) = \mathrm{Supp}(\mathcal{J}(K)/\mathcal{G}_{\mathbb{T}}(K))$. Similarly, $\varpi(\mathcal{J}_{s_0})(K)$ and $\varpi(\mathcal{J})(K)$ is isogenous. Since $E_P^{s_0}(K)$ is finite, $\varpi(\mathcal{J}_{s_0}(K))$ is isogenous to $\varpi(\mathcal{J}_{s_0})(K)$. Thus all $\varpi(\mathcal{J}_{s_0})(K)$, $\varpi(\mathcal{J}_{s_0}(K))$, $\varpi(\mathcal{J})(K)$ and $\varpi(\mathcal{J}(K))$ are isogenous. This implies finiteness of $E_P^\infty(K)$. Thus the sequence for $P = (\varpi) \in \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}$

$$0 \to \widehat{A}_P^{\mathrm{ord}}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \to J \xrightarrow{\varpi_{MW}} J \to 0 \tag{13.4}$$

is exact up to finite error by Proposition 12.2. Thus applying this fact to the following commutative diagram with exact rows (up to finite error) for $(\varpi) = P \in \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}(\overline{\mathbb{Q}}_p)$:

$$\begin{array}{ccccc}
\widehat{A}_P^{\mathrm{ord}}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\hookrightarrow} & J & \xrightarrow[\twoheadrightarrow]{\varpi} & J \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Sel}_K(\widehat{A}_P^{\mathrm{ord}}) & \xrightarrow{\hookrightarrow} & \mathrm{Sel}_K(\mathcal{J}) & \xrightarrow{\varpi} & \mathrm{Sel}_K(\mathcal{J}),
\end{array} \tag{13.5}$$

we get an exact sequence up to finite error for all $P \in \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}(\overline{\mathbb{Q}}_p)$:

$$0 \to Ш_K(\widehat{A}_P^{\mathrm{ord}}) \to Ш_K(\mathcal{J}) \to Ш_K(\mathcal{J}). \tag{13.6}$$

Note that $Z_K(\overline{\mathbb{Q}}_p) = \mathrm{Supp}(J^\vee)(\overline{\mathbb{Q}}_p)$ by definition. If $P \in \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}(\overline{\mathbb{Q}}_p)$ (which implies $P \in \Omega_{\mathbb{T}} - Z_K(\overline{\mathbb{Q}}_p)$), we have $\widehat{A}_P^{\mathrm{ord}}(K)$, $J[P]$ and $J \otimes_{\mathbb{T}} \mathbb{T}/P$ are all finite. Thus for $P = (\varpi) \in \Omega_{\mathbb{T}}$ outside $Z_{\mathbb{T},K}$, localizing the sequence (13.4) at P, every localized term of (13.4) vanishes, and hence $\dim_{\mathbb{T}/P\otimes\mathbb{Q}_p} \widehat{A}_P^{\mathrm{ord}}(K) = \dim_{H_P} A_P(K) = 0$ as desired (see Lemma 5.4). Thus under the assumption of the assertion (2), $P_1 \notin Z_{\mathbb{T},K}(\overline{\mathbb{Q}}_p)$, we can apply the above exact sequence (13.6) to $P = P_1$, and hence $Ш_K(\mathcal{J})$ is a torsion $\mathbb{T}$-module because $Ш_K(\widehat{A}_{P_1}^{\mathrm{ord}})$ is finite. Therefore, by (13.6), $Ш_K(\widehat{A}_P^{\mathrm{ord}})$ is finite for almost all $P \in \Omega_{\mathbb{T}} - Z_{\mathbb{T},K}$ proving (2).

Finally we prove the assertion (3) assuming $\mathrm{rank}_{\mathbb{T}} J^\vee = 1$ and $A_{P_0}(K)$ is infinite. We recall the proof of Lemma 13.2 (3). Then $J^\vee$ is pseudo isomorphic to $\mathbb{T} \oplus Y$ for a torsion $\mathbb{T}$-module Y without finite $\mathbb{T}$-submodules. Then $J[\varpi_0]^\vee$ is pseudo isomorphic to $\mathbb{T}/P_0 \oplus Y/P_0Y$. Since $Ш_K(\widehat{A}_{P_0}^{\mathrm{ord}})$ is finite, $E_{P_0}^\infty(K)$ is finite by Corollary 8.2, and the natural map

$$\widehat{A}_{P_0}^{\mathrm{ord}}(K)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \to J[\varpi_0]$$

has finite kernel and cokernel by Proposition 12.2. Thus

$$\mathrm{rank}_{\mathbb{Z}_p} Y/P_0Y + \mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}/P_0 = \mathrm{corank}_{\mathbb{Z}_p} \widehat{A}^{\mathrm{ord}}_{P_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p.$$

Since $\dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$, by Lemma 5.4,

$$\mathrm{corank}_{\mathbb{Z}_p} \widehat{A}^{\mathrm{ord}}_{P_0}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p = \mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}/P_0.$$

This implies $\mathrm{rank}_{\mathbb{Z}_p} Y/P_0Y = 0$, and hence $P_0 \notin Z_K(\overline{\mathbb{Q}}_p)$, and hence by (P0),

$$P_0 \notin Z_{\mathbb{T},K}(\overline{\mathbb{Q}}_p). \tag{13.7}$$

Applying the same argument to general $P = (\varpi) \in \Omega_{\mathbb{T}} - Z_K(\overline{\mathbb{Q}}_p)$ in place of P_0, we have the following commutative diagram with exact rows

$$\begin{array}{ccccc} J^\vee & \xrightarrow[\hookrightarrow]{\varpi} & J^\vee & \longrightarrow & (J[\varpi])^\vee \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{T} \oplus Y & \xrightarrow[\hookrightarrow]{\varpi} & \mathbb{T} \oplus Y & \xrightarrow{\twoheadrightarrow} & \mathbb{T}/P \oplus Y/PY. \end{array}$$

Note that Y/PY is finite as $P \notin Z_K(\overline{\mathbb{Q}}_p)$. Therefore $\mathrm{rank}_{\mathbb{Z}_p}(J[\varpi])^\vee = \mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}/P$. Since $\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J[\varpi]$ has finite kernel, if $\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ is infinite, it has $\mathbb{Z}_p$-corank equal to $\mathrm{rank}_{\mathbb{Z}_p} \mathbb{T}/P = \mathrm{corank}_{\mathbb{Z}_p} J[\varpi]$, and therefore, the map $\widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J[\varpi]$ has to have finite cokernel. We define $Ct'_{\mathbb{T}}$ to be the subset of all $P \in \omega_{\mathbb{T}}$ such that $A_P(K)$ is infinite, and put

$$Ct_{\mathbb{T}} := \{P^\sigma | P \in Ct'_{\mathbb{T}} - Z_{\mathbb{T},K}, \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\}.$$

By (13.7), we know $P_0 \in Ct_{\mathbb{T}}$.

By Lemma 13.3, $\mathcal{J}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \bigoplus_{P \in Ct'_{\mathbb{T}}} \widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Since $\mathrm{corank}_{\mathbb{Z}_p} J$ is infinity, by (13.3), $\dim_{\mathbb{Q}_p} \mathcal{J}(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \infty$; so, $Ct'_{\mathbb{T}}$ has to be an infinite set, and hence $Ct_{\mathbb{T}}$ is also infinite. Then for $P \in Ct_{\mathbb{T}}$, we conclude an exact sequence up to finite error:

$$0 \to \widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to J \xrightarrow{\varpi} J \to 0.$$

By this exact sequence, we conclude

$$\dim_{\mathbb{T}/P \otimes \mathbb{Q}_p} \widehat{A}^{\mathrm{ord}}_P(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{rank}_{\mathbb{T}} J^\vee = 1$$

for all $P \in Ct_{\mathbb{T}}$. Again applying the snake lemma to the diagram (13.5), we conclude the exactness of the sequence (13.6). Applying finiteness of $Ш_K(\widehat{A}^{\mathrm{ord}}_{P_0})$ to (13.6) (as $P_0 \in Ct_{\mathbb{T}}$), we conclude that $Ш_K(\mathcal{J})$ is $\mathbb{T}$-torsion, and hence for almost all $P \in Ct_{\mathbb{T}}$, $Ш_K(\widehat{A}^{\mathrm{ord}}_P)$ is finite. The last assertion for E^∞_P follows from Lemma 8.2. □

Remark 13.5. By the techniques invented in this work, we cannot prove that $\Omega_{\mathbb{T}} - Ct_{\mathbb{T}}$ is a finite set (so, we need a new idea for that). Indeed, for any Zariski dense subset $\mathfrak{Z}$ (i.e., an infinite subset) of $\mathrm{Spec}(\overline{\mathbb{Q}}_p)$, we have an exact sequence

$$0 \to \mathbb{T} \xrightarrow{i} \prod_{P \in \mathfrak{Z}} (\mathbb{T}/P) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\pi} \mathrm{Coker}(i) \to 0.$$

Take the Pontryagin dual exact sequence:

$$0 \to \mathrm{Coker}(i)^\vee \xrightarrow{\pi^\vee} \bigoplus_{P \in \mathfrak{Z}} (\mathbb{T}/P) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{i^\vee} \mathbb{T}^\vee \to 0.$$

This is because the $\mathbb{Q}_p$-vector space $(\mathbb{T}/P) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is self-dual by the trace pairing. Thus $\mathcal{J}(K)/\mathcal{G}_{\mathbb{T}}(K)$ could be something like $\mathrm{Coker}(i)^\vee$ for any infinite subset $\mathfrak{Z} = Ct_{\mathbb{T}}$ of $\Omega_{\mathbb{T}}$.

14. Parameterization of congruent abelian varieties

Let $B_{/\mathbb{Q}}$ be a $\mathbb{Q}$-simple abelian variety of GL(2)-type (as in the introduction). We assume that $O_B = \mathrm{End}(B_{/\mathbb{Q}}) \cap H_B$ is the integer ring of its quotient field H_B. Then the compatible system of two dimensional Galois representations $\rho_B = \{\rho_{B,\mathfrak{l}}\}_{\mathfrak{l}}$ realized on the Tate module of B has its L-function $L(s, B)$ equal to $L(s, f)$ for a primitive form $f \in S_2(\Gamma_1(C))$ for the conductor $C = C_B$ of ρ_B (see [KW09, Theorem 10.1]). Thus B is isogenous to A_f over $\mathbb{Q}$ (by a theorem of Faltings). The abelian variety A_f is known to be $\mathbb{Q}$-simple as H_{A_f} is generated by $\mathrm{Tr}(\rho_B(\mathrm{Frob}_l))$ for primes l outside Np. Let π_f be the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to f.

Fix a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h}_{\alpha,\delta,\xi})$. If $(\alpha, \delta, \xi) \neq (0, 1, \omega_d)$, for $P \in \Omega_{\mathbb{T}}$, the minimal (nearly ordinary) form $\mathbf{f} := f_P$ (in the sense of [H09, (L1–3)] and [H10, §1.1]) in π_f may not be primitive. We use the notation introduced in [H10, §1.1] for adelic automorphic forms without recalling its definition. Assume that P is principal (i.e. (A)) and f_P is on $\widehat{\Gamma}_r$. Then we define $A_{\mathbf{f}} = J_r[\mathfrak{a}_r]^\circ$ as in (A). If $H_{A_{\mathbf{f}}} = H_{A_f} = H_B$, $A_{\mathbf{f}}$ is $\mathbb{Q}$-simple and is isogenous to A_f.

Lemma 14.1. *Let the notation be as above. If the conductor of f is divisible by Np, the abelian variety $A_{\mathbf{f}}$ is isogenous to B over $\mathbb{Q}$ and $H_{A_{\mathbf{f}}} = H_{A_f} = H_B$. If the conductor of f is equal to N prime to p and $\mathbf{f}|U(p) = \varphi(p)\mathbf{f}$, $A_{\mathbf{f}}$ is isogenous to $B \otimes_{O_B} O_B[\varphi(p)]$ as abelian varieties of* GL(2)*-type, which is in turn $\mathbb{Q}$-isogenous to $B \times B$ just as abelian varieties.*

Proof. Since $a_l := \mathrm{Tr}(\rho_B(\mathrm{Frob}_l)) \in H_{A_{\mathbf{f}}}$ for all $l \nmid Np$, we have $H_B \subset H_{A_{\mathbf{f}}}$. Write $\pi_f = \widehat{\otimes}_v \pi_v$ and $\pi_p = \pi(\varphi, \beta)$ or $\sigma(\varphi, \beta)$ with p-adic unit $i_p(\varphi(p))$. Note that the $\mathbf{f}$ is characterized by

$$\mathbf{f} \in H^0(\widehat{\Gamma}^1_1(Np^r), \pi) \subset \mathcal{S}_2(\widehat{\Gamma}^1_1(Np^r)),\ \mathbf{f}|T(l) = a_l\mathbf{f} \text{ for all } l \nmid Np,\ \mathbf{f}|U(p) = \varphi(p)\mathbf{f} \\ \text{and } \pi((\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}))\mathbf{f} = \varphi(d)\beta(a)\mathbf{f} \text{ for } a, d \in \mathbb{Z}_p^\times, \tag{14.1}$$

writing $T(l)$ for $U(l)$ if $l|N$ (see [H89, §2]). Here $\mathcal{S}_2(\widehat{\Gamma}^1_1(Np^r))$ for the open compact subgroup $\widehat{\Gamma}^1_1(Np^r)$ defined in (3.2) is the space of adelic cusp form defined in [H10, §1.1] taking ψ in [H10, §1.1] to be the identity character. Moreover for the member $\rho_{\mathbf{f}}$ of ρ_B associated to the place $\mathfrak{p}_A$ induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we have (cf. [H89, §2])

$$\rho_{\mathbf{f}}|_{I_p} \cong \begin{pmatrix} \nu_p\psi & * \\ 0 & \varphi \end{pmatrix} \text{ with } \beta = |\cdot|_p^{-1}(i_p^{-1} \circ \psi) \ (\psi \text{ has finite order over } I_p) \tag{14.2}$$

for the inertia subgroup $I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q})$, regarding φ, ψ as characters of I_p by local class field theory. Then $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_B) \Leftrightarrow \rho_B^\sigma \cong \rho_B \Leftrightarrow (\pi^{(\infty)})^\sigma \cong \pi^{(\infty)}$, where $\pi^{(\infty)} = \widehat{\otimes}_{l<\infty}\pi_l$. This shows the minimal field of definition of $\pi^{(\infty)}$ is H_B (a result of Waldspurger), and by (14.2), H_B contains the values of $\varphi|_{I_p}$. Thus $H_{A_{\mathbf{f}}} = H_B(\varphi)$ generated over H_B by the values of φ, as the central character ψ_P of π has values in H_B over $\mathbb{A}^{(\infty)}$ (which follows from the fact that $\det \rho_B = \psi_P\nu$ for the compatible system ν of the cyclotomic characters). Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If φ or β is non-trivial over $\mathbb{Z}_p^\times$ or $A_{\mathbf{f}}$ is potentially multiplicative at p (i.e., the conductor of f is divisible by p), the nearly ordinary vector $\mathbf{f}$ is characterized by the above properties (14.1) without $\mathbf{f}|U(p) = \varphi(p)\mathbf{f}$. Thus in this case, $\mathbf{f}^\sigma \in \pi^\sigma \cong \pi$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_B)$ implies $\mathbf{f}^\sigma = \mathbf{f}$. In particular, $H_{A_{\mathbf{f}}} = H_B$ as desired. If f has conductor N, $\mathbf{f}$ is p-stabilized (i.e., $\mathbf{f}(z) = f(z) - \beta(p)f(pz)$), then $H_{A_{\mathbf{f}}} = H_B(\varphi(p))$. Since $\varphi(p)$ satisfies $X^2 - a_pX + \psi_P(p)p = 0$ for the $T(p)$ eigenvalue a_p of f, we have $[H_{A_{\mathbf{f}}} : H_B] \leq 2$, and $A_{\mathbf{f}}$ is isogenous to $B \otimes_{O_B} O_B[\varphi(p)]$ (as an abelian variety of GL(2)-type).

If the central character ψ_P is trivial, H_B is totally real, and $H_B(\varphi(p))$ is totally imaginary; so, $A_{\mathbf{f}}$ is isogenous to $B \times B$ if the conductor of B is prime to p. Even if the central character is not trivial, choosing a square root $\zeta := \sqrt{\psi_P(p)}$, $T(p)\zeta^{-1}$ is self adjoint on $S_2(\Gamma_0(N), \psi_P)$ (e.g., [MFM, Theorem 4.5.4]), and hence $a_p\zeta^{-1}$ is totally real, but for the root $\varphi(p)\zeta^{-1}$ of $X^2 - a_p\zeta^{-1}X + p$, $\mathbb{Q}(\varphi(p)\zeta^{-1})$ is totally imaginary as with $|a_p| \leq 2\sqrt{p}$ combined with $|\beta(p)|_p < |\varphi(p)|_p = 1$. This shows that $H_{A_{\mathbf{f}}}$ is a quadratic extension of H_B, and hence $A_{\mathbf{f}}$ is isogenous to $B \times B$. □

Let A be another $\mathbb{Q}$-simple abelian variety of GL(2)-type. Thus A is isogenous to A_g for a primitive form $g \in S_2(\Gamma_1(C_A))$ of conductor C_A. Let π_g be the automorphic representation of g, and write $\mathbf{g}$ for the minimal nearly p-ordinary form in π_g. Without losing generality, we may (and do) assume that $O_A = \mathrm{End}(A_{/\mathbb{Q}}) \cap H_A$ is the integer ring of H_A. Note that $H_B \cong Q(f) \subset \overline{\mathbb{Q}}$ and $H_A \cong \mathbb{Q}(g)$. Suppose A is congruent to B modulo p with $(B[\mathfrak{p}_B] \otimes_{\kappa(\mathfrak{p}_B)} \overline{\mathbb{F}}_p)^{ss} \cong (A[\mathfrak{p}_A] \otimes_{\kappa(\mathfrak{p}_A)} \overline{\mathbb{F}}_p)^{ss}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. Here, for any ring R and a prime ideal $\mathfrak{p}$ of R, $\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p}$.

Write $O_{\mathfrak{p}_A}$ for the $\mathfrak{p}_A$-adic completion of O_A, and let $T_{\mathfrak{p}_A}A = \varprojlim_n A[\mathfrak{p}_A^n](\overline{\mathbb{Q}}_p)$ (the $\mathfrak{p}_A$-adic Tate module of A). We call that A is of $\mathfrak{p}_A$-type (α, δ, ξ) if we have an exact sequence of I_p-modules $0 \to V(\nu\epsilon^{-\delta}, \xi^{-1}) \to T_{\mathfrak{p}_A}A \to V(\epsilon^{\alpha}, \xi^{-1}) \to 0$ with $V(\nu\epsilon^{-\delta}, \xi^{-1}) \cong V(\epsilon^{\alpha}, \xi^{-1}) \cong O_{\mathfrak{p}_A}$ as $O_{\mathfrak{p}_A}$-modules, where ϵ is a character of $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$ with values in μ_{p^∞}, $[u, \mathbb{Q}_p]$ with $u \in \mathbb{Z}_p^\times$ (resp. $[\zeta, \mathbb{Q}_p]$ for $\zeta \in \mu$) acts on $V(\nu\epsilon^{-\delta}, \xi^{-1})$ by $u^{-1} \cdot \epsilon^{-\delta}(u)$ (resp. by $\zeta^{-1} \cdot \xi^{-1}(\zeta, 1)$) and on $V(\epsilon^{\alpha}, \xi^{-1})$ by $\epsilon(u)^{\alpha}$ (resp. by $\xi^{-1}(1, \zeta)$). Here $[x, \mathbb{Q}_p]$ is the local Artin symbol. If $\xi(\zeta, \zeta') = \xi(\zeta)$ for $(\zeta, \zeta') \in \mu^2$ and $\alpha = 0$, this is just a $\mathfrak{p}_A$-ordinarity.

Choosing g (resp. f) well in the Galois conjugacy class of g (resp. f), we may assume that $\mathfrak{p}_A$ and $\mathfrak{p}_B$ are both induced by the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Lemma 14.2. *Let the notation be as above. Suppose that C_A/C_B is in $\mathbb{Z}[\frac{1}{p}]^\times$ and that B (resp. A) is of $\mathfrak{p}_B$-type (resp. $\mathfrak{p}_A$-type) (α, δ, ξ). Write $C_B = Np^r$. Then there exists a connected component* $\mathrm{Spec}(\mathbb{T})$ *of* $\mathrm{Spec}(\mathbf{h}_{\alpha,\delta,\xi}(N))$ *such that for some primes $P, Q \in \mathrm{Spec}(\mathbb{T})$, $\mathbf{f} = f_P$ and $\mathbf{g} = f_Q$.*

Proof. Let $\overline{\rho}$ be the two dimensional Galois representation into $\mathrm{GL}_2(\mathbb{F})$ realized on $B[\mathfrak{p}_B]$ for $\mathbb{F} = O_B/\mathfrak{p}_B$. Write N for the prime-to-p part of C_B (and hence of C_A). Replacing $\overline{\rho}$ by its semi-simplification, we may assume that $\overline{\rho}$ is semi-simple. Since $(B[\mathfrak{p}_B] \otimes_{\kappa(\mathfrak{p}_B)} \overline{\mathbb{F}}_p)^{ss} \cong (A[\mathfrak{p}_A] \otimes_{\kappa(\mathfrak{p}_A)} \overline{\mathbb{F}}_p)^{ss}$, $L(s, A) = L(s, g)$ and $L(s, B) = L(s, f)$ imply $f \mod \mathfrak{p}_B = g \mod \mathfrak{p}_B$. Since $f_B := \mathbf{f}$ is nearly p-ordinary with nearly ordinary character given by $[u\zeta, \mathbb{Q}_p] \mapsto \epsilon_B(u)^{\alpha}\xi^{-1}(1, \zeta)$ ($u \in \Gamma$ and $\zeta \in \mu$) for a character $\epsilon_B : \mathbb{Z}_p^\times \to \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ and has central character $z \mapsto \epsilon_B(z)^{\alpha-\delta}\xi^{-1}(z, z)\chi(z)$ for $z \in \widehat{\mathbb{Z}}^\times$, f_B generates an automorphic representation whose p-component π_p is given by the principal series $\pi(\phi, \varphi)$ (or the Steinberg representation $\sigma(\phi, \varphi)$) with $\varphi(u\zeta) = \epsilon_B(u)^{\alpha}\xi^{-1}(1, \zeta)$ and $\phi(u\zeta) = |u|_p\epsilon_B(u)^{-\delta}\xi^{-1}(\zeta, 1)$. Moreover, $f_B|U(p) = \varphi(p)f_B$ with $\mathrm{ord}_p(\varphi(p)) = 0$. See [H89, §2] for these facts (in particular, the p-component of f_B is proportional to the nearly ordinary vector v in π_p fixed by the p-component of $\widehat{\Gamma}_{H,r}$ characterized by $\pi_p((\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}))v = \phi(a)\varphi(d)v$ for $a, d \in \mathbb{Q}_p^\times$ and $U(p)(v) = \varphi(p)v$).

The form $f_A := \mathbf{g}$ associated to A has similar property whose p-component is given by $\pi(\phi', \varphi')$ (or the Steinberg representation $\sigma(\phi', \varphi')$) with $\varphi' \mod \mathfrak{p}_A = \varphi \mod \mathfrak{p}_B$ and $\phi' \mod \mathfrak{p}_A = \phi \mod \mathfrak{p}_B$. More precisely, we have $\varphi'(u\zeta) = \epsilon_A(u)^{\alpha}\xi^{-1}(1, \zeta)$ and $\phi'(u\zeta) = |u|_p\epsilon_A(u)^{-\delta}\xi^{-1}(\zeta, 1)$ for a character $\epsilon_A : \mathbb{Z}_p^\times \to \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$. Thus $\mathbf{g} = f_A$ (resp. $\mathbf{f} = f_B$) is lifted to a p-adic analytic family (of type (α, δ, ξ)) parameterized by an irreducible component $\mathrm{Spec}(\mathbb{I})$ (resp. $\mathrm{Spec}(\mathbb{J})$) of $\mathrm{Spec}(\mathbf{h}_{\alpha,\delta,\xi}(N))$. Since $\mathbf{f} \mod \mathfrak{p}_B = \mathbf{g} \mod \mathfrak{p}_B$, the algebra homomorphisms $\lambda_? : \mathbf{h}_{\alpha,\delta,\xi}(N) \to \overline{\mathbb{Q}}_p$ realized as $\mathbf{f}|T(n) = \lambda_{\mathbf{f}}(T(n))\mathbf{f}$ and $\mathbf{g}|T(n) = \lambda_{\mathbf{g}}(T(n))\mathbf{g}$ satisfy $\lambda_{\mathbf{f}} \equiv \lambda_{\mathbf{g}} \mod \mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ of $\mathbf{h}_{\alpha,\delta,\xi}(N)$. Then, $P = \mathrm{Ker}(\lambda_{\mathbf{f}})$ and $Q = \mathrm{Ker}(\lambda_{\mathbf{g}})$ belong to the connected component $\mathrm{Spec}(\mathbb{T})$ given by $\mathbb{T} = \mathbf{h}_{\alpha,\delta,\xi}(N)_{\mathfrak{m}}$, since the local rings of $\mathbf{h}_{\alpha,\delta,\xi}(N)$ corresponds one-to-one to the maximal congruence classes modulo $\mathfrak{P}$ ($\mathfrak{P} := \{x \in \overline{\mathbb{Q}}_p : |x|_p < 1\}$) of Hecke eigenforms of prime-to-p level N (and of type (α, δ, ξ)) just because the set of maximal ideals $\overline{\Sigma}$ of $\mathbf{h}_{\alpha,\delta,\xi}(N)$ is made of $\mathrm{Ker}(\lambda)$ for $\lambda \in \Sigma = \mathrm{Hom}_{\mathrm{alg}}(\mathbf{h}_{\alpha,\delta,\xi}(N), \overline{\mathbb{F}}_p)$. The maximal ideal $\mathfrak{m}$ is given by $\mathrm{Ker}(\lambda_{\mathbf{f}} \mod \mathfrak{P}) = \mathrm{Ker}(\lambda_{\mathbf{g}} \mod \mathfrak{P})$ for $\mathfrak{P} = \{x \in \overline{\mathbb{Q}}_p : |x|_p < 1\}$. □

The following result is just a combination of the above Lemma 14.2 and Theorem 5.6.

Corollary 14.3. *Let the notation and the assumptions be as in* Lemma 14.2 *and* Theorem 5.6 *(in particular, we assume* (F)*). Assume that the abelian variety B has conductor N prime to p. Let $f \in S_2(\Gamma_0(N), \chi)$ be the primitive form with conductor N prime to p (so, $\xi = \mathbf{1}$) whose L-function gives $L(s, B)$. Write $\chi|\cdot|_{\mathbb{A}}^{-1}$ for the central character of the automorphic representation generated by f. Write $f|T(p) = a_p f$. If $p \nmid 6D_\chi N\varphi(N)$ and $(a_p \mathrm{mod}\, \mathfrak{p}_B) \notin \Omega_{B,p} := \{0, 1, \pm\sqrt{\chi(p)}\}$, then $\mathbb{T}$ is a regular integral domain and $\mathbf{f}$ and $\mathbf{g}$ belongs to* $\mathrm{Spec}(\mathbb{T})$.

Again we can replace the condition: $p \nmid 6D_\chi N\varphi(N)$ by $p \nmid 2D_\chi N\varphi(N)$ in the case where $\mathbf{h}_{\alpha,\delta,\xi}(N)$ is Λ-free (see Proposition 18.2 for such cases).

15. A generalized version of Theorem B

Let $B_{/\mathbb{Q}}$ be a $\mathbb{Q}$-simple abelian variety of GL(2)-type of conductor N such that $O_B = \mathrm{End}(B_{/\mathbb{Q}}) \cap H_B$ is the integer ring of its quotient field H_B. We suppose the following minimalist condition:

(M) $\text{Ш}_K(B)$ is finite and $\dim_{H_B} B(K) \otimes_{\mathbb{Z}} \mathbb{Q} \le 1$.

Let $\rho_B = \{\rho_{B,\mathfrak{l}}\}$ be the two dimensional compatible system of Galois representations associated to B. Then ρ_B comes from a Hecke eigenform $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N), \chi)$ by [KW09, Theorem I.10.1]; so, $L(s, B) = L(s, \rho_B) = L(s, f)$. Fix an embedding $O_B \hookrightarrow \overline{\mathbb{Q}}$ and write $\mathfrak{p}_B$ for the prime ideal of O_B induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Then we realize the Hecke algebra $h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi])$ inside $\mathrm{End}_{\mathbb{C}}(S_2(\Gamma_0(N), \chi))$ which is generated over $\mathbb{Z}[\chi]$ by all Hecke operators $T(n)$ and $U(l)$. Then this Hecke algebra is free of finite rank over $\mathbb{Z}$, and hence its reduced part (modulo the nilradical) has a well defined discriminant D_χ over $\mathbb{Z}$.

Definition 15.1. *Let $S = S_B$ be the set of prime factors of $6D_\chi N\varphi(N)$ for the conductor N of ρ_B, where D_χ is the discriminant of the reduced part of $h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi])$.*

We could include $p = 3$ defining $S = S_B$ to be the set of prime factors of $2D_\chi N\varphi(N)$ if $\mathbf{h}_{\alpha,\delta,\mathbf{1}}$ is Λ-free (see remarks after Proposition 4.1 and see also Proposition 18.2). We write $\mathfrak{p}_B$-type of B as $(\alpha, \delta, \mathbf{1})$. The prime $p \notin S_B$ is *admissible* for B over K if

1. B has good reduction modulo p (so, $p \nmid N$);
2. $(a_p \bmod \mathfrak{p}_B) \notin \Omega_{B,p} := \{0, 1, \pm\sqrt{\chi(p)}\}$ (so, B has potential partially $\mathfrak{p}_B$-ordinary reduction modulo p);
3. For the local ring $\mathbb{T}$ of $\mathbf{h}_{\alpha,\delta,\mathbf{1}}$ with the arithmetic point P_0 for which A_{P_0} isogenous to B, the generic rank $\mathrm{rank}_{\mathbb{T}} J^\vee = \dim_{H_B} B(K) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Primes satisfying (1) and (2) has Dirichlet density 1 (e.g., [F02, §2.3] or [H13b, Section 7]). The condition (3) is the deep assumption of matching dimension (difficult to verify if $B(K)$ is infinite) which tells us $P_0 \notin Z_K$ for the mysterious finite set $(\Omega_{\mathbb{T}} \cap Z_K(\overline{\mathbb{Q}}_p)) \subset \mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ (see Lemma 13.2 (3)). When $B(K)$ is finite, almost all primes outside S_B is admissible by the $\mathbb{T}$-torsion of the ind Λ-Selmer group (see Corollary 10.5 (3)). When $B(K)$ is infinite, we do not know if most primes outside S_B for which B is of type $(1, 1, \mathbf{1})$ is admissible for B or not. Since $p \notin S_B$, the local ring $\mathbb{T}$ of $\mathbf{h}$ carrying A_{P_0} is unique by Theorem 5.6.

Since B has conductor prime to p, ρ_B is unramified at p, and ξ has to be the identity character $\mathbf{1}$ of $\mu \times \mu$ (on the other hand, (α, δ) can be freely chosen). Here is a general version of Theorem B:

Theorem 15.2. *Assume* (F) *for $(\alpha, \delta, \mathbf{1})$, and let K be a number field. Let $p \notin S_B$ be a prime admissible for B and N be the conductor of B. Suppose the minimalist condition* (M) *and that B is isogenous to A_{P_0} for $P_0 \in \Omega_{\mathbb{T}}$. Consider the set $\mathcal{A}_{B,p}$ made up of all $\mathbb{Q}$-isogeny classes of $\mathbb{Q}$-simple abelian varieties $A_{/\mathbb{Q}}$ of $\mathfrak{p}_A$-type $(\alpha, \delta, \mathbf{1})$ congruent to B modulo p over $\mathbb{Q}$ with prime-to-p conductor N. Then, infinite members $A \in \mathcal{A}_{B,p}$ have finite $\text{Ш}_K(A)_{\mathfrak{p}_A}$ and $\dim_{H_A} A(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ equal to $\dim_{H_B} B(K) \otimes_{\mathbb{Z}} \mathbb{Q}$. If further $\widehat{B}^{\mathrm{ord}} \cong \widehat{A}_{P_0}^{\mathrm{ord}}$ for $P_0 \in \Omega_{\mathbb{T}}$ with $\mathrm{Sel}_K(B)_{\mathfrak{p}_B} = 0$ and all prime factors of p in K has residual degree 1, then $\mathrm{Sel}_K(A)_{\mathfrak{p}_A}$ is finite for all $A \in \mathcal{A}_{B,p}$ without exception.*

Proof. Suppose that p is outside S_B, by Theorem 5.6, $\mathbb{T}$ is a regular integral domain $\mathbb{I}$. Thus for any $P \in \Omega_{\mathbb{T}}$, we have $P = (\varpi)$ for $\varpi \in \mathbb{I}$ and (ϖ, A_P) satisfies (A).

Since $B[\mathfrak{p}_B^\infty]$ is an ordinary Barsotti–Tate group by our assumption, $A[\mathfrak{p}_A^\infty]$ is potentially ordinary by the congruence modulo p between A and B. Here we say $A[\mathfrak{p}_A^\infty]$ "potentially ordinary" if $H_0(k, A[\mathfrak{p}_A^\infty](\overline{\mathbb{Q}}_p))$ has non-trivial p-divisible rank and $A[\mathfrak{p}_A^\infty]$ over $\mathbb{Q}_p$ extends to a Barsotti–Tate group with non-trivial étale quotient over the integer ring of a finite extension k of $\mathbb{Q}_p$. Choosing the embedding $O_A \hookrightarrow \overline{\mathbb{Q}}$ well, we may assume that $\mathfrak{p}_A$ is induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Then by Lemma 14.2, A is isogenous to a modular abelian variety A_P for $P \in \Omega_{\mathbb{T}}$ of a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h}_{\alpha,\delta,\mathbf{1}}(N))$ for the big p-adic Hecke algebra $\mathbf{h}_{\alpha,\delta,\mathbf{1}}(N)$. Since B is of GL(2)-type, we have $B \sim A_{P_0}$ (an isogeny) for $P_0 \in \Omega_{\mathbb{T}}$ with $P_0 = (\varpi_0)$. Thus we conclude, up to isogeny,

$$\mathcal{A}_{B,p} = \{A_Q | Q \in \Omega_{\mathbb{T}}\}$$

by the theorem of Khare–Wintenberger [KW09, Theorem I.10.1] (combined with the proof of the Tate conjecture for abelian varieties by Faltings).

Since O_A is the integer ring of H_A, we can factor $O_{A,p} = O_A \otimes_{\mathbb{Z}} \mathbb{Z}_p$ into the product $O_{A,p} = O^{\mathrm{ord}}_{A,p} \oplus O^{\mathrm{ss}}_{A,p}$ so that for the idempotent e of the factor $O^{\mathrm{ord}}_{A,p}$, $eA[p^\infty]$ is the maximal p-ordinary Barsotti–Tate group which becomes étale and multiplicative after étale extension. Since $O_{A,p}$ acts on $\widehat{A}$, we can define $\widehat{A}^{\mathrm{ord}} := e(\widehat{A})$. Since A is isogenous to A_P, $\widehat{A}^{\mathrm{ord}}$ is isogenous to $\widehat{A}^{\mathrm{ord}}_P$; so, $\text{Ш}_K(\widehat{A}^{\mathrm{ord}})$ is isogenous to $\text{Ш}_K(\widehat{A}^{\mathrm{ord}}_P)$. From Remark **??**, we know $P_0 \notin Z_K(\overline{\mathbb{Q}}_p)$. Then finiteness of $\text{Ш}_K(\widehat{A}^{\mathrm{ord}})$ for infinitely many members of $\mathcal{A}_{B,p}$ and the assertion for the Mordell–Weil rank follows from the assumption on the generic rank and Theorem 13.4.

Suppose $\mathrm{Sel}_K(\widehat{B}^{\mathrm{ord}}) = 0$ and K_v for all $v|p$ has residue field $\mathbb{F}_p$. Then $|\varphi(\mathrm{Frob}_v)-1|_p = |a_p-1|_p = 1$ as $p \notin \Omega_{B,p}$. Thus by Schneider [Sc83, Proposition 2, Lemma 3] (see also [Sc82, Proposition 2]), we have, for all $v|p$,

$$|H^1(K_v[\mu_{p^\infty}]/K_v, \widehat{A}^{\mathrm{ord}}_r(K_v[\mu_{p^\infty}])| = |\widehat{A}^{\mathrm{ord}}_r(\mathbb{F}_p)|^2 = |\widehat{A}_r(\mathbb{F}_p)|^2. \tag{15.1}$$

Note that $|\widehat{A}_r(\mathbb{F}_p)|^2 = 1$ by our assumption. Strictly speaking, Schneider assumes in [Sc83, §7] that A_r has ordinary good reduction, but his argument works well without change replacing $(A_r(p) := A_r[p^\infty], A_r)$ there by $(A_r[p^\infty]^{\mathrm{ord}}, \widehat{A}^{\mathrm{ord}}_r)$. Indeed, he later takes care of the general case of formal Lie groups in [Sc87, Theorem 1] (including the case of the ordinary part of the formal group of A_r). So, $E^\infty(K_v)_{\mathbb{T}} = E_{\mathrm{Sel}}(K_v)_{\mathbb{T}} = 0$ for all $v|p$ (see Theorem 17.2 and Corollary 10.3 for more details of this fact). Then from Corollary 10.5 (2), we conclude $\mathrm{Sel}_K(A_P)_{\mathfrak{p}_A}$ is finite for all $P \in \Omega_{\mathbb{T}}$. □

Remark 15.3. If we start with an elliptic curve E as in Theorem B, by its modularity, we find a modular factor $B \subset J_1(Np^r)$ isogenous to E. Choose $(\alpha, \delta, \mathbf{1}) = (1, 1, \mathbf{1})$. The finiteness of $\text{Ш}_{\mathbb{Q}}(E)$ implies the finiteness of $\text{Ш}_{\mathbb{Q}}(B)$; so, the above theorem implies the statements of Theorem B.

Here is a conjecture:

Conjecture 15.4. *Suppose and $\xi(a, a) = 1$ for all $a \in \mathbb{Z}_p^\times$. Fix a totally real field K. Let* $\mathrm{Spec}(\mathbb{I})$ *be a primitive irreducible component of* $\mathrm{Spec}(\mathbf{h})$. *If $\alpha/\delta = 1$, we assume that the root number is $\epsilon := \pm 1$ for K. Then,*

1. *if $\alpha/\delta = 1$, we have $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = \frac{1-\epsilon}{2}$ for almost all $P \in \Omega_{\mathbb{I}}$,*
2. *if $\alpha/\delta \neq 1$, we have $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for almost all $P \in \Omega_{\mathbb{I}}$.*

As we remarked after stating Theorem A, if we could prove $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} \equiv \frac{1-\epsilon}{2} \mod 2$ for almost all $P \in \Omega_{\mathbb{I}}$, Conjecture 15.4 (1) holds once we find a good point P_0 with A_{P_0} satisfying the assumptions of Theorem 15.2.

16. p-Local cohomology of formal Lie groups

We prove a technical lemma on Galois cohomology for proving vanishing of the error terms when $l = p$ in Theorem 17.2. Just for finiteness of the error term, as will be explained in the proof of the theorem, it follows from the computation of the universal norm by P. Schneider in [Sc83, Proposition 2 and Lemma 3, §7] and [Sc87, Theorem 1], and therefore, perhaps, for the first reading, the reader may want to skip this section.

Let K be a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. Write $K_s = K[\mu_{p^s}]$ and X^{ur} for the maximal unramified extension of $X = K, K_s$ and $\widehat{X}^{ur}$ is the completion of X^{ur}. Let A be an abelian variety defined over K. Suppose that $\mathrm{End}(A_{/K})$ contains a reduced commutative algebra O_A. Assume

- (A1) $A_{/K_r}$ has semi-stable reduction over the integer ring W_r of K_r;
- (A2) The formal Lie group of the Néron model of A over W_r has a maximal multiplicative factor $\mathcal{A}$ (see [Sc87, §1] for the maximal multiplicative factor);
- (A3) Writing $O_{\mathcal{A}}$ for the p-adic closure of the image of O_A in $\mathrm{End}(\mathcal{A}_{/W_r})$, we have $\mathcal{A} \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{A}$ over $\widehat{W}^{ur}_r$ as formal O_A-modules, where $\mathfrak{A}$ is an $O_{\mathcal{A}}$-lattice in $O_{\mathcal{A}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (i.e., $\mathfrak{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong O_{\mathcal{A}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$) and $\widehat{W}^{ur}_r$ is the p-adic completion of the integer ring W^{ur}_r of K^{ur}_r.

We now study the $\mathrm{Gal}(K_s/K)$-module structure and the cohomology of $\mathcal{A}(W_s)$. The Barsotti-Tate group $\widehat{A}^{\mathrm{ord}}[p^\infty]_{/\mathbb{Q}_p}$ has a filtration $\mathcal{A}[p^\infty] \hookrightarrow \widehat{A}^{\mathrm{ord}}[p^\infty] \twoheadrightarrow \widehat{A}^{\mathrm{ord}}[p^\infty]^{pet}$, where $\widehat{A}^{\mathrm{ord}}[p^\infty]^{pet}$ becomes unramified over $\mathbb{Q}_p[\mu_{p^r}]$. On $T_p\mathcal{A}[p^\infty]$, $\mathrm{Gal}(K_r^{ur}/K^{ur})$ acts by a character $\nu_p\psi$ with values in $O_{\mathcal{A}}^\times$, where ν_p is the p-adic cyclotomic character. The character ψ factors through $\mathrm{Gal}(K_r^{ur}/K^{ur}) \cong \mathrm{Gal}(\mathbb{Q}_p[\mu_{p^r}]/\mathbb{Q}_p)$. Identifying ψ with the corresponding character of $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^r}]/\mathbb{Q}_p)$, we twist the Galois action on the group functor $R \mapsto A(R)$ so that

$$\sigma \cdot x := \psi^{-1}(\sigma|_{\mathbb{Q}[\mu_{p^r}]})\sigma(x) \tag{16.1}$$

for $\mathbb{Q}_p[\mu_{p^r}]$-algebras R, where $\sigma \in \mathrm{Aut}(R_{/\mathbb{Q}_p})$. Since $\psi(\sigma)^{-1} \in \mathrm{Aut}(A_{/\mathbb{Q}_p[\mu_{p^r}]})$ gives a descent datum (see [GME, §1.11.3, (DS2)]), we can twist A by this cocycle, and get another abelian variety $A_{\mu/\mathbb{Q}_p}$ (see [Mi72, (a)]).

Similarly, on $T_pA[p^\infty]^{pet}$, $\mathrm{Gal}(K^{ur}[\mu_{p^r}]/K^{ur}) \cong \mathrm{Gal}(\mathbb{Q}_p[\mu_{p^r}]/\mathbb{Q}_p)$ acts by a character φ with values in $O_A^\times$. Identifying φ with the corresponding character of $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^r}]/\mathbb{Q}_p)$, via the new action $\sigma\cdot x := \varphi^{-1}(\sigma|_{\mathbb{Q}[\mu_{p^r}]})\sigma(x)$, we get another abelian variety $A_{et/\mathbb{Q}_p}$. Thus the Galois action on $A_{et/\mathbb{Q}_p}[p^\infty]^{pet}$ is unramified over $\mathbb{Q}_p$.

For a scheme $X_{/S'}$ and finite flat morphism $S' \to S$, we write $\mathrm{Res}_{S'/S}X$ for the Weil restriction of scalars; so, $\mathrm{Res}_{S'/S}X$ is a scheme over S such that $\mathrm{Res}_{S'/S}X(T) = X(S' \times_S T)$ for all S-schemes T. We describe the twisted abelian variety $A_?$ $(? = \mu, et)$ as a factor of $\mathrm{Res}_{K_r/K}A$. Here is a known facts from [NMD, §7.6]:

- (Res1) If S'/S is finite flat, $\mathrm{Res}_{S'/S}X$ exists [NMD, Thereom 4],
- (Res2) If X is a separated scheme over S, the natural map $X \to \mathrm{Res}_{S'/S}(X \times_S S')$ corresponding to the projection $T \times_S S' \to T$ is a closed immersion [NMD, page 197],
- (Res3) If $X \hookrightarrow Y$ is a closed immersion, then $\mathrm{Res}_{S'/S}X \to \mathrm{Res}_{S'/S}Y$ is a closed immersion,
- (Res4) Let k'/k be a finite extension of fields. If $X_{/k'}$ for a field k' is an abelian scheme with Néron model $\widetilde{X}_{/O'}$ for a discrete valuation ring O' with quotient field k, $\mathrm{Res}_{O'/O}\widetilde{X}$ is the Néron model of $\mathrm{Res}_{k'/k}X$ [NMD, Proposition 6].

Let $\mathrm{Res}_{K_r/K}A$ be the restriction of scalars. Since $A_\mu \cong A \cong A_{et}$ over W_r, we find $\mathrm{Res}_{K_r/K}A \cong \mathrm{Res}_{K_r/K}A_\mu \cong \mathrm{Res}_{K_r/K}A_{et}$. Since $\mathrm{Res}_{K_r/K}A(R) = A(R \otimes_K K_r)$ for each K-algebra R, the inclusion $R \hookrightarrow R \otimes_K K_r$ given by $x \mapsto x \otimes 1$ produces a monomorphism of covariant functors $A(R) \to \mathrm{Res}_{K_r/K}A(R)$; so, we have a morphism of schemes (by Yoneda's lemma), $A \to \mathrm{Res}_{K_r/K}A$. Since A and $\mathrm{Res}_{K_r/K}A$ are projective, we find that $A \hookrightarrow \mathrm{Res}_{K_r/K}A$ is a closed immersion. In the same way, we have another closed immersion $A_\mu \hookrightarrow \mathrm{Res}_{K_r/K}A_\mu \cong \mathrm{Res}_{K_r/K}A$.

Since $K_r \otimes_K K_r \cong \prod_{\sigma\in\mathrm{Gal}(K_r/K)} K_r$ by sending $x \otimes y$ to $(x\sigma(y))_\sigma$, for any variety X defined over K_r, we have $\mathrm{Res}_{K_r/K}X \cong \prod_\sigma X^\sigma$, where $X^\sigma = X \otimes_{K_r,\sigma} K_r$. Thus $\tau \in \mathrm{Gal}(K_r/K)$ acts on $\mathrm{Res}_{K_r/K}X$ by a permutation: $x = (x_\sigma)_\sigma \mapsto \tau\cdot x := (x_{\sigma\tau})_\sigma$, and $\mathrm{Gal}(K_r/K) \hookrightarrow \mathrm{Aut}(\mathrm{Res}_{K_r/K}X)$. Thus $O_A[\mathrm{Gal}(K_r/K)] \subset \mathrm{End}(\mathrm{Res}_{K_r/K}A_\mu)$ by embedding $\mathrm{Gal}(K_r/K)$ in this way. For $x = (x_\sigma)_\sigma \in \mathrm{Res}_{K_r/K}X(\overline{\mathbb{Q}}_p)$, we have $x^\tau = \tau|_{K_r}\cdot(x_\sigma^\tau)_\sigma$. Then the image of A in $\mathrm{Res}_{K_r/K}A_\mu$ is given by $1_\psi(\mathrm{Res}_{K_r/K}A_\mu)$, where $1_\psi = [K_r : K]^{-1}\sum_\sigma \psi^{-1}(\sigma)\sigma \in O_A[\mathrm{Gal}(K_r/K)]$. Since $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ acts on $x \in \mathrm{Res}_{K_r/K}A_\mu$ by $(x_\sigma)_\sigma \mapsto (x_\sigma^\tau)_{\sigma\tau}$, writing the Galois action on A_μ as $x \mapsto x^{\sigma_\mu}$ the action of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ on $x \in A(\overline{\mathbb{Q}}_p)$ is $x \mapsto \psi(\sigma|_{K_r})(\sigma)(x^{\sigma_\mu})$, where $\psi(\sigma|_{K_r})$ is regarded as an automorphism of A_μ. By the same argument, writing the Galois action on A_{et} as $x \mapsto x^{\sigma_{et}}$, the action of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ on $x \in A(\overline{\mathbb{Q}}_p)$ is $x \mapsto \varphi(\sigma|_{K_r})(\sigma)(x^{\sigma_{et}})$. In particular, $A_{et}[p^\infty]^{\text{ét, ord}}(K_\infty^{ur})$ is unramified, and the action of $\mathrm{Gal}(K_\infty/K)$ on ${A_\mu}^{\circ,\mathrm{ord}}[p^\infty](K_\infty^{ur})$ is via the p-adic cyclotomic character. Here ${A_\mu}^{\circ,\mathrm{ord}}$ is the formal Lie group whose Barsotti–Tate group is the potentially connected part of the Barsotti–Tate group of A_μ. This formal Lie group descends to W and is isomorphic $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{A}$ over $\widehat{W}^{ur}$ for the integer ring W of K. Thus we have an identity $\mathcal{A} \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{A}(\psi)$ over $\widehat{W}_r^{ur}$ and $\mathcal{A}[p^\infty] \cong \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathfrak{A}(\psi)$ over W_r^{ur}, where $\mathfrak{A}(\psi) \cong \mathfrak{A}$ as $O_{\mathcal{A}}$-modules on which $\mathrm{Gal}(K_\infty/K)$ acts by ψ. Note that the second identity is valid over W_r^{ur} as this is the identity of Barsotti–Tate groups. From this, we get

Lemma 16.1. *Assume $p > 2$. Let $a \in O_{\mathcal{A}}$ be given by the action of* Frob. *Then we have, for $s \geq r$,*

$$H^1(\mathrm{Gal}(K_s/K), \mathcal{A}[p^s](W_s)) \cong (\mathfrak{A}/(p^{s-1}, \nu_p\psi(\sigma) - 1)\mathfrak{A})[a-1]$$

which is finite and bounded independent of $s \geq r$.

Proof. The Frobenius element Frob acts on $\mathcal{A}[p^s]$ via multiplication by a. Note that, for $s \geq r$

$$\begin{aligned}\mathcal{A}[p^s](W_s) &= (\mu_{p^s}(W_s^{ur}) \otimes_{\mathbb{Z}} \mathfrak{A}(\psi))[a-1] \\ &= \{x \in \mu_{p^s}(W_s^{ur}) \otimes_{\mathbb{Z}} \mathfrak{A}(\psi) | (a-1)x = 0\} \cong (\mathfrak{A}(\psi)/p^s\mathfrak{A}(\psi))[a-1]\end{aligned}$$

as $\mathrm{Gal}(K_\infty/K)$-modules. Since $\mathrm{Gal}(K_\infty/K)$ acts on μ_{p^∞} by ν_p, we conclude

$$H^1(\mathrm{Gal}(K_s/K), \mathcal{A}[p^s](W_s)) \cong (\mathfrak{A}/(p^{s-1}, \nu_p\psi(\sigma_s) - 1)\mathfrak{A})[a-1]$$

as desired. □

17. Finiteness of the p-local error term

We assume (F) and $p > 2$. Here $K_{/\mathbb{Q}_p}$ is a finite extension with p-adic integer ring W. Put $K_s = K[\mu_{p^s}]$ with integer ring W_s.

We studied the Λ-BT group $\mathcal{G}_{1,0,\omega_d}$ associated to the tower $\{X_1(Np^r)\}_r$ in [H14, §5], which is defined over $\mathbb{Z}_p[\mu_{p^\infty}]$. Here $\omega_d(a,d) = \omega(d)$. For the general tower $\{X_r\}_r$ determined by the fixed data (α, δ, ξ), J_r is a factor of $\mathrm{Res}_{F_\xi/\mathbb{Q}} J_1(Np^r)$ again over $\mathbb{Q}[\mu_{p^r}]$ if $r \geq \epsilon$, since $F_\xi \subset \mathbb{Q}[\mu_{p^r}]$. Thus taking the tower of regular model $X_{r/\mathbb{Z}_p[\mu_{p^r}]}$ made out of the regular model $X_1(Np^r)_{/\mathbb{Z}_{(p)}[\mu_{p^r}]}$ (via the corresponding Weil restriction of scalars) and considering $J_{r/\mathbb{Z}_p[\mu_{p^r}]} := \mathrm{Pic}^0_{X_r/\mathbb{Z}_p[\mu_{p^r}]}$, over $\mathbb{Z}_{(p)}[\mu_{p^\infty}]$, $\mathcal{G} = \mathcal{G}_{\alpha,\delta,\xi/\mathbb{Z}_{(p)}[\mu_{p^\infty}]} := J_\infty^{\mathrm{ord}}[p^\infty]_{/\mathbb{Z}_{(p)}[\mu_{p^\infty}]}$ is a Λ-direct factor of $\mathcal{G}_{1,0,\omega_d}^{[F_\xi:\mathbb{Q}]}$ over $\mathbb{Z}_{(p)}[\mu_{p^\infty}]$. Thus $\mathcal{G}_{/\mathbb{Z}_{(p)}[\mu_{p^\infty}]}$ is a Λ-BT group in the sense of [H14, §3] (replacing (CT) by (ct) in [H14, Remark 5.5] if $\xi = \mathbf{1}$). Though it is assumed that $p > 3$ in [H14, §3], the result there is valid for $p = 2, 3$. This is because the ordinary or nearly ordinary part is trivial if $Np \leq 3$ (and the assumption $p > 3$ is imposed to have $Np \geq 4$ for the representability of the elliptic moduli problem). We take its connected component $\mathcal{G}^\circ_{/\mathbb{Z}_p[\mu_{p^\infty}]}$ and put $\mathcal{G}^\circ_{s/\mathbb{Z}_p[\mu_{p^\infty}]} = \mathcal{G}^\circ[\gamma^{p^{s-\epsilon}} - 1]$ which is a connected Barsotti–Tate group defined over $\mathbb{Z}_p[\mu_{p^s}]$. Write $G_{s/\mathbb{Z}_p[\mu_{p^s}]}$ for the formal Lie group associated to the connected Barsotti–Tate group $\mathcal{G}^\circ_{s/\mathbb{Z}_p[\mu_{p^s}]}$ [GME, 1.13.5].

We put $G_{\infty/\mathbb{Z}_p[\mu_{p^\infty}]} = \varprojlim_s G_s$, where the projection $G_{s+1} \to G_s$ is induced by the natural trace map $\pi^s_{s'} : \mathcal{G}_{s'} \to \mathcal{G}_s$ for $s' > s$. We study $\mathrm{Coker}(\widehat{J}_s^{\mathrm{ord}}(K) \xrightarrow{\varpi} \varpi(\widehat{J}_s^{\mathrm{ord}})(K))$. Identify $\widehat{B}_s^{\mathrm{ord}}$ with $\widehat{B}_r^{\mathrm{ord}}$ by π^r_s and $\widehat{A}_s^{\mathrm{ord}}$ with $\widehat{A}_r^{\mathrm{ord}}$ by $\pi^*_{s,r}$. Let $\mathcal{A}_s \cong \mathcal{A}_r$ be the connected formal Lie group over $\mathbb{Z}_p[\mu_{p^s}]$ associated to (the connected component of) the Barsotti–Tate group of $\widehat{A}_s[p^\infty]^{\mathrm{ord}} \cong \widehat{A}_r[p^\infty]^{\mathrm{ord}}$.

We first study $\mathrm{Coker}(G_s(W) \to \varpi(G_s)(W))$. We have an exact sequence of Barsotti–Tate groups over the integral base $\mathbb{Z}_p[\mu_{p^{s'}}]$ [H14, §5]:

$$0 \to \mathcal{A}_s[p^\infty] \to \mathcal{G}_s^\circ \to \mathcal{G}_s^\circ/\mathcal{A}_s[p^\infty] \to 0. \tag{17.1}$$

This produces to the following commutative diagram of formal Lie groups over $\mathbb{Z}_p[\mu_{p^{s'}}]$ with exact rows:

$$\begin{array}{ccccccc} \mathcal{A}_s & \xrightarrow{\hookrightarrow} & G_s & \xrightarrow{\twoheadrightarrow} & G_s/\mathcal{A}_s & = & \varpi(G_s) \\ {\scriptstyle \pi^*_{s,s'}}\downarrow\wr & & {\scriptstyle \pi^*_{s,s'}}\downarrow & & \downarrow & & \\ \mathcal{A}_{s'} & \xrightarrow{\hookrightarrow} & G_{s'} & \xrightarrow{\twoheadrightarrow} & G_{s'}/\mathcal{A}_{s'} & = & \varpi(G_{s'}). \end{array}$$

Since $\mathcal{G}_s^\circ/\mathcal{A}_s[p^\infty]$ is a Barsotti–Tate group over W_s by [H14, Theorem 5.4], $G_s/\mathcal{A}_s$ is a smooth formal group over W_s (e.g., [Sc87, Lemma 1]). Thus $G_s \cong (G_s/\mathcal{A}_s) \times_{W_s} \mathcal{A}_s$ as formal schemes (but not necessarily as formal groups). Anyway, this shows that $G_s(W_{s'}) \to \varpi(G_s)(W_{s'})$ is surjective for all $s' \geq s$. Therefore, we get an exact sequence

$$0 \to \mathcal{A}_s(W_{s'}) \to G_s(W_{s'}) \to \varpi(G_s)(W_{s'}) \to 0 \quad \text{for all } s' \geq s \geq r \text{ including } s' = \infty. \tag{17.2}$$

Since $\mathrm{Gal}(\overline{K}/K_r^{ur})$ acts on $\mathcal{A}_s$ by the p-adic cyclotomic character, we find $\mathcal{A}_s \cong \widehat{\mathbb{G}}_m^d$ over $\widehat{W}_r^{ur}$ for $d = \dim \mathcal{A}_r$. In Corollary in the introduction of [O00] (see also [H13a, Lemma 4.2]), Ohta shows that $T\mathcal{G}^\circ := \varprojlim_s T_p\mathcal{G}_s^\circ \cong \mathbf{h}$ (and hence $T\mathcal{G}_{\mathbb{T}}^\circ \cong \mathbb{T}$) canonically as $\mathbf{h}$-modules. Assuming (F), we have $T_p\mathcal{G}_s^\circ \cong \mathbf{h}_s$; so, $G_s \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{h}_s$ over $\widehat{W}_s^{ur}$. Define $\mathfrak{A} \subset \mathbf{h}_s$ by the annihilator of $G_s/\mathcal{A}_s$ and $\mathfrak{B} := \mathrm{Ker}(\mathbf{h}_s \to \mathrm{End}(A_{/K_s})$. Hence we have an exact sequence of formal groups:

$$0 \to \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{A} \to G_s \xrightarrow{\varpi} G_s \to \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{h}_s/\mathfrak{B} \to 0 \tag{17.3}$$

since $0 \to \mathfrak{A} \to \mathbf{h}_s \xrightarrow{\varpi} \mathbf{h}_s \to \mathbf{h}_s/\mathfrak{B} \to 0$ is an exact sequence of $\mathbb{Z}_p$-free modules. Thus we have $\mathcal{A}_s \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{A}$ over $\widehat{W}_r^{ur}$, and $\mathfrak{A}$ is an $\mathbf{h}_s$-ideal and is an $O_{\mathcal{A}_s}$-module. This shows that A_s, B_s, $\varpi(\widehat{J}_s^{\mathrm{ord}}) := \widehat{J}_s^{\mathrm{ord}}/\widehat{A}_s^{\mathrm{ord}}$ and J_s all satisfy (A1–3) in Section 16.

The action of the Frobenius $[p : \mathbb{Q}_p]$ on $\mathcal{A}_s[p^\infty](\overline{\mathbb{Q}}_p)$ is the multiplication by $a_p^{-1} \in O_{\mathcal{A}_s}^\times$ (where a_p is the image of $U(p)$ in $O_{\mathcal{A}_s}$). Thus $\mathcal{A}_s(\widehat{W}_s^{ur}) = \mathfrak{A} \otimes \widehat{\mathbb{G}}_m(\widehat{W}_s^{ur}) \cong \mathfrak{A} \otimes_{\mathbb{Z}_p} (1+\mathfrak{m}_{\widehat{W}_s^{ur}})$ on which the natural Galois action on $\widehat{\mathbb{G}}_m(\widehat{W}_s^{ur})$ is twisted by a character $\psi : \mathrm{Gal}(K_r/K) \cong \mathrm{Gal}(\widehat{K}_r^{ur}/\widehat{K}^{ur}) \to O_{\mathcal{A}_s}^\times$ induced by the nearly ordinary character ψ sending $[z, \mathbb{Q}_p]$ ($z \in \mathbb{Z}_p^\times$) to the image in $O_{\mathcal{A}_s}^\times$ of the Hecke operator in $\mathbf{h}_r$ of the class of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)$ in $\widehat{\Gamma}_r^0/\widehat{\Gamma}_r$. Write simply A for the abelian variety A_s. Let $O_A := \mathrm{End}(A_{/\mathbb{Q}})$, which is an order of the Hecke algebra generated over $\mathbb{Q}$ by Hecke operators $T(n)$ in $\mathrm{End}^0(A_{/\mathbb{Q}}) = \mathrm{End}(A_{/\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Recall the Galois representation ρ_A of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ realized on $T_p\widehat{A}_r^{\mathrm{ord}}$. Take the connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ such that $\mathbf{h}/\varpi\mathbf{h} = \mathbb{T}/\varpi\mathbb{T}$. Write symbolically $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)} \cong \begin{pmatrix} \nu_p\psi & * \\ 0 & \varphi \end{pmatrix}$ and $\rho_{\mathbb{T}} = \begin{pmatrix} \nu_p\boldsymbol{\psi} & * \\ 0 & \boldsymbol{\varphi} \end{pmatrix}$ for a deformation $\boldsymbol{\psi} : \mathrm{Gal}(K_\infty^{ur}/K) \to \mathbb{T}^\times$ of ψ. Here $\nu_p\psi$ and $\nu_p\boldsymbol{\psi}$ acts on $T_p\mathcal{A}_r[p^\infty]$ and on $T\mathcal{G}_{\mathbb{T}}^\circ$, respectively. Thus φ (resp. $\boldsymbol{\varphi}$) gives the action on $T_p\widehat{A}_r^{\mathrm{ord}}/T_p\mathcal{A}_r$ (resp. on $T\mathcal{G}_{\mathbb{T}}^{pet}$). Note that $T\mathcal{G}_{\mathbb{T}}^{pet} \cong \mathrm{Hom}_\Lambda(\mathbb{T}, \Lambda)$ as $\mathbb{T}$-modules (so, if $\mathbb{T}$ is Gorenstein, the above form $\rho_{\mathbb{T}}$ of 2×2 matrix is literally true). We write $\mathrm{Frob} \in \mathrm{Gal}(K_\infty^{ur}/K)$ for the Frobenius element inducing the generator of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F})$ and an appropriate power of the identity $\mathrm{id} = [p, \mathbb{Q}_p]$ on K_∞/K.

Proposition 17.1. *Suppose* (F). *Let $\mathcal{G}^\circ$ be the connected component of $\mathcal{G} = \mathcal{G}_{\alpha,\delta,\xi}$, and take a generator σ of* $\mathrm{Gal}(K_\infty/K)$. *Then we have $H^1(K_\infty/K, \mathcal{G}_{s,\mathbb{T}}^\circ(W_\infty)) = (\mathbb{T}_s/(\nu_p\boldsymbol{\psi}(\sigma)-1)\mathbb{T}_s)[\boldsymbol{\varphi}(\mathrm{Frob})-1]$. If either $|\nu_p\psi(\sigma)-1|_p = 1$ or $|\varphi(\mathrm{Frob})-1|_p = 1$, then we have the vanishing $H^1(K_\infty/K, \mathcal{G}_{\mathbb{T}}^\circ(W_\infty)) = 0$.*

Proof. As we saw, under (F), we have $\mathcal{G}_{s,\mathbb{T}}^\circ(W_\infty^{ur}) \cong \mu_{p^\infty}(W_\infty^{ur}) \otimes_{\mathbb{Z}_p} \mathbb{T}_s(\boldsymbol{\psi})$ as $\mathrm{Gal}(K_\infty^{ur}/K)$-modules, where $\mathrm{Gal}(K_\infty^{ur}/K)$ acts on $\mathbb{T}_s(\psi) \cong \mathbb{T}_s$ by $\boldsymbol{\psi}$. We apply Lemma 16.1 to the formal Lie group $\mathcal{A}$ with $\mathcal{A}[p^\infty] = \mathcal{G}_{s,\mathbb{T}}^\circ$. Note that a in the Lemma is the image of $\boldsymbol{\varphi}(\mathrm{Frob})$ in $O_{\mathcal{A}}$ by [H14, (6-1)]. From this, the cohomology of $\mathcal{G}_s$ vanishes if either $|\nu_p\psi(\sigma) - 1|_p = 1$ or $|\varphi(\mathrm{Frob}) - 1|_p = 1$. We have then $H^1(K_\infty/K, \mathcal{G}^\circ(W_\infty)) = \varinjlim_s H^1(K_\infty/K, \mathcal{G}_s^\circ(W_\infty)) = 0$. □

Theorem 17.2. *Let the notation be as in* Theorem 10.4. *Let K be a finite extension of $\mathbb{Q}_p$ for $p > 2$, and put $K_s = K[\mu_{p^s}]$ $(s = 1, 2, \ldots, \infty)$. If A_r does not have split multiplicative reduction over W_r, then the error term $E^\infty(K)_{\mathbb{T}}$ is finite. If further A_r has good reduction over $W_1 = W[\mu_p]$ with $|\varphi(\mathrm{Frob}) - 1|_p = 1$, then $E^\infty(K)_{\mathbb{T}}$ vanishes.*

Proof. Let us first sketch the proof. As before, we write symbolically $\varpi(J_s)$ for the abelian variety quotient J_s/A_s, since $\widehat{J_s/A_s}^{\mathrm{ord}} = \widehat{J}_s^{\mathrm{ord}}/\widehat{A}_s^{\mathrm{ord}} = \varpi(\widehat{J}_s^{\mathrm{ord}})$ by definition. Thus $A_s(F) \hookrightarrow J_s(F) \xrightarrow{\varpi} \varpi(J_s)(F)$ is exact for any algebraic extension $F_{/K}$, and hence $\widehat{A}_s^{\mathrm{ord}}(F) \hookrightarrow J_{s,\mathbb{T}}^{\mathrm{ord}}(F) \xrightarrow{\varpi} \varpi(J_{s,\mathbb{T}}^{\mathrm{ord}})(F)$ is exact. We first assume that $J_{s,\mathbb{T}}^{\mathrm{ord}}$ is contained in an abelian subvariety of J_s having good reduction over W_∞ (so, we may assume that the subabelian variety has good reduction over W_s). Then the sequence

$$0 \to A_s[p^\infty]^{\mathrm{ord}} \to \mathcal{G}_s \to \varpi(\mathcal{G}_s) \to 0 \tag{17.4}$$

is exact as Barsotti–Tate groups over W_s (see [H14, §5] and a remark after Corollary 6.4). Since the complex of Néron models $A_{s/W_s} \to J_{s/W_s} \to \varpi(J_{s/W_s})$ is exact up to p-finite errors [NMD, Proposition 7.5.3], the exactness of (17.4) shows the sequence $\widehat{A}_{s/W_s}^{\mathrm{ord}} \hookrightarrow \widehat{J}_{s/W_s}^{\mathrm{ord}} \twoheadrightarrow \varpi(J_{s/W_s}^{\mathrm{ord}})$ is exact as fppf sheaves over W_s. Since $0 \to \widehat{A}^{\mathrm{ord}}_{s/W_s}(\mathbb{F}) \to \widehat{J}^{\mathrm{ord}}_{s/W_s}(\mathbb{F}) \to \varpi(J^{\mathrm{ord}}_{s/W_s})(\mathbb{F}) \to H^1(\mathbb{F}, \widehat{A}^{\mathrm{ord}}) = 0$ is exact, (17.2) shows that $\widehat{J}^{\mathrm{ord}}_s(K_\infty) \twoheadrightarrow \varpi(\widehat{J}^{\mathrm{ord}}_s)(K_\infty)$ is onto.

By (15.1), we have

$$|H^1(K_\infty/K_r, \widehat{A}^{\mathrm{ord}}_r(K_\infty))| = |A_r[p^\infty]^{\mathrm{ord}}(\mathbb{F})|^2.$$

Since we have an exact sequence $\widehat{A}^{\mathrm{ord}}_r(K_\infty) \hookrightarrow \widehat{J}^{\mathrm{ord}}_{s,\mathbb{T}}(K_\infty) \twoheadrightarrow \varpi(\widehat{J}^{\mathrm{ord}}_{s,\mathbb{T}}(K_\infty))$, by cohomology sequence of this short exact sequence, we have the claimed finiteness.

Let us now sketch the proof in the non-split multiplicative case (over W). We have a similar exact sequence of the formal Lie groups, and applying the formal version [Sc87, Theorem 1] (particularly in the non-split multiplicative case), we get the finiteness for the connected part. The surjectivity (up to finite error) for the special fiber (of Néron models) will be shown below. Thus if A_r has either good or non-split multiplicative reduction over W_r, we still have finiteness of $H^1(K_\infty/K_r, \widehat{A}^{\mathrm{ord}}_r(K_\infty))$ as above. Then by the inflation-restriction exact sequence:

$$\begin{aligned} &H^1(K_r/K, \widehat{A}^{\mathrm{ord}}_r(K_r)) \to H^1(K_\infty/K, \widehat{A}^{\mathrm{ord}}_r(K_\infty)) \\ &\qquad\to H^0(K_r/K, H^1(K_\infty/K_r, \widehat{A}^{\mathrm{ord}}_r(K_\infty)) \to H^2(K_r/K, \widehat{A}^{\mathrm{ord}}_r(K_r)),\end{aligned}$$

finiteness of $H^j(K_r/K, \widehat{A}^{\mathrm{ord}}_r(K_r))$ $(j = 1, 2)$ and $H^1(K_\infty/K_r, \widehat{A}^{\mathrm{ord}}_r(K_\infty))$ tells us finiteness of the cohomology $H^1(K_\infty/K, \widehat{A}^{\mathrm{ord}}_r(K_\infty))$, from which we conclude the finiteness of $E^\infty(K)_{\mathbb{T}}$. If $r = 1$, $p \nmid [K_1 : K]$ and

$$H^q(K_1/K, \widehat{A}^{\mathrm{ord}}_r(K_1)) = 0 \quad \text{for } q > 0.$$

Then, still assuming $r = 1$, we conclude

$$H^1(K_\infty/K, \widehat{A}^{\mathrm{ord}}_r(K_\infty)) \cong H^0(K_r/K, H^1(K_\infty/K_r, \widehat{A}^{\mathrm{ord}}_r(K_\infty))) \tag{17.5}$$

If in addition $|\varphi(\mathrm{Frob}) - 1|_p = 1$ and A_r has good reduction over W_r, from $|H^1(K_\infty/K_r, \widehat{A}^{\mathrm{ord}}_r(K_\infty))| = |A_r[p^\infty](\mathbb{F})|^2 = 0$, the groups in (17.5) vanish so, $E^\infty(K)_{\mathbb{T}} = 0$. In any case, $H^1(K_\infty/K, \widehat{A}^{\mathrm{ord}}_r(K_\infty))$ is finite. Similarly, by the formal group version [Sc87, Theorem 1], we conclude the finiteness of $H^1(K_\infty/K, \mathcal{A}_r(W_\infty))$.

We now give details of the proof in the general case. We first look into the identity connected components over W_∞. By (17.2),

$$0 \to \mathcal{A}_r(W_\infty) \to G_\infty(W_\infty) \to \varpi(G_\infty)(W_\infty) \to 0$$

is exact. Taking its Galois cohomology sequence, we get another exact sequence

$$0 \to \mathcal{A}_r(W) \to G_\infty(W) \xrightarrow{\varpi_\infty} \varpi(G_\infty)(W) \to H^1(K_\infty/K, \mathcal{A}_r(W_\infty)).$$

Since the cohomology group $H^1(K_\infty/K, \mathcal{A}_r(W_\infty))$ is finite (cf. [Sc87, Theorem 1]), we find that

$$\mathrm{Coker}(G_\infty(W_\infty)^{\mathrm{Gal}(K_\infty/K)} \xrightarrow{\widehat{\varpi}_\infty} \varpi(G_\infty)(W_\infty)^{\mathrm{Gal}(K_\infty/K)})$$

is finite.

As for the special fiber (of the Néron models), we have the exact sequence:

$$0 \to \widehat{A}^{\mathrm{ord}}_r(\mathbb{F}) \to \widehat{J}^{\mathrm{ord}}_s(\mathbb{F}) \to \varpi(\widehat{J}^{\mathrm{ord}}_s)(\mathbb{F}) \to H^1(\mathbb{F}, \widehat{A}^{\mathrm{ord}}_r).$$

If $\varphi(\mathrm{Frob}) \neq \pm 1$, A_r has good reduction (not just semi-stable one) over W_r; so, by Lang's theorem [L56], $H^1(\mathbb{F}, \widehat{A}^{\mathrm{ord}}_r) \subset H^1(\mathbb{F}, A_r) = 0$. Even if $\varphi(\mathrm{Frob}) = \pm 1$, from the exact sequence $0 \to A^0_r(\overline{\mathbb{F}}_p) \to A_r(\overline{\mathbb{F}}_p) \to \pi_0(A_{r/\overline{\mathbb{F}}_p}) \to 0$ for the connected component A^0 of $A_{/\overline{\mathbb{F}}_p}$, we find $H^1(\mathbb{F}, A_r) \cong H^1(\mathbb{F}, \pi_0(A_r))$ as $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F})$ has cohomological dimension 1. Thus $H^1(\mathbb{F}, \widehat{A}^{\mathrm{ord}}_r)$ is finite. After passing to the limit, we find

$$|\,\mathrm{Coker}(J^{\mathrm{ord}}_\infty(\mathbb{F}) \xrightarrow{\widetilde{\varpi}_\infty} \varpi(J^{\mathrm{ord}}_\infty)(\mathbb{F}))| \le |\pi_0(A_{r/\overline{\mathbb{F}}_p})| < \infty.$$

We have the following exact sequence:

$$0 \to \mathcal{G}^\circ(W_\infty)_{\mathbb{T}} \to \mathcal{G}(K_\infty)_{\mathbb{T}} \xrightarrow{\mathrm{red}} J_\infty(\mathbb{F})[p^\infty]^{\mathrm{ord}}_{\mathbb{T}} \to 0.$$

Indeed, the maximal étale quotient $\mathcal{G}^{\text{ét}}$ of $\mathcal{G}_{/W_\infty}$ is a Λ-BT group by [H14, Proposition 6.3]; so, its closed points lifts to a W_∞-point as W_∞ is henselian. (Note that $\mathcal{G}^{\text{ét}}_{/W_\infty}$ may not be an étale Barsotti–Tate group for finite s.) Taking the fixed point of $\mathrm{Gal}(K^{ur}_\infty/K)$, we have

$$0 \to \mathcal{G}^\circ(W^{ur}_\infty)^{\mathrm{Gal}(K^{ur}_\infty/K)}_{\mathbb{T}} \to \mathcal{G}(K)_{\mathbb{T}} \xrightarrow{\text{red}} J_\infty(\mathbb{F})[p^\infty]^{\text{ord}}_{\mathbb{T}} \to H^1(K^{ur}_\infty/K, \mathcal{G}^\circ(W_\infty)_{\mathbb{T}}).$$

Then by Proposition 17.1, $\mathrm{Coker}(\mathcal{G}(K)_{\mathbb{T}} \xrightarrow{\text{red}} J_\infty(\mathbb{F})[p^\infty]^{\text{ord}}_{\mathbb{T}}) = 0$ (assuming either $|\varphi(\mathrm{Frob})-1|_p = 1$ or $|\psi\nu_p(\sigma)-1|_p = 1$), and in particular, $\mathrm{Coker}(J^{\text{ord}}_\infty(K)_{\mathbb{T}} \xrightarrow{\text{red}} J_\infty(\mathbb{F})[p^\infty]^{\text{ord}}_{\mathbb{T}}) = 0$.

We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
G_\infty(W_\infty)^{\mathrm{Gal}(K_\infty/K)}_{\mathbb{T}} & \xrightarrow{\hookrightarrow} & J^{\text{ord}}_\infty(K)_{\mathbb{T}} & \xrightarrow{\text{red}_J} & J_\infty(\mathbb{F})[p^\infty]^{\text{ord}}_{\mathbb{T}} \\
\widehat{\varpi}_\infty\downarrow & & \varpi_\infty\downarrow & & \downarrow\widetilde{\varpi}_\infty \\
\varpi(G_\infty)(W_\infty)^{\mathrm{Gal}(K_\infty/K)}_{\mathbb{T}} & \xrightarrow{\hookrightarrow} & \varpi(J^{\text{ord}}_\infty)(K)_{\mathbb{T}} & \longrightarrow & \varpi(J_\infty)(\mathbb{F})[p^\infty]^{\text{ord}}_{\mathbb{T}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Coker}(\widehat{\varpi}_\infty) & \longrightarrow & \mathrm{Coker}(\varpi_\infty) & \longrightarrow & \mathrm{Coker}(\widetilde{\varpi}_\infty).
\end{array} \tag{17.6}$$

For the Frobenius endomorphism ϕ $(= \boldsymbol{\varphi}(\mathrm{Frob}))$, we have

$$J_\infty(\mathbb{F})[p^\infty]^{\text{ord}}_{\mathbb{T}} = J_\infty(\overline{\mathbb{F}}_p)[p^\infty]^{\text{ord}}_{\mathbb{T}}[\phi-1].$$

Since $\phi \equiv \varphi(\mathrm{Frob}) \mod \mathfrak{m}_{\mathbb{T}}$, if $\varphi(\mathrm{Frob}) \not\equiv 1 \mod \mathfrak{m}_W$ $(\Leftrightarrow |\varphi(\mathrm{Frob})-1|_p = 1)$, $J_\infty(\mathbb{F})[p^\infty]^{\text{ord}}_{\mathbb{T}} = 0$, and $\mathrm{Coker}(\widehat{\varpi}_\infty) \to \mathrm{Coker}(\varpi_\infty) \to 0$ is exact; so, $\mathrm{Coker}(\varpi_\infty)$ is finite.

We need to argue more if $|\varphi(\mathrm{Frob})-1|_p < 1$. We apply $X \mapsto X^\vee := \mathrm{Hom}(X, \mathbb{Q}_p/\mathbb{Z}_p)$ to the above diagram. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is $\mathbb{Z}_p$-injective, $X \mapsto X^\vee$ is an exact contravariant functor, all arrows of (17.6) are reversed, but exactness is kept. Since $\mathrm{Coker}(\varpi_\infty) \hookrightarrow H^1(K, A_r)_p$, its Pontryagin dual module $\mathrm{Coker}(\varpi_\infty)^\vee$ is a $\mathbb{Z}_p$-module of finite type. Since this module killed by arithmetic prime (ϖ), we need to show the vanishing of the (ϖ)-localization $\mathrm{Coker}(\varpi_\infty)^\vee_{(\varpi)} = 0$. Note that we have a surjective morphism of Λ-module: $\mathrm{Coker}(\text{red}) \twoheadrightarrow \mathrm{Coker}(\text{red}_J)$ and that $\mathrm{Coker}(\text{red})$ is killed by $(\nu_p\boldsymbol{\psi}(\sigma)-1)|(\gamma t-1)$ by Proposition 17.1. Since (ϖ) is prime to $\gamma t-1$, we have the vanishing of the localization $\mathrm{Coker}(\text{red}_J)^\vee_{(\varpi)} = 0$. From the diagram obtained by applying $X \mapsto X^\vee$, the localized sequence $0 = \mathrm{Coker}(\widetilde{\varpi}_\infty)^\vee_{(\varpi)} \to \mathrm{Coker}(\varpi_\infty)^\vee_{(\varpi)} \to \mathrm{Coker}(\widehat{\varpi}_\infty)^\vee_{(\varpi)}$ is exact. Since $\mathrm{Coker}(\widehat{\varpi}_\infty)^\vee$ is finite under $\psi(\sigma) \neq 1$ and $\varphi(\mathrm{Frob}) \neq 1$, we conclude $\mathrm{Coker}(\widehat{\varpi}_\infty)^\vee_{(\varpi)} = 0$; so, $\mathrm{Coker}(\varpi_\infty)^\vee_{(\varpi)} = 0$ by the finiteness of $\mathrm{Coker}(\widehat{\varpi}_\infty)$. Since $\mathrm{Coker}(\varpi_\infty)^\vee_{(\varpi)} = 0$ is finite $\mathbb{Z}_p$-module, dualizing back, this shows finiteness of $\mathrm{Coker}(\varpi_\infty)$ as desired. □

18. Twisted family

We briefly describe, in down-to-earth terms, how to create the p-adic analytic family of modular forms associated to an irreducible component of $\mathbf{h}_{\alpha,\delta,\xi}$ from a p-ordinary family coming from an irreducible component of $\mathrm{Spec}(\mathbf{h}_{0,1,\phi_{ord}})$. We show that as a Λ-algebra $\mathbf{h}^{\text{ord}} := \mathbf{h}_{0,1,\phi_{ord}}$ is isomorphic to $\mathbf{h}_{\alpha,\delta,\xi}$ (for a specific choice ξ depending on ϕ) by $T(l) \mapsto \varphi(l_l)T(l)$ regarding l as an idele l_l in $(\mathbb{A}^{(p\infty)})^\times$ supported on $\mathbb{Q}_l^\times$. Here φ is a suitably chosen character of $(\mathbb{A}^{(\infty)})^\times/\mathbb{Q}^\times$ with values in $\Lambda^\times$. For simplicity, we assume $p > 2$ and that $\phi_{ord} = \mathbf{1}$ which implies $\xi = \mathbf{1}$ (leaving the general case to attentive readers). Recall open compact subgroups $\widehat{\Gamma}_0(Np^r)$ in (3.2).

Let $U(N) := \widehat{\mathbb{Z}}^\times \cap (1 + N\widehat{\mathbb{Z}})$ for $\widehat{\mathbb{Z}} = \prod_l \mathbb{Z}_l = \varprojlim_N \mathbb{Z}/N\mathbb{Z}$, where l runs over all primes and the projective limit is with respect to the divisibility order. By the isomorphism $\mathbb{A}^\times/\mathbb{Q}^\times U(N)\mathbb{R}_+^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times$ sending primes $l \in \mathbb{Q}_l^\times \hookrightarrow \mathbb{A}^\times$ outside N to the class of $(l \mod N)$, we regard a Dirichlet character $\phi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ as an idele character. We use the symbol $\phi_{\mathbb{A}}$ to denote the idele character lifted from ϕ. Let $\mathcal{S}(\widehat{\Gamma}_0(Np^s), \varepsilon, \epsilon) = \mathcal{S}_2(\widehat{\Gamma}_0(Np^s), \varepsilon, \epsilon)$ for a character $\varepsilon, \epsilon : \mathbb{Z}_p^\times \to \mu_{p^{s-1}} \times \mu_{p-1}$ be the space of cusp forms $\mathbf{f} : \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ satisfying the following four properties:

(S1) $f(g\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \varepsilon_{\mathbb{A}}(a_p)\epsilon_{\mathbb{A}}(d_p)f(x)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(Np^s)$;

(S2) $\mathbf{f}(gz) = \varepsilon_{\mathbb{A}}\epsilon_{\mathbb{A}}(z)f(g)$ for $z \in \mathbb{A}^\times$ (regarded as a scalar matrix in $\mathrm{GL}_2(\mathbb{A})$;

(S3) $\mathbf{f}(gr) = \mathbf{f}(g)j(r,i)^{-2}$ with $i = \sqrt{-1}$ for $r \in \mathrm{SO}_2(\mathbb{R})$, where $\tau \in \mathfrak{H}$ (the upper half complex plane) and $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = |ad - bc|^{-1/2}(c\tau + d)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$;

(S4) the function $\mathfrak{H} \ni \tau = g_\infty(i) \mapsto \mathbf{f}(g^{(\infty)}g_\infty)j(g_\infty, i)^2 \in \mathbb{C}$ induced on the upper half plane for a fixed finite part $g^{(\infty)}$ is a holomorphic function $\mathbf{f}_{g^{(\infty)}} : \mathfrak{H} \to \mathbb{C}$ for each choice of $g^{(\infty)}$.

Under the above three conditions (S1–3), fixing the finite part $g^{(\infty)}$, (S4) means that the value $\mathbf{f}(g^{(\infty)}g_\infty)j(g_\infty, i)^2$ only depends on $\tau = g_\infty(i)$ (as easily verified; see for example [MFG, §3.1.5]), and we get a function $\mathbf{f}_{g^{(\infty)}}(\tau) := \mathbf{f}(g^{(\infty)}g_\infty)j(g_\infty, i)^2$, which is required to be holomorphic in τ.

The space of cusp forms associated to $\mathbf{h}_{\alpha,\delta,\mathbf{1}}$ of level Np^s is given by $\mathcal{S}(\widehat{\Gamma}_0(Np^s), \varepsilon, \epsilon)$ with (ε, ϵ) satisfying $\varepsilon^\delta \epsilon^\alpha = 1$. As in (S1–4), ε_p and ϵ_p has values in $\mu_{p^{s-1}}$. Then (S1) implies that $\mathbf{f}$ is right $\widehat{\Gamma}_s$-invariant. In other words, putting $\mathcal{S}(\widehat{\Gamma}_s) := \bigoplus_{\varepsilon,\epsilon} \mathcal{S}(\widehat{\Gamma}_0(Np^s), \varepsilon, \epsilon)$, $\mathcal{S}(\widehat{\Gamma}_s)$ is made of cusp forms satisfying the following condition:

(s1) $f(g\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = f(x)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}_s$,

in addition to (S3) and (S4). Since $X_s - \{\text{cusps}\} \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\widehat{\Gamma}_s\mathrm{SO}_2(\mathbb{R})$, by sending $\mathbf{f} \to \mathbf{f}_1(\tau)d\tau \in H^0(X_s, \Omega_{X_s/\mathbb{C}})$, we get an isomorphism

$$\mathcal{S}(\widehat{\Gamma}_s) \cong H^0(X_s, \Omega_{X_s/\mathbb{C}}).$$

As explained in Section 3, X_s is canonically defined over $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$ as a moduli space classifying ellptic curves with level structure described there, for any $\mathbb{Z}_{(p)}$-algebra A, we have well defined $H^0(X_s, \Omega_{X_s/A})$. Pulling back this integral structure to $\mathcal{S}(\widehat{\Gamma}_s)$, we get the corresponding A-integral space $\mathcal{S}(\widehat{\Gamma}_s, A)$.

Let $\kappa : \mathbb{Z}_p^\times = (1 + p\mathbb{Z}_p) \times \mu_{p-1} \to \mathbb{Z}_p[[T]]^\times$ be the Λ-valued character sending $(1+p)^z \in 1 + p\mathbb{Z}_p$ to $t^z := (1+T)^z = \sum_{n \geq 0} \binom{z}{n} T^n$ and the entire μ_{p-1} to the identity 1. Then we have

$$\{(\varepsilon, \epsilon) : \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mu_{p^{s-1}} | \varepsilon^\delta \epsilon^\alpha = 1\} = \{(\kappa^\alpha, \kappa^{-\delta}) \mod (t - \zeta) : \zeta \in \mu_{p^{s-1}}\}.$$

Writing $\kappa_\zeta := \kappa \mod (t - \zeta)$ and taking $(\varepsilon, \epsilon) = (\kappa_\zeta^\alpha, \kappa_\zeta^{-\delta})$, (S1-2) is summarized into

$$\mathbf{f}(zxu) = \kappa_{\zeta,\mathbb{A}}^{\alpha-\delta}(z)\kappa_{\zeta,\mathbb{A}}(a_p^\alpha d_p^{-\delta})\mathbf{f}(x)$$

for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}_0(Np^r)$ and z in the center $Z(\mathbb{A})$ of $\mathrm{GL}_2(\mathbb{A})$.

Consider the space $S_k(\Gamma_0(Np^s), \phi)$ of classical holomorphic cusp firms with Neben character ϕ for $\phi : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$. A modular form $f \in S_k(\Gamma_0(Np^s), \phi)$ satisfies

$$f(\frac{az+b}{cz+d}) = \phi(a \bmod p^r)f(x)(cz+d)^k$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(Np^s)$. Since $ad \equiv 1 \mod p^s$, $\kappa_{\zeta,\mathbb{A}}(a_p^\alpha d_p^{-\delta}) = \kappa_\zeta(a^{\alpha+\delta}) = \kappa_\zeta(d^{-\alpha-\delta})$, and hence $\mathbf{f} \mapsto \mathbf{f}_1$ induces an isomorphism of vector spaces

$$\mathcal{S}(\widehat{\Gamma}_0(Np^s), \kappa_\zeta^\alpha, \kappa_\zeta^{-\delta}) \cong S_2(\Gamma_0(Np^s), \kappa_\zeta^{\alpha+\delta}),$$

but unless $(\alpha, \delta, \mathbf{1}) = (0, 1, \mathbf{1})$, the Hecke operator action on the left-hand-side is a twisted one of the right-hand-side depending on the data $(\alpha, \delta, \mathbf{1})$.

Here is a description of the twist. For a character $\varphi : \mathbb{Z}_p^\times \to \mu_{p^{s-1}}$ and $\mathbf{f} \in \mathcal{S}(\widehat{\Gamma}_0(Np^s), \mathbf{1}, \phi)$, define $\mathbf{f} \otimes \varphi \in \mathcal{S}(\widehat{\Gamma}_0(Np^s), \varphi_\mathbb{A}, \phi\varphi_\mathbb{A})$ by $\mathbf{f} \otimes \varphi(g) = \varphi_\mathbb{A}(\det(g))\mathbf{f}(g)$. Thus we get an isomorphism $\otimes\varphi : \mathcal{S}(\widehat{\Gamma}_0(Np^s), \mathbf{1}, \phi) \cong \mathcal{S}(\widehat{\Gamma}_0(Np^s), \varphi_\mathbb{A}, \varphi_\mathbb{A}\phi)$. By definition, we get the following fact.

Lemma 18.1. *If $\mathbf{f}$ as above satisfies $\mathbf{f}|T(n) = \lambda(T(n))\mathbf{f}$ (a Hecke eigenform) with $T(l) = U(l)$ if $l|Np$, we have $(\mathbf{f} \otimes \varphi)|T(l) = \varphi_\mathbb{A}(l_l)\lambda(T(l))(\mathbf{f} \otimes \varphi)$ for all primes l prime to p, and for $U(p)$, we have $(\mathbf{f} \otimes \varphi)|U(p) = \lambda(U(p))(\mathbf{f} \otimes \varphi)$. Thus this operation $\mathbf{f} \mapsto \mathbf{f} \otimes \varphi$ preserves "ordinarity".*

Here l_l is an idele whose l-component is l but is trivial outside l; so, $\varphi_\mathbb{A}(l_l) = \varphi(l)$ as long as $l \neq p$. The formula $(\mathbf{f} \otimes \varphi)|U(p) = \lambda(U(p))(\mathbf{f} \otimes \varphi)$ is consistent with $(\mathbf{f} \otimes \varphi)|T(l) = \varphi_\mathbb{A}(l_l)\lambda(T(l))(\mathbf{f} \otimes \varphi)$ because φ factors through the p-adic cyclotomic character whose value at p is equal to 1 (i.e., $\varphi_\mathbb{A}(p_p) = 1$).

By the lemma, out of the abelian subvariety $A_{\mathbf{f}}$ attached to $\mathbf{f}$, we get an abelian variety $A_{\mathbf{f}\otimes\varphi}$ of J_s (for a suitable choice of (α,δ)) which is the φ-twist of $A_{\mathbf{f}}$ (see (16.1)); i.e., we have an identity of l-adic Tate modules $T_lA_{\mathbf{f}\otimes\varphi} \cong (T_lA_{\mathbf{f}})\otimes\varphi$ as Galois modules regarding φ as a Galois character via $\mathbb{Z}_p^\times = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$.

If $\mathbf{f}$ is a Hecke eigenform, the modular form $\mathbf{f} : \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ and its right translations $R(g)(\mathbf{f})(x) = \mathbf{f}(xg) = \mathbf{f}(xg)$ for $g \in \mathrm{GL}_2(\mathbb{A})$ generate an irreducible automorphic representation $\pi = \pi_f$ of $\mathrm{GL}_2(\mathbb{A})$. Similarly, the modular form $\mathbf{f}\otimes\varphi : \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ and its right translation $R(g)(\mathbf{f}\otimes\varphi)(x) = (\mathbf{f}\otimes\varphi)(xg) = \mathbf{f}(xg)\varphi_{\mathbb{A}}(\det(xg))$ for $g \in \mathrm{GL}_2(\mathbb{A})$ generate an irreducible automorphic representation $\pi_{\mathbf{f}\otimes\varphi}$. Plainly, we have $\pi_{\mathbf{f}\otimes\varphi} \cong \pi\otimes\varphi_{\mathbb{A}}$. Inside $\pi_{\mathbf{f}\otimes\varphi}$, we find a unique new vector $(\mathbf{f}\otimes\varphi)^\circ$ which corresponds to a classical primitive Hecke eigenform $(\mathbf{f}\otimes\varphi)^\circ_1 =: f_\varphi \in S_2(\Gamma_0(C(\pi\otimes\varphi)), \phi\varphi^2)$ for the conductor $C(\pi\otimes\varphi_{\mathbb{A}})$ of $\pi\otimes\varphi_{\mathbb{A}}$. The form f_φ is usually not equal to the classical form $(\mathbf{f}\otimes\varphi)_1$ corresponding to the adelic form $\mathbf{f}\otimes\varphi$ even if $\mathbf{f}$ is new (as their Neben types are plainly different). As explained in [H09, §3.1], $\mathbf{f}\otimes\varphi$ often has level smaller than the level of the primitive form f_φ. Unless the p-component π_p is super-cuspidal, $\pi_p\otimes\varphi$ has a non-zero $U(p)$-eigenvector with non-zero eigenvalue. Indeed, if $\pi_p = \pi(\alpha,\beta)$, there is a non-zero eigenspace in $\pi\otimes\varphi_{\mathbb{A}}$ on which $U(p)$ acts by $\alpha\varphi_{\mathbb{A}}(p_p)$ (resp. $\beta\varphi_{\mathbb{A}}(p_p)$) (if $\alpha(p) \neq \beta(p)$, the eigenspaces of each of the above value is one-dimensional). If π_p is special, we have one dimensional eigenspace with non-zero eigenvalue. Even if φ is highly ramified at p, the eigenvalues of $U(p)$ for $\mathbf{f}\otimes\varphi$ and $\mathbf{f}$ are equal.

Take a family for the standard tower (i.e., $(\alpha,\delta) = (0,1)$)

$$\mathcal{F}_{\mathbb{I}} := \{f_P \in S_2(\Gamma_0(Np^{r(P)}), \varepsilon_P)\}_{P\in\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)} \cong \{\mathbf{f}_P \in \mathcal{S}(\widehat{\Gamma}_0(Np^{r(P)}), \mathbf{1}, \varepsilon_P)\}_{P\in\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)}$$

which is an ordinary p-adic analytic family, and $\varepsilon_P : \mathbb{Z}_p^\times/\mu \to \mu_{p^{r(P)-1}}(\overline{\mathbb{Q}})$. If $P\cap\Lambda = (t-\zeta_P)$ with $\varepsilon_P(1+p) = \zeta_P \in \mu_{p^{r(P)-1}}$, we can rewrite

$$\{\mathbf{f}_P \in \mathcal{S}(\widehat{\Gamma}_0(Np^{r(P)}), \mathbf{1}, \varepsilon_P)\}_{P\in\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)} = \{\mathbf{f}_P \in \mathcal{S}(\widehat{\Gamma}_0(Np^{r(P)}), \mathbf{1}, \kappa_{\zeta_P}^{-1})\}_{P\in\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)}$$

as the family of $(\alpha,\delta) = (0,1)$. Pick a positive integer b prime to p. Then we consider the twisted family $\mathcal{F}_{\mathbb{I}}(b) = \{f_P\otimes\kappa_{\zeta_P}^{1/b}\}$. Since $\mathbf{f}_P \in \mathcal{S}(\widehat{\Gamma}_0(Np^{r(P)}), 1, \kappa_{\zeta_P}^{-1})$, we have $\mathbf{f}_P\otimes\kappa_{\zeta_P}^{1/b} \in \mathcal{S}(\widehat{\Gamma}_0(Np^{r(P)}), \kappa_{\zeta_P}^{1/b}, \kappa_{\zeta_P}^{1/b-1})$. Though

$$\mathcal{S}(\widehat{\Gamma}_0(Np^{r(P)}), \kappa_{\zeta_P}^{1/b}, \kappa_{\zeta_P}^{1/b-1}) \cong S_2(\Gamma_0(Np^{r(P)}, \kappa_{\zeta_P}) \text{ by } \mathbf{f}\mapsto\mathbf{f}_1$$

as vector spaces, the Hecke operator action comes from the left-hand-side, which is the twist by $\kappa_{\zeta_P}^{1/b}$ of the standart action on the right-hand-side. Thus in this case, the family is for the exotic tower of $(\alpha,\delta) = (1,b-1) = b(\frac{1}{b}, 1-\frac{1}{b})$ as elements in $\mathbf{P}^1(\mathbb{Z}_p)$. If one starts with $f_{P_0} \in S_2(\Gamma_0(Np))$ whose L-function has root number ±1, the L-function $f_P\otimes\varepsilon_P^{-1/2}$ has the same root number. Therefore, the most interesting case is when $b = 2$ and $(\alpha,\delta) = (1,1)$. This process can be reversed by tensoring back $\kappa_{\zeta_P}^{-1/b}$. Though we have assumed that $\xi = \mathbf{1}$, introducing the twist by a character $\varphi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mu_{p-1}$ in addition to the twist by $\kappa^{1/b}$, we get one-to-one onto correspondence of families of modular forms of $\mathbf{h}_{0,1,\xi_{ord}}$ with $\xi_{ord}(a,d) = \phi(d)$ and $\mathbf{h}_{1,b-1,\xi_\varphi}$, where $\xi_\varphi(a,d) = \phi(d)\varphi(a)\varphi(d)$. We leave the details of the argument for the non-trivial φ to attentive readers. This shows, writing $\Lambda = \mathbb{Z}_p[[T]]$ with $t = 1+T$,

Proposition 18.2. *Let the notation as above, and suppose $p > 2$. Then the algebra $\mathbf{h}_{1,b-1,\xi_\varphi}$ is isomorphic to $\mathbf{h}_{0,1,\xi_{ord}}$ ($\xi_{ord}(a,d) = \phi(d)$) as $\mathbb{Z}_p$-algebras by $T(l^n) \mapsto t^{-1/b\log_p(l^n)/\log_p(\gamma)}\varphi_{\mathbb{A}}(l_l^n)T(l^n)$ for primes l, where $\gamma = 1+p^\epsilon$ and $\log_p$ is the p-adic logarithm and we have written $T(l^n) = U(l^n)$ for $l|Np$. The Λ-algebra structure of $\mathbf{h}_{1,b-1,\xi_\varphi}$ is obtained by transferring the Λ-algebra structure of $\mathbf{h}_{0,1,\xi_{ord}}$ by this isomorphism. In particular, the algebra $\mathbf{h}_{1,b-1,\xi_\varphi}$ is free of finite rank over Λ for all primes $p > 2$.*

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