Exponential decay of reconstruction error from binary measurements of sparse signals

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Joint work with R. Baraniuk, S. Foucart, Y. Plan, and M. Wootters

Outline

- Introduction
  - Mathematical Formulation & Methods
- Practical CS
  - Other notions of sparsity
  - Heavy quantization
  - Adaptive sampling

The mathematical problem

1. Signal of interest $f \in \mathbb{C}^d (= \mathbb{C}^{N \times N})$
2. Measurement operator $\mathscr{A} : \mathbb{C}^d \to \mathbb{C}^m \; (m \ll d)$
3. Measurements $y = \mathscr{A} f + \xi$

$$\begin{bmatrix} \\ y \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & \mathscr{A} & \\ & & \end{bmatrix} \begin{bmatrix} \\ f \\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \xi \\ \\ \end{bmatrix}$$

4. *Problem:* Reconstruct signal f from measurements y

Sparsity

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Assume f is *sparse*:

- ✧ In the coordinate basis: $\|f\|_0 \stackrel{\text{def}}{=} |\operatorname{supp}(f)| \le s \ll d$
- ✧ In orthonormal basis: $f = Bx$ where $\|x\|_0 \le s \ll d$

In practice, we encounter *compressible* signals.

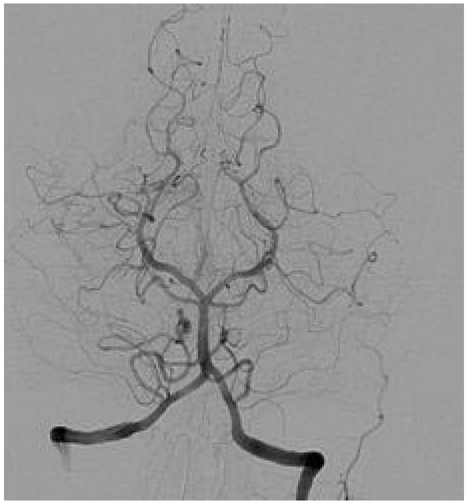
- ✦ f_s is the best s-sparse approximation to f

Many applications...

- Radar, Error Correction
- Computational Biology, Geophysical Data Analysis
- Data Mining, classification
- Neuroscience
- Imaging
- Sparse channel estimation, sparse initial state estimation
- Topology identification of interconnected systems
- ...

Sparsity...

Sparsity in coordinate basis: f=x



Reconstructing the signal f from measurements y

- ℓ_1-minimization [Candès-Romberg-Tao]

Let A satisfy the *Restricted Isometry Property* and set:

$$\hat{f} = \operatorname*{argmin}_g \|g\|_1 \quad \text{such that} \quad \|\mathscr{A} f - y\|_2 \leq \varepsilon,$$

where $\|\xi\|_2 \leq \varepsilon$. Then we can stably recover the signal f:

$$\|f - \hat{f}\|_2 \lesssim \varepsilon + \frac{\|x - x_s\|_1}{\sqrt{s}}.$$

This error bound is optimal.

Restricted Isometry Property

- $\mathscr{A}$ satisfies the Restricted Isometry Property (RIP) when there is $\delta < c$ such that

$$(1-\delta)\|f\|_2 \le \|\mathscr{A} f\|_2 \le (1+\delta)\|f\|_2 \quad \text{whenever } \|f\|_0 \le s.$$

- $m \times d$ Gaussian or Bernoulli measurement matrices satisfy the RIP with high probability when

$$m \gtrsim s \log d.$$

- Random Fourier and others with fast multiply have similar property: $m \gtrsim s \log^4 d$.

Other recovery methods

Greedy Algorithms

- If A satisfies the RIP, then $A^* A$ is “close” to the identity on sparse vectors
- Use proxy $p = A^* y = A^* Ax \approx x$
- Threshold to maintain sparsity: $\hat{x} = H_s(p)$
- Repeat
- (Iterative Hard Thresholding)

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$$\|\mathbf{x} - \Delta(\mathbf{y})\| \le \gamma$$

provided the oversampling factor satisfies

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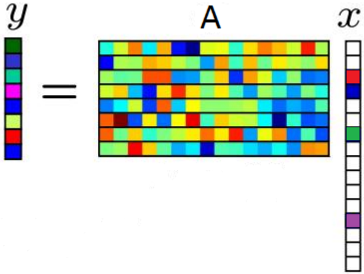
for g rapidly decreasing to zero when λ increases.

A visual



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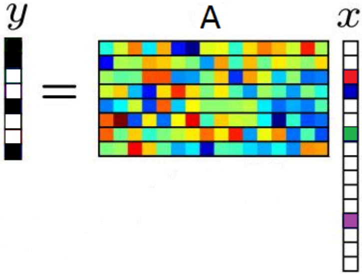

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$$\|\mathbf{x} - \Delta_{\mathrm{SOCP}}(\mathbf{y})\|_2 \lesssim \lambda^{-1/12} \quad \text{whenever } \|\mathbf{x}\|_0 \leq s, \|\mathbf{x}\|_2 = 1.$$

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- Power decay is optimal since

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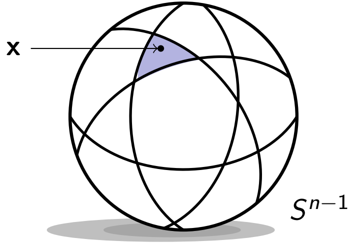

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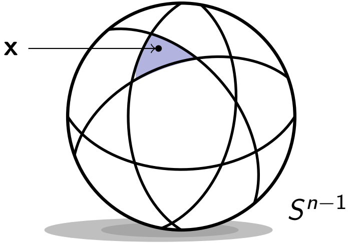

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- Remedy: adaptive choice of dithers $\tau_1, \ldots, \tau_m$ in

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- Rely on an *order-one quantization/recovery scheme*: for any $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_0 \leq s$, $\|\mathbf{x}\|_2 \leq R$, take $q \asymp s \ln(n/s)$ one-bit measurements and estimate both the direction and the magnitude of $\mathbf{x}$ by producing $\widehat{\mathbf{x}}$ such that

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- Software step needed to compute the thresholds $\tau_i = \langle \mathbf{a}_i, \mathbf{x}^t \rangle$.

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- If $q \geq c\delta^{-4} s \ln(n/s)$, then w/hp

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- Cons: slow, post-quantization error not handled.

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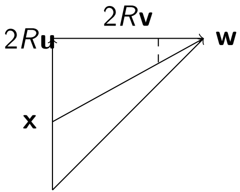

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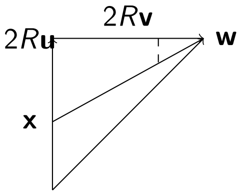

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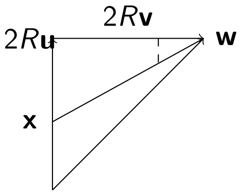

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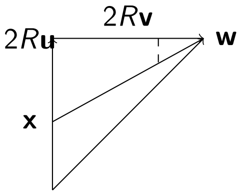

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- Cons: dithers $\langle \mathbf{a}_i, \mathbf{w} \rangle$ are adaptive.

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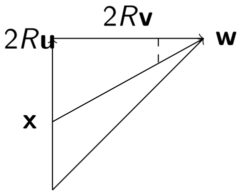

- Use other half to estimate the direction of $\mathbf{x} - \mathbf{w}$ applying hard thresholding again.
- Plane geometry to estimate direction and magnitude of $\mathbf{x}$.
- Cons: dithers $\langle \mathbf{a}_i, \mathbf{w} \rangle$ are adaptive.
- Pros: deterministic, fast, handles pre/post-quantization errors.

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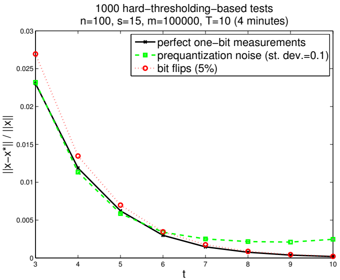
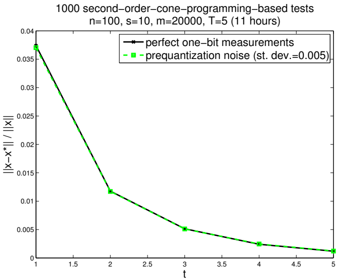
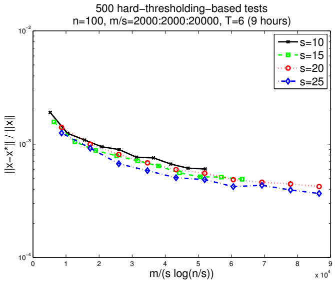
$$\|\mathbf{x} - \mathbf{x}^T\|_2 \leq R\, 2^{-T} = R \exp(-c\lambda)$$
for the hard-thresholding scheme.

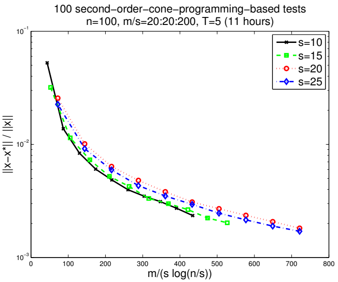
Numerical Illustration





Numerical Illustration, ctd





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- Sign Product Embedding Property: if $q \geq C\delta^{-6} s \ln(n/s)$, then with w/hp

$$\left| \frac{\sqrt{\pi/2}}{q} \langle \mathbf{A}\mathbf{w}, \mathrm{sign}(\mathbf{A}\mathbf{x}) \rangle - \langle \mathbf{w}, \mathbf{x} \rangle \right| \leq \delta$$

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- Simultaneous (ℓ_2, ℓ_1)-Quotient Property: w/hp, every $\mathbf{e} \in \mathbb{R}^q$ can be written as

$$\mathbf{e} = \mathbf{A}\mathbf{u} \quad \text{with} \quad \begin{cases} \|\mathbf{u}\|_2 & \leq & d\|\mathbf{e}\|_2/\sqrt{q}, \\ \|\mathbf{u}\|_1 & \leq & d'\sqrt{s_*}\|\mathbf{e}\|_2/\sqrt{q}, \end{cases}$$

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- Restricted Isometry Property: if $q \geq C\delta^{-2} s \ln(n/s)$, then with w/hp

$$\left| \frac{1}{q}\|\mathbf{A}\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2 \right| \leq \delta \|\mathbf{x}\|_2^2$$

  for all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_0 \leq s$.

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Thank you!

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