All questions are of equal value. There is plenty of working space, and a blank page at the end.

Good luck!

Full name: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Total: ____________________
Problem 1. Let $l^\infty$ be the space of all bounded sequences of real numbers $(x_n)_{n=1}^{\infty}$, with the sup norm

$$||x||_\infty = \sup_{n=1}^{\infty} |x_n|.$$ 

Show that $(l^\infty, ||||_\infty)$ is a Banach space. (You may assume that this space satisfies the conditions for a normed vector space).
Problem 2. Let \((a_n)_{n=1}^{\infty}\) be a bounded sequence of real numbers. Prove that there exists a bounded sequence \((b_n)_{n=1}^{\infty}\) such that

\[ b_{n-1} + 4b_n + b_{n+1} = a_n \quad (*) \]

for all \(n = 1, 2, \ldots\), where we take \(b_0\) to equal 0. [You may assume the result of Problem 1].

Hint: Use the Contraction Mapping theorem. You may need to rewrite the recurrence (*).
Problem 3.

Let $T_1, T_2, \ldots$ be a sequence of continuous linear transformations from a Banach space $X$ to a normed vector space $Y$. Assume that none of the $T_i$ are identically zero; in other words, for every $i$ there exists a $x \in X$ such that $T_ix \neq 0$. Show that there exists a single $x \in X$ (which does not depend on $i$) such that $T_ix \neq 0$ for every $i$.

Hint: use the Baire Category theorem.
Problem 4.

(a) Show that the product of two totally bounded sets is totally bounded.

(b) Show that every bounded set in $\mathbb{R}^n$ is totally bounded.
Problem 5. Suppose $f : X \rightarrow Y$ is a continuous map from a metric space $X$ to a metric space $Y$.

(a) Is the inverse image of a closed set under $f$ always closed? Justify your answer.

(b) Is the inverse image of a compact set under $f$ always compact? Justify your answer.