Mathematics 121 Midterm
Terence Tao
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All questions are of equal value. There is plenty of working space, and a blank page at the end.

Good luck!

Full name: ______________________

Student ID: ______________________

Signature: ______________________

Problem 1. ____________

Problem 2. ____________

Problem 3. ____________

Problem 4. ____________

Problem 5. ____________

Total: ______________________

Problem 1. Let l^∞ be the space of all bounded sequences of real numbers $(x_n)_{n=1}^\infty$, with the sup norm

$$\|x\|_\infty = \sup_{n=1}^{\infty} |x_n|.$$

Show that $(l^\infty, \|\|_\infty)$ is a Banach space. (You may assume that this space satisfies the conditions for a normed vector space).

Problem 2. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Prove that there exists a bounded sequence $(b_n)_{n=1}^{\infty}$ such that

$$b_{n-1} + 4b_n + b_{n+1} = a_n \qquad (*)$$

for all $n = 1, 2, \ldots$, where we take b_0 to equal 0. [You may assume the result of Problem 1].

Hint: Use the Contraction Mapping theorem. You may need to rewrite the recurrence (*).

Problem 3.

Let $T_1, T_2, \ldots$ be a sequence of continuous linear transformations from a Banach space X to a normed vector space Y. Assume that none of the T_i are identically zero; in other words, for every i there exists a $x \in X$ such that $T_i x \neq 0$. Show that there exists a single $x \in X$ (which does not depend on i) such that $T_i x \neq 0$ for every i.

Hint: use the Baire Category theorem.

Problem 4.

(a) Show that the product of two totally bounded sets is totally bounded.

(b) Show that every bounded set in $\mathbf{R}^n$ is totally bounded.

Problem 5. Suppose $f : X \to Y$ is a continuous map from a metric space X to a metric space Y.

(a) Is the inverse image of a closed set under f always closed? Justify your answer.

(b) Is the inverse image of a compact set under f always compact? Justify your answer.