HOMEWORK 9

1. Let $\mathcal{A}$ be an invertible linear operator on a finite dimensional inner product space. Prove that $\mathcal{A}^*$ is also invertible and $(\mathcal{A}^{-1})^* = (\mathcal{A}^*)^{-1}$.

2. Let $\mathcal{A}$ be a linear operator on an inner product space. Prove that $N(\mathcal{A}^* \circ \mathcal{A}) = N(\mathcal{A})$.

3. Let $\mathcal{A}$ be a linear operator on a finite dimensional inner product space. Prove that $rank(\mathcal{A}^*) = rank(\mathcal{A})$.

4. Let $\mathcal{A}$ be a linear operator on a finite dimensional inner product space. Prove that $R(\mathcal{A}^*)^\perp = N(\mathcal{A})$ and $R(\mathcal{A}^*) = N(\mathcal{A})^\perp$.

5. Let $\mathcal{A}$ be a linear operator on an inner product space. Let $W \subset V$ be a subspace such that $\mathcal{A}(W) \subset W$. Prove that $\mathcal{A}^*(W^\perp) \subset W^\perp$.

6. Let $\mathcal{A} : V \to W$ be a linear map of inner product spaces such that $\langle \mathcal{A}(v), \mathcal{A}(v') \rangle = \langle v, v' \rangle$ for all $v, v' \in V$. Prove that $\mathcal{A}$ is injective.

7. Let W and W' be two finite dimensional subspaces of an inner product space such that $\dim(W) < \dim(W')$. Prove that there is a nonzero vector $x \in W'$ orthogonal to W.

8. Let $\mathcal{A}$ be a normal linear operator on a finite dimensional inner product space. Prove that $N(\mathcal{A}^*) = N(\mathcal{A})$ and $R(\mathcal{A}^*) = R(\mathcal{A})$.

9. Let $\mathcal{A}$ be a normal linear operator on a finite dimensional real inner product space V. Suppose that the characteristic polynomial of $\mathcal{A}$ splits. Prove that there is an orthonormal basis for V consisting of eigenvectors of $\mathcal{A}$.

10(*). Let $\mathcal{A}$ be a normal linear operator on a finite dimensional complex inner product space V. Let $W \subset V$ be a subspace such that $\mathcal{A}(W) \subset W$. Prove that $\mathcal{A}^*(W) \subset W$.