Upload your solutions to gradescope for the following questions by 11:59pm LA time on Sunday 17 May.

- Late exams will not be accepted.
- Your scans must be readable and good quality. Use good lighting and a scanning app.
- Questions 1, 2, 3 must begin on a new page and questions must be allocated correctly on Gradescope.
- Write your solutions **linearly**. We should be able to easily read your solutions and do not want to hunt around the page for it.

1. Let $T : V \rightarrow W$ be a linear map between finite dimensional vector spaces.
   (a) (3 points) Suppose further that $B$ is a basis for $V$. Prove that $T$ is an isomorphism if and only if $T(B)$ is a basis for $W$.
   (b) (2 points) Give an example of a linear map that is not the zero map, that is diagonalisable, but is not an isomorphism.

2. Let $V = \{ M \in \text{Mat}_{2 \times 2}(\mathbb{C}) \mid \text{tr}(M) = 0 \}$ be the space of $2 \times 2$ matrices with zero trace (see the front page). Let
   
   $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

   Define a linear map $T : V \rightarrow V$ by $T(M) = HM - MH$.
   (a) (2 points) Find the characteristic polynomial and eigenvalues of $T$.
   (b) (2 points) For each eigenvalue, determine a nonzero eigenvector of $T$ (note that the eigenvectors should live in $V$, i.e. you eigenvectors should be matrices!).
   (c) (1 point) Is $T$ diagonalisable?

3. Suppose that $T : V \rightarrow V$ is a linear operator on a finite dimensional vector space $V$.
   (a) (2 points) If 0 is an eigenvalue for $T$, show that $T$ is not an isomorphism.
   (b) (3 points) Suppose 0 is not an eigenvalue of $T$, show that $T$ is an isomorphism. *Hint: again think about the kernel and you will also find the rank-nullity theorem useful.*

\[ 1. \text{ (a) } T \text{ is an isomorphism} \]

\[ \begin{align*}
\Rightarrow & \{ T \text{ is one-to-one i.e. } \ker(T) = \{ 0 \} \\
& \text{ and } T \text{ is onto i.e. } \text{Im}(T) = W \\
\Rightarrow & \{ \text{T(B) is linear independent} \text{ i.e. } \text{Span} \{ T(B) \} = W \\
\Rightarrow & \{ T(B) \text{ is a basis of } W \}
\end{align*} \]
Proof of (1): If \( \ker(T) = \{ 0 \} \), then for \( v_1, \ldots, v_n \in \mathcal{B} \), if

\[
\sum_{i=1}^{n} a_i T(v_i) + \cdots + an T(v_n) = 0,
\]

then

\[
\sum_{i=1}^{n} a_i v_i + \cdots + an v_n = 0.
\]

If \( T(\mathcal{B}) \) are L.I., then any \( v = a_1 v_1 + \cdots + an v_n \in \mathcal{V} \), we have

\[
T(v) = a_1 T(v_1) + \cdots + an T(v_n) = 0
\]

\[
\Rightarrow a_1, \ldots, an = 0 \Rightarrow v = 0
\]

\[
\therefore \ker(T) = \{ 0 \}.
\]

Rubric: One point for splitting into (1), (2), one point for proving (1), one point for proving (2).
(b) \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[
T(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

then \([T]_\beta\) for standard basis \(\beta\) is

\[
[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

It's diagonalizable but not an isomorphism.

Rubric: 1 point for example, 1 point for explanation

3. (a) Because \( \exists v \in \text{Ker} \), s.t. \( T(v) = 0 \)

\[ \Rightarrow \ker(T) \neq \{0\}, \quad T \text{ is not one-to-one} \]

\[ \therefore T \text{ is not an isomorphism} \]

(b) Because \( \ker(T) = \{0\} \Rightarrow T \text{ is one-to-one} \)

By dimension thm.

\[
\text{nullity} = 0 \Rightarrow \text{rank} + \text{nullity} = \text{dim} \text{Im}(T) = \text{dim} \text{Im}(T)
\]

Therefore, \( \text{Im}(T) = \mathbb{V} \)
\[ T \text{ is an onto} \]
\[ \Rightarrow T \text{ is an isomorphism} \]

2. (a) Find a basis for \( V \).
\[ \beta = \{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \} \]
\[ [T]_{\beta} = \begin{pmatrix} m_1 & m_2 & m_3 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
\[ p_T(t) = (t-2)(t+2) + t \]
\[ \lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 0 \]

(b) \[ \lambda_1 = 2 \]
\[ \nu_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]
\[ \lambda_2 = -2 \]
\[ \nu_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]
\[ \lambda_3 = 0 \]
\[ \nu_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
eigenvalues

\[
\begin{align*}
N_1 &= M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
N_2 &= M_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
N_3 &= M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

( ) Yes. We find $\tilde{Q}^{\top}P$ is a diagonal matrix.