### 4.14 Basis

**Definition:** A subset $\beta \subset V$ is a basis if

1. it is a spanning set
2. linearly independent

**Explanation:**

i) $\forall v \in V$, $v \in \text{span}(\beta)$

$\exists v_1, \ldots, v_n \in \beta$, and $c_1, \ldots, c_n \in \mathbb{F}$, s.t.

$v = c_1 v_1 + \ldots + c_n v_n$

ii) If for $v_1, \ldots, v_n \in \beta$, $c_1, \ldots, c_n \in \mathbb{F}$, then $c_1 = \ldots = c_n = 0$

**Examples:**

i) $\mathbb{F}^n$ 

$e_1 = (1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1)$

$\{e_1, \ldots, e_n\}$ is a standard basis for $\mathbb{F}^n$

**Check:**

i) $\forall v = (v_1, \ldots, v_n) \in \mathbb{F}^n$, we have

$v = v_1 e_1 + \ldots + v_n e_n \in \text{span}(\{e_1, \ldots, e_n\})$

ii) If $c_1 e_1 + \ldots + c_n e_n = (c_1, \ldots, c_n) = 0$, then

$c_1 = \ldots = c_n = 0$. \hfill \checkmark
(2) $M_{m \times n}(F)$, $E_{ij} = i(\begin{smallmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{smallmatrix})$

$\{ E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n \}$ is the standard basis

(3) $P_n(F)$, $\{ 1, x, ..., x^n \}$ is the standard basis

(4) $\{ (0), (0), (1) \}$ is not a basis because it's not linear independent.

(5) $\{ (0), (1) \}$ is not a basis because it's not a spanning set.

* Need to be "enough" and "not redundant"

**Theorem (algorithm)** Suppose $S \subseteq V$ is a finite spanning set. Then $\exists$ basis $B \subseteq S$

**Proof:**

i) $S = \emptyset$ or $\{ 0 \}$, $V = \{ 0 \}$, $B = \emptyset$.

ii) $S \neq \emptyset$, let $S = \{ u_1, ..., u_n \}$

Go through the elements one by one (on lecture notes).
Examples:

(1) \( S = \{(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1), (7, 2, 0)\} \quad V = \mathbb{R}^3 \)

**Solution:** \( u_1 = (2, -3, 5), \beta = \{u_1\}, \)

\( u_2 = (8, -12, 20) = 4u_1, \text{ skip} \)

\( u_3 = (1, 0, -2) \in \text{span}\{u_1\}, \beta = \{u_1, u_3\} \)

\( u_4 = (0, 2, -1) \in \text{span}\{u_1, u_3\}, \beta = \{u_1, u_3, u_4\} \)

\( u_5 = (7, 2, 0) \in \text{span}\{u_1, u_3, u_4\} \), skip

\( \therefore \beta = \{u_1, u_3, u_4\} \text{ is a basis} \)

(2) \( \{x+1, x^2, x^2+x+1, x^2+2x\} \quad V = P_3(\mathbb{R}) \)

\( u_1 = x+1, \beta = \{u_1\} \)

\( u_2 = x^2, \beta = \{u_1, u_2\} \)

\( u_3 = x^2+x+1 \in \text{span}\{u_1, u_2\}, \text{ skip} \)

\( u_4 = x^2+2x \in \text{span}\{u_1, u_2\} \)

\( \therefore \beta = \{u_1, u_2, u_4\} \text{ is a basis} \)
**Theorem (Replacement theorem)**

Suppose $G$ is a spanning set of vector space $V$, $L$ is a linearly independent set, s.t. $\# G = m$ and $\# L = n$, then

1. $n \leq m$
2. $\exists H \subseteq G$ s.t. $\# H = m - n$ and $\text{span}\left\{ L \cup H \right\} = V$

$G : \quad \{ u_1, \ldots, u_m \}$

$L : \quad \{ v_1, \ldots, v_n \}$

**Proof:** By induction (on $n$)

i) $n = 0$, $L = \emptyset$, take $H = G$, $V$

ii) Suppose that is true for some $n \geq 0$, we prove it’s true for $n+1$

Suppose $L = \{ v_1, \ldots, v_{n+1} \}$, $G = \{ u_1, \ldots, u_m \}$

$L$ is linearly independent $\Rightarrow$
\{v_1, \ldots, v_m\} is linearly independent.

By induction hypothesis, \(n \leq m\) and

\(\exists H \subseteq L, \#H = m-n, \text{s.t. } \text{LVH generates } V\).

Suppose \(H = \{u_1, \ldots, u_{m-n}\}\), then

\[\forall v_{m-n+1} \in \text{span}\{v_1, \ldots, v_m, u_1, \ldots, u_{m-n}\}\]

\[\exists\text{ scalars } a_1, \ldots, a_m, b_1, \ldots, b_{m-n}\text{ such that}\]

\[v_{m-n+1} = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_{m-n} u_{m-n}\]

If \(n = m\) then \(v_{m-n+1} = a_1 v_1 + \cdots + a_m v_m \in \text{span}\{v_1, \ldots, v_m\}\)

contradicts with \(L\) linearly independence.

\[\therefore n > m \Rightarrow n \geq m+1\]

and \(b_1, \ldots, b_{m-n}\) cannot all be 0's (for the same reason). Suppose \(b_i \neq 0\)

\[\therefore v_{m-n+1} = (-b_1^{-1} a_1) v_1 + (-b_1^{-1} a_2) v_2 + \cdots + (-b_1^{-1} a_m) v_m +
\]

\[+ (-b_1^{-1} b_1) u_1 + (-b_1^{-1} b_2) u_2 + \cdots + (-b_1^{-1} b_{m-n}) u_{m-n}\]

\[\in \text{span}\{v_1, \ldots, v_m, u_1, \ldots, u_{m-n}\}\]

\[\therefore H = \{u_1, \ldots, u_{m-n}\}\text{ and then LVH generates vector space } V.\]
Example \( V = \{ (x,y,z) \in \mathbb{R}^3 \mid x+2y+3z = 0 \} \)

Find a basis for vector space \( V \).

Solution: \( \mathbf{v}_1 = (1,1,-1), \ \mathbf{v}_2 = (1,4,-3) \)

Then \( \beta = \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is

- linearly independent
- a spanning set of \( V \)
- \( \beta \) is a basis for \( V \).

We don’t need to first find a spanning set and then conduct the algorithm. We can just find the exact number of vectors and show they satisfy the conditions.