6.4 Problem Set 9

#4. Let $V$ be a real inner product space and let $r: V \rightarrow V^*$ be the map defined as a linear function in $V^*$

$$r(x) = \langle \phi_x : = (\langle \cdot, x \rangle) \rangle$$

In class we showed that if $V$ is finite dimensional, then $r$ is an isomorphism.

(a) Assume that $V$ is infinite dimensional. Prove that $r$ is injective.

(b) Let $V = \mathbb{R}[x]$ and let

$$W = \{ (a_0, a_1, \ldots) | a_i \in \mathbb{R} \}$$

be the vector space of all infinite sequences. Show that the map $f: V^* \rightarrow W$ given by $f(\phi) = (\phi(x^n))_{n \geq 0}$ is an isomorphism.
(c) Use this to demonstrate that \( r \) is not necessarily surjective, i.e. find an element \( x \in V^* \) such that \( (x \neq r(c)) \) for \( \forall x \in R^\infty \).

**Solution:** (a) If \( V \) is an infinite dimensional space, then \( r(x) = \langle -, x \rangle \)

\[ r(x) = 0 \implies \forall y \in V, \langle y, x \rangle = 0 \]

\[ \therefore \langle x, x \rangle = 0 \implies x = 0 \]

\[ \ker(r) = \{0\} \quad r \text{ is injective} \]

(b) (1) \( f \) is linear

\[ f(q_1 + q_2) = f(q_1) + f(q_2) \]

\[ f(cq) = cf(q) \]

(2) \( f \) is one-to-one
\[ f(q) = 0 \Rightarrow q(x^n) = 0 \quad n = 1, 2, \ldots \]

\[ \text{\because } q(p) = 0 \quad \text{for } \forall p \in \mathbb{R}[x] \]

\[ \ker q = \{0\} \quad \checkmark \]

\[ p = \sum_{i=0}^{\infty} a_i x^i \]

2. \( f \) is onto

for any sequence

\[ \mathcal{S} = (b_0, b_1, \ldots, b_n, \ldots) \in \mathcal{W} \]

define \( q : V^\ast \rightarrow \mathcal{W} \), for any

\[ p(x) = \sum_{i=0}^{n} a_i x^i, \]

\[ q(p) = \sum_{i=0}^{\infty} a_i b_i \in \mathbb{R} \]

then we have

\[ q(x^n) = b_n \Rightarrow f(q) = \mathcal{S}. \]

\[ \therefore f \text{ is onto} \quad \checkmark \]

Therefore we proved \( f : V^\ast \rightarrow \mathcal{W} \)
is an isomorphism.

(c) We want to find $\varphi \in \mathbb{K}^*$, such that there is no $x \in \mathbb{K}$ representing $\varphi$. ($\varphi(y) = \langle y, x \rangle, A^y$)

\[ \langle \cdot, \cdot \rangle \text{ on } \mathbb{K}[x,j]: \]
\[ \langle t^i, t^j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

From (b) we know that
\[ \varphi \leftrightarrow \mathcal{F} = \{ \varphi(t^n) \}_{n=1}^{\infty} \]
we actually want to construct such a sequence $\mathcal{F}$ s.t. $\varphi$ cannot be represented by any $x$.

If $\varphi = \langle -, x \rangle$, $x(t) = \sum_{i=0}^{n} a_i t^i$
then \( \varphi(t^n) = \langle t^n, x \rangle \)

\[ = a_n \]

\[ \varphi = (a_0, a_1, \ldots, a_m) \]

We just need to consider \( \varphi \) with infinitely many nonzero terms. Then there is no \( p \in \mathbb{R}[x] \), such that

\[ \langle (t^2, 2t^2) \rangle = (1, 1, 0), (2, 2, 1) \]

\[ = 2 \]

For example, \( \varphi = (1, 1, 1, \ldots, 1) \)

then \( p \) needs to be \( \sum_{j=0}^{\infty} t^j \),

but this is not a polynomial.

5. \( V \) finite dimensional \( \implies \)

For any \( T : V \rightarrow V \), define \( \hat{T} : V^* \rightarrow V^* \)

\[ \hat{T}(\phi) = \phi \circ T. \quad \forall \phi : V^* \rightarrow \mathbb{R} \]
$x^\perp : V \rightarrow V$ by $x^\perp = r^{-1} \circ x \circ r$.

Prove that $T^* = \hat{T}^\perp$

**Proof.** Verify $\langle T x, y \rangle = \langle x, \hat{T}^\perp y \rangle$

$\hat{T}^\perp = r^{-1} \circ \hat{T} \circ r$

$\hat{T}^\perp (y) = r^{-1} \circ \hat{T} \circ r(y) = \varphi \in V^*$

$= r^{-1} \circ \varphi \circ T$

$= r^{-1} (r(y) \circ T)$

the "representing vector" of $r(y) \circ T \in V^*$.

\[ \therefore \quad r(y) \ (T(x)) = \langle x, \hat{T}^\perp(y) \rangle \]

We know $r(y) = \langle -, y \rangle$

\[ \therefore \quad r(y) \ (T(x)) = \langle T(x), y \rangle \]

We proved that
\[ \langle T(x), y \rangle = \langle x, \hat{T}^*(y) \rangle \]

which means

\[ T^* = \hat{T}^\perp \]