MATH 132 PROBLEM SET 3 - SELECTED SOLUTIONS

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Problem 1. Consider the system of differential equations
\[
\frac{dA}{dt} = B(t), \quad \frac{dB}{dt} = -A(t).
\]
Solve this system using the initial conditions \( A(0) = A_0 \) and \( B(0) = B_0 \) by considering the complex-valued function \( Q(t) = A(t) + iB(t) \).

Proof. Taking the derivative of \( Q(t) \), we find
\[
\frac{dQ}{dt} = \frac{dA}{dt} + i\frac{dB}{dt} = B(t) - iA(t) = -i(A(t) + iB(t)) = -iQ(t).
\]
This differential equation has solution \( Q(t) = Ce^{-it} \) where \( C \) is a possibly complex constant. By the initial condition we have \( Q(0) = A_0 + iB_0 = C \), so separating into real and imaginary parts gives
\[
Q(t) = (A_0 + iB_0)(\cos(-t) + i\sin(-t)) = (A_0 \cos t + B_0 \sin t) + i(B_0 \cos t - A_0 \sin t),
\]
where we have used that \( \cos(-t) = \cos t \) and \( \sin(-t) = -\sin t \). Hence the solutions to the system are
\[
A(t) = A_0 \cos t + B_0 \sin t, \quad B(t) = B_0 \cos t - A_0 \sin t.
\]
\(\square\)

Problem 3. Show directly that for any \( z \neq 1 \),
\[
1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.
\]
Proof. Put \( S_n := 1 + z + z^2 + \cdots + z^n \). Then \( S_n - zS_n = (1 - z)S_n = 1 - z^{n+1} \), so for \( z \neq 1 \),
\[
S_n = \frac{1 - z^{n+1}}{1 - z}.
\]
\(\square\)

Problem 4. Use complex exponentials and the identity from Problem 3 to derive the formula
\[
1 + \cos \theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin((n + 1/2)\theta)}{2\sin(\theta/2)}.
\]
Proof. Note that \( \cos n\theta = \text{Re}(e^{i\theta}) \). Using the formula of Problem 3 with \( z = e^{i\theta} \), we have
\[
1 + e^{i\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{e^{i(n+1)\theta/2} - e^{-i(n+1)\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \cdot \frac{e^{-i\theta/2} - e^{i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} = \frac{e^{i(n+1)\theta/2}}{e^{i\theta/2}} \frac{\sin((n + 1)\theta/2)}{\sin(\theta/2)}.
\]
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On the other hand, $e^{(n+1)\theta/2}/e^{\theta/2} = e^{in\theta/2}$, which has real part $\cos(n\theta/2)$. Hence

$$1 + \cos \theta + \cdots + \cos n\theta = \Re(1 + e^{i\theta} + \cdots + e^{in\theta}) = \frac{\cos(n\theta/2) \sin((n+1)\theta/2)}{\sin(\theta/2)}.$$ 

To finish the proof we use the trigonometric identity

$$\cos u \sin v = \frac{\sin(u + v) - \sin(u - v)}{2},$$

which gives

$$\cos(n\theta/2) \sin((n+1)\theta/2) = \frac{\sin((n+1/2)\theta) - \sin(-\theta/2)}{2} = \frac{\sin((n+1/2)\theta) + \sin(\theta/2)}{2},$$

which combined with our previous work yields

$$1 + \cos \theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin((n+1/2)\theta)}{2 \sin(\theta/2)}.$$

\Box

**Problem 5.** Let $a$ denote one of the $n$th roots of unity which is not 1. Show that

$$1 + 2a + 3a^2 + \cdots + na^{n-1} = \frac{n}{a - 1}.$$

**Proof.** One way of doing this problem is differentiation of the identity from Problem 3, which gives

$$1 + 2z + \cdots + nz^{n-1} = \frac{nz^{n+1} - (n+1)z^n + 1}{(1-z)^2}.$$

Plugging in $z = a$ where $a$ is an $n$th root of unity, we have

$$1 + 2a + 3a^2 + \cdots + na^{n-1} = \frac{na - (n+1) + 1}{(1-a)^2} = \frac{n}{a - 1}.$$

\Box

**Problem 8.** Suppose $f(z) = u(x,y) + iv(x,y)$ is expressed in polar coordinates as $f = A(r,\theta) + iB(r,\theta)$. Derive the polar Cauchy-Riemann equations for $A$ and $B$ assuming $u$ and $v$ satisfy the usual Cartesian ones.

**Proof.** Applying the multivariable chain rule to $A = u$ gives

$$A_r = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = u_x \cos \theta + u_y \sin \theta$$

$$A_\theta = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = u_x (-r \sin \theta) + u_y (r \cos \theta),$$

and, likewise, we have for $B = v$,

$$B_r = v_x \cos \theta + v_y \sin \theta$$

$$B_\theta = v_x (-r \sin \theta) + v_y (r \cos \theta).$$

From the Cartesian Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ we find

$$A_r = v_y \cos \theta - v_x \sin \theta,$$

from which it is easy to see $rA_r = B_\theta$. Similarly, we have $-rB_r = A_\theta$. \Box
Problem 9. Let \( g(z) \) be a complex function which is defined and differentiable everywhere. Define \( f(z) = \frac{g(z)}{z} \). Show using the definition of complex derivative that \( f \) is differentiable everywhere and find a formula of \( f'(z) \) in terms of \( g'(z) \).

Proof. Fix a point \( z \in \mathbb{C} \) and consider the difference quotient
\[
\frac{f(z+h) - f(z)}{h} = \frac{g(z + h) - g(z)}{h} \cdot \frac{g(z + h) - g(z)}{h} = \frac{g(z + h) - g(z)}{h},
\]
where we note that \( h = h \). Since \( g \) is differentiable everywhere it is certainly differentiable at \( z \), and also \( h \to 0 \) if and only if \( h \to 0 \). Thus, applying the definition of derivative above, we have
\[
f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \left( \frac{g(z + h) - g(z)}{h} \right) = g'(z).
\]

Problem 10. Prove the result of Problem 9 using the Cauchy-Riemann equations.

Proof. Writing \( g = u + iv \) we have by the Cauchy-Riemann equations
\[
\frac{\partial u}{\partial x} (x, y) = \frac{\partial v}{\partial y} (x, y), \quad \frac{\partial v}{\partial x} (x, y) = -\frac{\partial u}{\partial y} (x, y)
\]
for all \( z = x + iy \in \mathbb{C} \). On the other hand, \( f(z) = \frac{g(z)}{z} = u(x, -y) - iv(x, -y) \). Putting \( \bar{u}(x, y) = u(x, -y) \) and \( \bar{v}(x, y) = -v(x, -y) \), we need to show \( \bar{u} \) and \( \bar{v} \) also satisfy the Cauchy-Riemann equations. Indeed,
\[
\frac{\partial \bar{u}}{\partial x} (x, y) = u_x(x, -y),
\]
while the chain rule gives
\[
\frac{\partial \bar{v}}{\partial y} (x, y) = (-v_y(x, -y))(-1) = v_y(x, -y).
\]
Since the Cauchy-Riemann equations are true at \( z = x - iy \) the above two quantities are equal. Similar reasoning works for the other Cauchy-Riemann equation.

Problem 14. Find a nontrivial homogeneous polynomial in \( x \) and \( y \) of degree four which is harmonic.

Proof. The easiest way to construct such a polynomial is to examine the real and imaginary parts of \( f(z) = z^4 = (x + iy)^4 \), which are automatically harmonic since \( f \) is analytic.

\[
(x + iy)^4 = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3),
\]
so any constant multiple of \( x^4 - 6x^2y^2 + y^4 \) and \( 4x^3y - 4xy^3 \) work.

Problem 15. Derive, from first principles, expressions for the real and imaginary parts of the function
\[
f(z) = \sin z
\]
in terms of the usual circular and hyperbolic functions.
Proof. Write \( z = x + iy \) and observe

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{ix}e^{-y} - e^{-ix}e^y}{2i} = \frac{1}{2i} \left[ (\cos x + i \sin x)e^{-y} - (\cos(-x) + i \sin(-x))e^y \right].
\]

Since \( \cos(-x) = \cos x \) and \( \sin(-x) = -\sin x \), we have

\[
\sin z = \frac{1}{2i} \left[ (\cos x + i \sin x)e^{-y} - (\cos x - i \sin x)e^y \right]
= i \cos x \frac{e^y - e^{-y}}{2} + \sin x \frac{e^{-y} + e^y}{2} = \sin x \cosh y + i \cos x \sinh y,
\]
as desired. \qed