MATH 132 PROBLEM SET 1 - SELECTED SOLUTIONS

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Problem 7. Let z_1 and z_2 denote arbitrary but nonzero complex numbers. Show that the product of the inverses is the inverse of the product, i.e.,

$$\frac{1}{z_1 z_2} = \frac{1}{z_1}\frac{1}{z_2}.$$

Proof. The grindy approach Chayes alluded to is straightfoward, so I'll give a proof using the algebraic approach. It is sufficient to show that the equation $(z_1 z_2)w = 1$ is satisfied by $w = z_1^{-1} z_2^{-1}$ as multiplicative inverses in a field are unique. In other words, if we show $z_1^{-1} z_2^{-1}$ is *a* multiplicative inverse, then it must be equal to $(z_1 z_2)^{-1}$ as there's *only one* multiplicative inverse. To this end, apply commutativity and associativity of multiplication a few times to get

$$(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = 1 \cdot 1 = 1.$$

Hence $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$, as desired. □

Problem 9. Prove that if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$.

Proof. Note that the statement is true for real numbers. For complex numbers write $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Then $z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$. Since $z_1 z_2 = 0$, we obtain the following two equations by setting the real and imaginary parts equal to zero:

$$a_1 a_2 - b_1 b_2 = 0$$

and

$$a_1 b_2 + a_2 b_1 = 0.$$

If $z_1 = 0$ then we are done. Otherwise, either $a_1 \neq 0$ or $b_1 \neq 0$. In the former case we find $a_2 = b_1 b_2 / a_1$ from the first equation, so the second equation becomes

$$a_1 b_2 + \frac{b_1^2 b_2}{a_1} = \Big(a_1 + \frac{b_1^2}{a_1}\Big) b_2 = 0.$$

Thus either $b_2 = 0$ or $a_1^2 = -b_1^2$. Note that the second option is impossible since a strictly positive number cannot equal a nonpositive number, meaning we must have $b_2 = 0$. This implies that $a_2 = b_1 b_2 / a_1 = 0$, so $z_2 = a_2 + b_2 i = 0$.

Roughly the same argument works in the $b_1 \neq 0$ case. Rearrange to find $b_2 = a_1 a_2 / b_1$, substitute this into the second equation, and use the zero-product property of real numbers as before to find $z_2 = 0$. □

Problem 15. Let $\underline{q} = (q_0, \vec{q})$ and $\underline{p} = (p_0, \vec{p})$ denote two quaternions. Derive a formula for the quaternionic product $\underline{q}\,\underline{p}$ in terms of the standard vector products.

Proof. If we write $\underline{q} = q_0 + \vec{q}$ where $\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ and likewise for $\underline{p}$ we have the expansion

$$\underline{q}\,\underline{p} = (q_0 + \vec{q})(p_0 + \vec{p}) = q_0p_0 + q_0\vec{p} + p_0\vec{q} + \vec{q}\vec{p}.$$

Note that multiplication of quaternionic units satisfies the same algebraic relations as the cross-product of the standard basis vectors, except $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$. From this we find

$$\begin{aligned}\vec{q}\,\vec{p} &= (q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) \\ &= -(q_1p_1 + q_2p_2 + q_3p_3) + (q_2p_3 - p_2q_3)\mathbf{i} + (q_3p_1 - p_3q_1)\mathbf{j} + (q_1p_2 - p_1q_2)\mathbf{k} \\ &= -\vec{q}\cdot\vec{p} + \vec{q}\times\vec{p}.\end{aligned}$$

Hence

$$\underline{q}\,\underline{p} = (q_0p_0 - \vec{q}\cdot\vec{p}) + q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p}.$$

□

Problem 16. Show that there is a continuous two-parameter family of quaternion solutions to $\underline{q}^2 = -1$.

Proof. We use the formula derived in Problem 15 to find

$$\underline{q}^2 = (q_0^2 - \vec{q}\cdot\vec{q}) + (2q_0\vec{q} + \vec{q}\times\vec{q}) = (q_0^2 - \|\vec{q}\|^2) + 2q_0\vec{q}$$

where we've used the fact $\vec{q}\times\vec{q} = 0$. By comparing the constant and vectorial components of $\underline{q}^2 = -1$ we get the equations $q_0^2 - \|\vec{q}\|^2 = -1$ and $2q_0\vec{q} = \vec{0}$, the zero vector. Observe $\vec{q} \neq \vec{0}$ since the first equation would then reduce to $q_0^2 = -1$, which is impossible for real q_0. Hence $2q_0\vec{q} = \vec{0}$ implies $q_0 = 0$, meaning $\|\vec{q}\|^2 = 1$.

Geometrically the solutions to $\underline{q}^2 = -1$ is the set of $\underline{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ which lie on the unit sphere as vectors in $\mathbb{R}^3$. In particular,

$$\{\underline{q} : \underline{q}^2 = -1\} = \{\cos\theta\sin\varphi\,\mathbf{i} + \sin\theta\sin\varphi\,\mathbf{j} + \cos\varphi\,\mathbf{k} : \theta \in [0, 2\pi) \text{ and } \varphi \in [0, \pi]\}.$$

The angles θ, φ are the aforementioned two parameters of the family. There are other parameterizations that you can use, but these are probably the most convenient. □

Problem 17. Let M and N be integers expressible as the sum of squares of two integers:

$$\begin{aligned}M =& A^2 + B^2 \\ N =& C^2 + D^2.\end{aligned}$$

Use complex analysis to show that MN can also be written as the sum of squares of two integers.

Proof. Define $z_M = A + iB$ and likewise $z_N = C + iD$. Then $M = |z_M|^2 = z_M\overline{z_M}$ while $N = |z_N|^2 = z_N\overline{z_N}$. Hence $MN = |z_Nz_M|^2$, yet a simple calculation shows

$$z_Nz_M = (AC - BD) + (AD + BC)i,$$

so $MN = (AC - BD)^2 + (AD + BC)^2$ by the definition of complex modulus, as desired. □

Problem 18. Show that $|z| \leq |\operatorname{Re} z| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|$.

Proof. Note that $|z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$ is simply the triangle inquality; if $z = a + bi$ then it is clear that $a^2 + b^2 \leq a^2 + 2|a||b| + b^2 = (|a| + |b|)^2$, and thus

$$|z| = \sqrt{a^2 + b^2} \leq \sqrt{a^2 + 2|a||b| + b^2} = |a| + |b|.$$

For the second inquality, it suffices to show that $a^2 + 2|a||b| + b^2 \leq 2(a^2 + b^2)$, since taking square roots would then give $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|$. Equivalently, we would like to prove $2|a||b| \leq a^2 + b^2$. This can be deduced from the inequality $(|a| - |b|)^2 \geq 0$ or by applying AM-GM. □

Problem 20. If z is a complex number with $\operatorname{Re}(z) > 1$, show that $0 \leq \operatorname{Re}(1/z) \leq 1$.

Proof. Let $z = a + bi$ where $a > 1$. Observe that

$$\operatorname{Re}\left(\frac{1}{a+bi}\right) = \operatorname{Re}\left(\frac{a-bi}{a^2+b^2}\right) = \frac{a}{a^2+b^2}.$$

This quantity is clearly at least 0 since a is positive. On the other hand, $a < a^2$ as $1 < a$. Since $a^2 \leq a^2 + b^2$ as well, we find that

$$\frac{a}{a^2+b^2} < \frac{a^2}{a^2+b^2} \leq \frac{a^2+b^2}{a^2+b^2} = 1.$$

□

Problem 21. Consider $\mathbb{Q}[\sqrt{5}/2]$. Find (a) the law of multiplication for distinct elements of $\mathbb{Q}[\sqrt{5}/2]$ and (b) a formula for the inverse of an element $w = a + (\sqrt{5}/2)b$.

Proof. (a) Let $w = a + (\sqrt{5}/2)b$ and $v = c + (\sqrt{5}/2)d$. Then

$$wv = (ac + \frac{5bd}{4}) + \frac{\sqrt{5}}{2}(bc + ad).$$

(b) To find the inverse $v = w^{-1}$, use the multiplication law above. By comparing coefficients we obtain the two equations

$$ac + \frac{5b}{4}d = 1$$

and

$$bc + ad = 0.$$

We can quickly solve this system for c and d by using matrix algebra:

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a & 5b/4 \\ b & a \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{a^2 - 5b^2/4} \begin{bmatrix} a & -5b/4 \\ -b & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{a^2 - 5b^2/4} \begin{bmatrix} a \\ -b \end{bmatrix}.$$

Thus $c = a/(a^2 - 5b^2/4)$ and $d = -b/(a^2 - 5b^2/4)$.

Some of you might worry about the possibility that $a^2 - 5b^2/4 = 0$ for some choice of a and b, which would break our formulas. This is not a problem though, since the solution $a = b = 0$ is eliminated by the assumption $w \neq 0$, and the only other possible solutions satisfy $a/b = \pm\sqrt{5}/2$. Since a, b are rational numbers this cannot be the case, so the assumption $a^2 - 5b^2/4 \neq 0$ is fair. □