Problem 7. Let $z_1$ and $z_2$ denote arbitrary but nonzero complex numbers. Show that the product of the inverses is the inverse of the product, i.e.,

$$\frac{1}{z_1 z_2} = \frac{1}{z_1} \cdot \frac{1}{z_2}.$$  

Proof. The grindy approach Chayes alluded to is straightforward, so I’ll give a proof using the algebraic approach. It is sufficient to show that the equation $(z_1 z_2)^{-1} w = 1$ is satisfied by $w = z_1^{-1} z_2^{-1}$ as multiplicative inverses in a field are unique. In other words, if we show $z_1^{-1} z_2^{-1}$ is a multiplicative inverse, then it must be equal to $(z_1 z_2)^{-1}$ as there’s only one multiplicative inverse. To this end, apply commutativity and associativity of multiplication a few times to get

$$(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1}) (z_2 z_2^{-1}) = 1 \cdot 1 = 1.$$  

Hence $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$, as desired. □

Problem 9. Prove that if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$.

Proof. Note that the statement is true for real numbers. For complex numbers write $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Then $z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$. Since $z_1 z_2 = 0$, we obtain the following two equations by setting the real and imaginary parts equal to zero:

$$a_1 a_2 - b_1 b_2 = 0$$

and

$$a_1 b_2 + a_2 b_1 = 0.$$  

If $z_1 = 0$ then we are done. Otherwise, either $a_1 \neq 0$ or $b_1 \neq 0$. In the former case we find $a_2 = b_1 b_2 / a_1$ from the first equation, so the second equation becomes

$$a_1 b_2 + \frac{b_1^2 b_2}{a_1} = \left(a_1 + \frac{b_1^2}{a_1}\right) b_2 = 0.$$  

Thus either $b_2 = 0$ or $a_1^2 = -b_1^2$. Note that the second option is impossible since a strictly positive number cannot equal a nonpositive number, meaning we must have $b_2 = 0$. This implies that $a_2 = b_1 b_2 / a_1 = 0$, so $z_2 = a_2 + b_2 i = 0$.

Roughly the same argument works in the $b_1 \neq 0$ case. Rearrange to find $b_2 = a_1 a_2 / b_1$, substitute this into the second equation, and use the zero-product property of real numbers as before to find $z_2 = 0$. □

Problem 15. Let $q = (q_0, \vec{q})$ and $p = (p_0, \vec{p})$ denote two quaternions. Derive a formula for the quaternionic product $q p$ in terms of the standard vector products.
Proof. If we write \( q = q_0 + \bar{q} \) where \( \bar{q} = q_1 i + q_2 j + q_3 k \) and likewise for \( p \) we have the expansion
\[
q \cdot p = (q_0 + \bar{q})(p_0 + \bar{p}) = q_0 p_0 + q_0 \bar{p} + p_0 \bar{q} + \bar{q} \bar{p}.
\]
Note that multiplication of quaternionic units satisfies the same algebraic relations as the cross-product of the standard basis vectors, except \( i^2 = j^2 = k^2 = -1 \). From this we find
\[
\bar{q} \bar{p} = (q_1 i + q_2 j + q_3 k)(p_1 i + p_2 j + p_3 k)
= -(q_1 p_1 + q_2 p_2 + q_3 p_3) + (q_2 p_3 - p_2 q_3)i + (q_3 p_1 - p_3 q_1)j + (q_1 p_2 - p_1 q_2)k
= -\bar{q} \cdot \bar{p} + \bar{q} \times \bar{p}.
\]
Hence
\[
q \cdot p = (q_0 p_0 - \bar{q} \cdot \bar{p}) + q_0 \bar{p} + p_0 \bar{q} + \bar{q} \times \bar{p}.
\]
\[\square\]

Problem 16. Show that there is a continuous two-parameter family of quaternion solutions to \( q^2 = -1 \).

Proof. We use the formula derived in Problem 15 to find
\[
q^2 = (q_0^2 - \bar{q} \cdot \bar{q}) + (2q_0 \bar{q} + \bar{q} \times \bar{q}) = (q_0^2 - \|\bar{q}\|^2) + 2q_0 \bar{q}
\]
where we’ve used the fact \( \bar{q} \times \bar{q} = 0 \). By comparing the constant and vectorial components of \( q^2 = -1 \) we get the equations \( q_0^2 - \|\bar{q}\|^2 = -1 \) and \( 2q_0 \bar{q} = \bar{0} \), the zero vector. Observe \( \bar{q} \neq \bar{0} \) since the first equation would then reduce to \( q_0^2 = -1 \), which is impossible for real \( q_0 \). Hence \( 2q_0 \bar{q} = \bar{0} \) implies \( q_0 = 0 \), meaning \( \|\bar{q}\|^2 = 1 \).

Geometrically the solutions to \( q^2 = -1 \) is the set of \( q = q_1 i + q_2 j + q_3 k \) which lie on the unit sphere as vectors in \( \mathbb{R}^3 \). In particular,
\[
\{ q : q^2 = -1 \} = \{ \cos \theta \sin \varphi i + \sin \theta \sin \varphi j + \cos \varphi k : \theta \in [0, 2\pi) \text{ and } \varphi \in [0, \pi) \}.
\]
The angles \( \theta, \varphi \) are the aforementioned two parameters of the family. There are other parameterizations that you can use, but these are probably the most convenient. \[\square\]

Problem 17. Let \( M \) and \( N \) be integers expressible as the sum of squares of two integers:
\[
M = A^2 + B^2 \\
N = C^2 + D^2.
\]
Use complex analysis to show that \( MN \) can also be written as the sum of squares of two integers.

Proof. Define \( z_M = A + iB \) and likewise \( z_N = C + iD \). Then \( M = |z_M|^2 = z_M \overline{z_M} \) while \( N = |z_N|^2 = z_N \overline{z_N} \). Hence \( MN = |z_N z_M|^2 \), yet a simple calculation shows
\[
z_N z_M = (AC - BD) + (AD + BC)i,
\]
so \( MN = (AC - BD)^2 + (AD + BC)^2 \) by the definition of complex modulus, as desired. \[\square\]

Problem 18. Show that \(|z| \leq |\text{Re } z| + |\text{Im } z| \leq \sqrt{2}|z|\).
Proof. Note that $|z| \leq |\text{Re}(z)| + |\text{Im}(z)|$ is simply the triangle inequality; if $z = a + bi$ then it is clear that $a^2 + b^2 \leq a^2 + 2|a||b| + b^2 = (|a| + |b|)^2$, and thus

$$|z| = \sqrt{a^2 + b^2} \leq \sqrt{a^2 + 2|a||b| + b^2} = |a| + |b|.$$

For the second inequality, it suffices to show that $a^2 + 2|a||b| + b^2 \leq 2(a^2 + b^2)$, since taking square roots would then give $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}|z|$. Equivalently, we would like to prove $2|a||b| \leq a^2 + b^2$. This can be deduced from the inequality $(|a| - |b|)^2 \geq 0$ or by applying AM-GM.

Problem 20. If $z$ is a complex number with $\text{Re}(z) > 1$, show that $0 \leq \text{Re}(1/z) \leq 1$.

Proof. Let $z = a + bi$ where $a > 1$. Observe that

$$\text{Re} \left( \frac{1}{a + bi} \right) = \text{Re} \left( \frac{a - bi}{a^2 + b^2} \right) = \frac{a}{a^2 + b^2}.$$

This quantity is clearly at least 0 since $a$ is positive. On the other hand, $a < a^2$ as $1 < a$. Since $a^2 \leq a^2 + b^2$ as well, we find that

$$\frac{a}{a^2 + b^2} < \frac{a^2}{a^2 + b^2} \leq \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

Problem 21. Consider $\mathbb{Q}[\sqrt{5}/2]$. Find (a) the law of multiplication for distinct elements of $\mathbb{Q}[\sqrt{5}/2]$ and (b) a formula for the inverse of an element $w = a + (\sqrt{5}/2)b$.

Proof. (a) Let $w = a + (\sqrt{5}/2)b$ and $v = c + (\sqrt{5}/2)d$. Then

$$wv = (ac + \frac{5bd}{4}) + \frac{\sqrt{5}}{2}(bc + ad).$$

(b) To find the inverse $v = w^{-1}$, use the multiplication law above. By comparing coefficients we obtain the two equations

$$ac + \frac{5b}{4}d = 1$$

and

$$bc + ad = 0.$$

We can quickly solve this system for $c$ and $d$ by using matrix algebra:

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a & 5b/4 \\ b & a \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{a^2 - 5b^2/4} \begin{bmatrix} a & -5b/4 \\ -b & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{a^2 - 5b^2/4} \begin{bmatrix} a \\ -b \end{bmatrix}.$$

Thus $c = a/(a^2 - 5b^2/4)$ and $d = -b/(a^2 - 5b^2/4)$.

Some of you might worry about the possibility that $a^2 - 5b^2/4 = 0$ for some choice of $a$ and $b$, which would break our formulas. This is not a problem though, since the solution $a = b = 0$ is eliminated by the assumption $w \neq 0$, and the only other possible solutions satisfy $a/b = \pm\sqrt{5}/2$. Since $a, b$ are rational numbers this cannot be the case, so the assumption $a^2 - 5b^2/4 \neq 0$ is fair.