

Stretched Kostka, Littlewood-Richardson, and Schubert Coefficients

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Joint work with Alexey Garber and Igor Pak.

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Motivation

In algebraic combinatorics, the *stretching* of various structure constants such as the *stretched Littlewood–Richardson (LR) coefficients* $c_{N\lambda, N\mu}^{N\nu}$ has been of recent interest.

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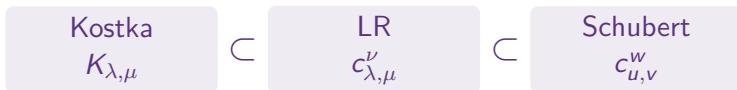
In algebraic combinatorics, the *stretching* of various structure constants such as the *stretched Littlewood–Richardson (LR) coefficients* $c_{N\lambda, N\mu}^{N\nu}$ has been of recent interest.

- Gelfand and coauthors suggested that these constants count the number of integer points in polytopes, giving a connection to *Ehrhart theory*.
- Stretching was studied in the context of the *saturation property*:

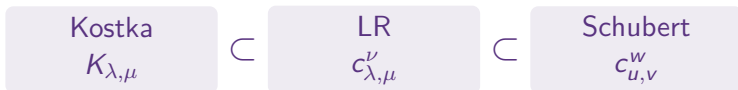
$$c_{\lambda, \mu}^{\nu} > 0 \iff c_{N\lambda, N\mu}^{N\nu} > 0 \text{ for any } N \geq 1,$$

whose proof (Knutson–Tao 1999) resolved the *Horn problem* on spectra of Hermitian matrices.

Overview

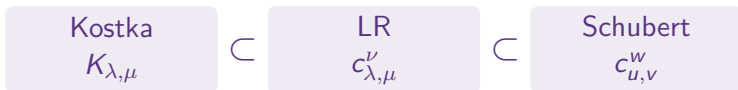


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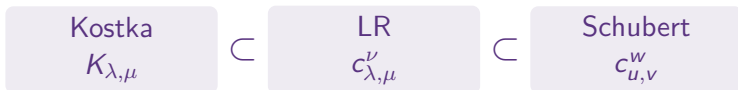


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Question 1

Can we generalize this result to Schubert structure constants for permutations $u, v, w \in S_n$?

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Question 2

In the case of Kostka numbers, how much does the value at $N = 1$ constrain the whole polynomial?

Kostka numbers and Schur polynomials

A partition λ specifies a Young diagram. A *semistandard young tableau (SSYT)* has **weakly increasing rows** and **strictly increasing columns**.

Example

$$\lambda = (2, 1)$$

1	2
3	

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$$K_{(2,1),(1,1,1)} = 2,$$

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$$K_{(2,1),(1,1,1)} = 2,$$

$$\begin{aligned} s_{\lambda}(x_1, x_2, x_3) &= 2x_1x_2x_3 \\ &+ x_1^2x_2 + x_1^2x_3 + x_1x_2^2 \\ &+ x_1x_3^2 + x_2^2x_3 + x_2x_3^2. \end{aligned}$$

Stretching a partition

Definition - *Stretched Kostka polynomial*

$$N\lambda := (N\lambda_1, N\lambda_2, \dots), \quad f_{\lambda, \mu}(N) := K_{N\lambda, N\mu}.$$

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For $\lambda = (2, 1)$, $\mu = (1, 1, 1)$, we have $f_{\lambda, \mu}(2) = 3$:

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In general,

$$f_{\lambda, \mu}(N) = N + 1$$

since we can put between 0 and N (inclusive) twos into row one.

Littlewood–Richardson coefficients

Structure constants for Schur polynomials

$$s_{\lambda} s_{\mu} = \sum_{\nu} c_{\lambda, \mu}^{\nu} s_{\nu}, \quad c_{\lambda, \mu}^{\nu} \in \mathbb{Z}_{\geq 0}.$$

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- Have many combinatorial interpretations, e.g., $c_{\lambda, \mu}^{\nu}$ is the number of LR tableaux of shape ν/λ and content μ .
- Generalize Kostka numbers:

$$K_{\lambda, \mu} = c_{\sigma, \lambda}^{\tau}, \quad \text{where} \quad \tau_i = \sum_{j \geq i} \mu_j, \quad \sigma_i = \sum_{j > i} \mu_j.$$

- This construction commutes with stretching.

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- No known (unsigned) combinatorial interpretations.
- Recover the LR coefficients if u, v, w are Grassmannian:

The **descent set** of a permutation w is

$$\text{Des}(w) := \{i : w(i) > w(i+1)\}.$$

If $\text{Des}(w) = \{r\}$, we say w is ***r-Grassmannian***. In this case,

$$\mathfrak{S}_w = s_{\lambda(w)}(x_1, \dots, x_r).$$

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How do we generalize stretching to permutations (so that it commutes with this construction)?

Stretching a permutation

Definition - *Lehmer code*

$$\text{code}_i(w) := \#\{j > i : w(j) < w(i)\}.$$

Example

$$\text{code}(2413) = (1, 2, 0, 0) = (1, 2).$$

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Definition - *Code scaling*

$$N * w := \text{code}^{-1}(N \text{code}(w)), \quad f_{u,v}^w(N) := c_{N*u, N*v}^{N*w}.$$

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$$w = 2413 \xrightarrow{\text{code}} (1, 2) \xrightarrow{\times 2} (2, 4) \xrightarrow{\text{code}^{-1}} 361245 = 2 * w.$$

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If w is r -Grassmannian, then the corresponding partition is

$$\lambda(w) := (\text{code}_r(w), \text{code}_{r-1}(w), \dots, \text{code}_1(w)),$$

so code scaling stretches the corresponding partition.

Saturation property and hive polytopes

Theorem – Saturation Property (Knutson–Tao 1999)

$$c_{\lambda, \mu}^{\nu} > 0 \iff c_{N\lambda, N\mu}^{N\nu} > 0 \text{ for any } N \geq 1,$$

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Proof idea:

- Construct a *hive polytope* $P_{\lambda, \mu}^{\nu}$ such that

$$c_{N\lambda, N\mu}^{N\nu} = |N \cdot P_{\lambda, \mu}^{\nu} \cap \mathbb{Z}^d|.$$

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$P_{\lambda, \mu}^{\nu}$ can have rational vertices so Ehrhart theory only guarantees that $f_{\lambda, \mu}^{\nu}(N)$ is a **quasi-polynomial**.

Definition - Quasi-polynomial (polynomial on residue classes)

$f(N)$ is a quasi-polynomial if there exist a period m and polynomials $f_0(N), \dots, f_{m-1}(N)$ such that

$$f(N) = f_i(N) \quad \text{for } N \equiv i \pmod{m}.$$

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In 2004, King, Tollu, and Toumazet (KTT) made a series of conjectures regarding these quasi-polynomials:

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- $k = 3$ still open, even for Kostka.

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(S3) In particular, the series

$$\sum_{N \geq 1} c_{N*u, N*v}^{N*w} t^N$$

is a rational function.

- Weaker than (S3), still open.

Results Part One

Stretched Schubert coefficients are eventually
quasi-polynomial

Main Theorem

Theorem (Pak–Slonim April 2026)

For fixed u, v, w , the function

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is eventually quasi-polynomial. Consequently,

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Subsequent improvement: Alexandersson (July 2026) proved eventual polynomiality using Demazure operators and Schubert duality.

Properties that do not generalize

Proposition – Coefficients can be negative

Let $u = 3142, v = 1432, w = 4321 \in S_4$. Then,

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Proposition – “Eventually” is necessary

For any $n \geq 4$, there exist $u, v, w \in S_n$ such that

$$c_{N*u, N*v}^{N*w} = \begin{cases} 1 & \text{for } N = 2n - 6, \\ 0 & \text{for } N > 2n - 6. \end{cases}$$

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Proposition – “Eventually” is necessary

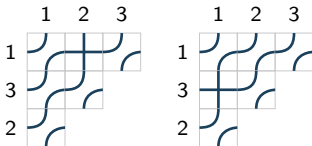
For any $n \geq 4$, there exist $u, v, w \in S_n$ such that

$$c_{N*u, N*v}^{N*w} = \begin{cases} 1 & \text{for } N = 2n - 6, \\ 0 & \text{for } N > 2n - 6. \end{cases}$$

The proof shows that the offset is at most $M = e^{O(n^3)}$.

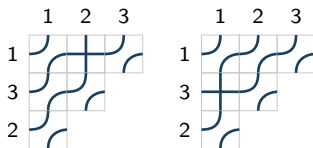
Pipe dreams

A *(reduced) pipe dream* uses **elbow** tiles and **cross** tiles; no two pipes cross more than once. Connect $1, 2, \dots, n$ along the top to $u(1), u(2), \dots, u(n)$ down the left side.



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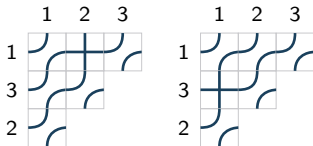
Theorem (Bergeron–Billey 1993)

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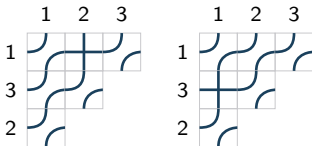
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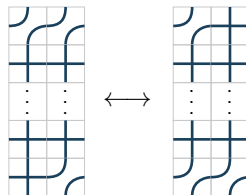
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Code scaling stretches the bottom pipe dream horizontally: $D_{\text{bot}}(N * u)$ has N times as many crosses in each row as $D_{\text{bot}}(u)$.

Ladder moves and encoding pipe dreams

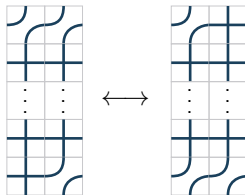
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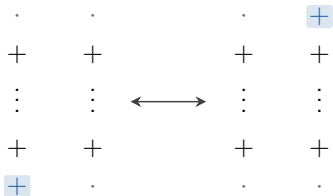
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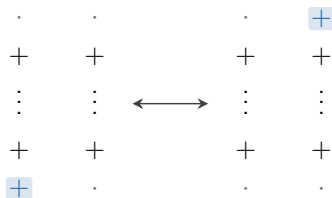
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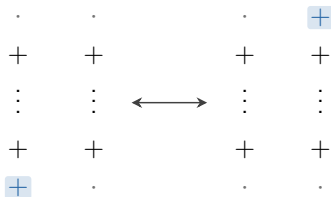
To each pipe dream, we associate a canonical *ladder sequence*

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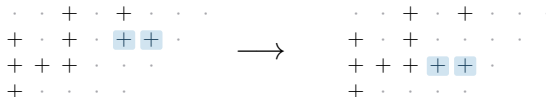
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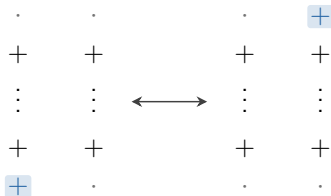
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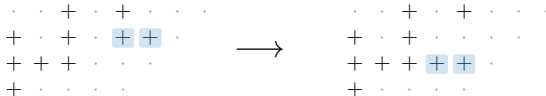
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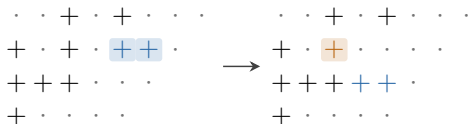
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- Record the number of crosses in each “patch”, i.e., that moved in the same way.



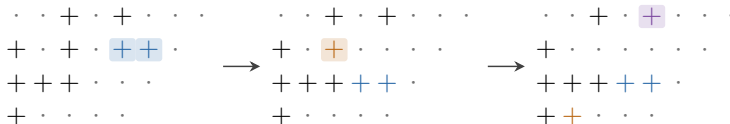
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Ladder sequence example



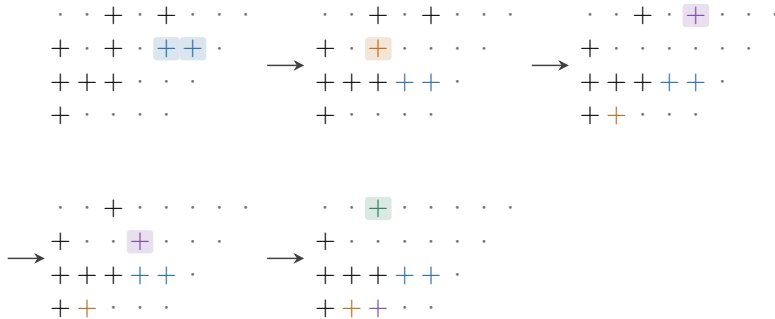
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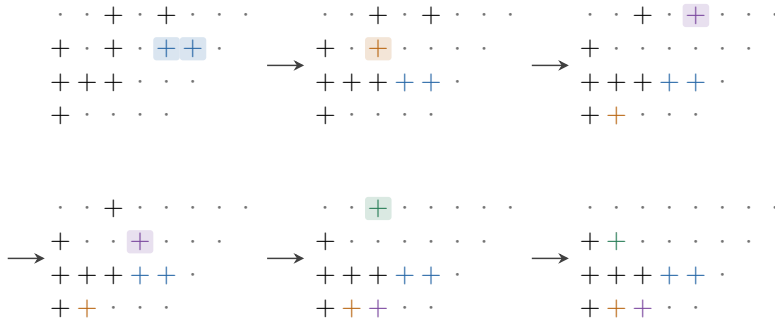
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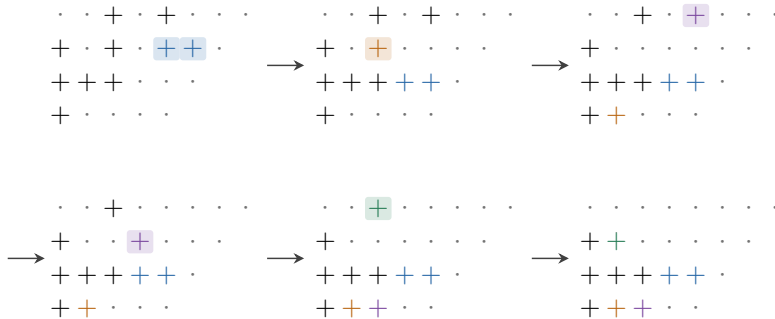
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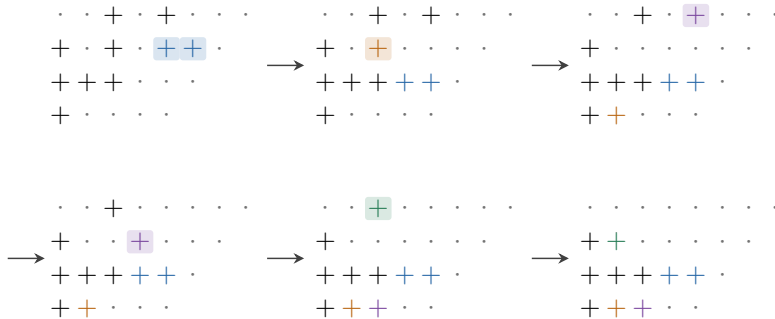
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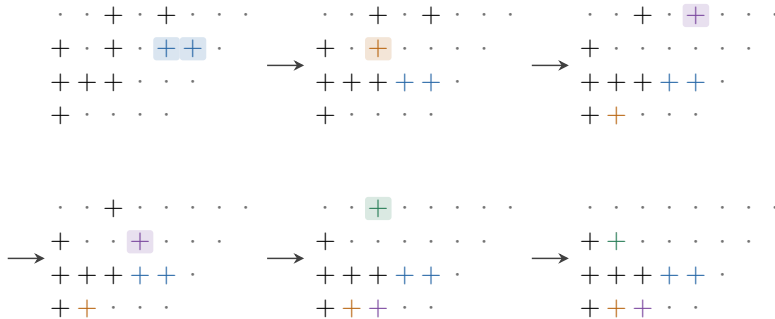
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- Valid ladder sequences satisfy inequalities involving the code.
- Stretching the permutation leaves the form of these inequalities fixed; only scales the code terms.

Triples of pipe dreams

Lemma (Postnikov–Stanley 2009)

For $u, v, w \in S_\infty$, we have

$$c_{u,v}^w = c_{u,v,w_\circ w} = \sum_{\sigma \in S_m} \text{sign}(\sigma) \sum_{(a,b,c) \in \Omega(\sigma)} K_{u,a} K_{v,b} K_{w_\circ w,c}.$$

$m := \max\{i : \text{code}_i(u) > 0 \text{ or } \text{code}_i(v) > 0 \text{ or } \text{code}_i(w) > 0\}$ and
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- Encode as ladder sequences $\Rightarrow$ counting integer points in a collection of polytopes.
- Stretching corresponds to affine dilations $A\mathbf{x} \leq N \cdot \mathbf{b} + \mathbf{c}$.

Results Part Two

There are finitely many stretched Kostka polynomials with a given Kostka number

Main Theorem

Theorem (Garber–Pak–Slonim 2026+)

Fix an integer k . There are finitely many stretched Kostka polynomials $f_{\lambda,\mu}(N)$ with $K_{\lambda,\mu} = k$.

In particular, if $k = 3$, then

$$f_{\lambda,\mu}(N) = 2N + 1 \quad \text{or} \quad f_{\lambda,\mu}(N) = \frac{(N+1)(N+2)}{2}.$$

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Lemma (Howard–McAlister 2011)

Let λ, μ be partitions such that

$$f_{\lambda,\mu}(N) := K_{N\lambda, N\mu}$$

has degree d . Then

$$\sum_{N \geq 0} f_{\lambda,\mu}(N) t^N = \frac{\sum_{i=0}^d h_i t^i}{(1-t)^{d+1}}, \quad h_i \in \mathbb{Z}_{\geq 0}.$$

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Expanding the generating function equation gives

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Remark: $K_{\lambda,\mu} \geq 1$ if and only if λ dominates μ .

Degree formula and bounding the length

Theorem (McAllister 2008)

Let (λ, μ) be primitive, $\ell := \ell(\mu)$, and pad λ with zeroes to length ℓ . Write $\lambda = (a_1^{v_1}, a_2^{v_2}, \dots, a_m^{v_m})$ in exponent notation. Then

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- In this case, $d \neq 0$ so (1) gives $d \geq \ell - 3$ so

$$\boxed{\ell \leq k + 2.}$$

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The corresponding hive polytopes are defined by a fixed system of inequalities

$$A_\ell \mathbf{x} \leq \mathbf{b}_{\lambda,\mu},$$

for some matrix A_ℓ independent of λ and μ .

Discrete geometry lemma

Lemma (Garber–Pak–Slonim 2026+)

Fix an integral matrix $A \in \text{Mat}_{n \times d}(\mathbb{Z})$ and $k \geq 1$. Up to integer translation, only finitely many polytopes

$$P_A(\mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\}, \quad \mathbf{b} \in \mathbb{Z}^n,$$

contain between 1 and k lattice points.

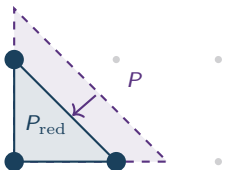
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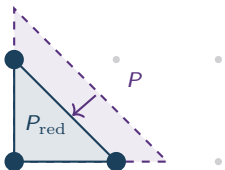
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Proof idea:

- Finitely many polytopes have the same reduction.
- Induct on dimension.
- Use Minkowski uniqueness with volume of facets.
- Denominator of vertices bounded in terms of A .

Summary

We showed:

- Stretched Schubert coefficients are eventually quasi-polynomial so $\sum_{N \geq 1} c_{N*u, N*v}^{N*w} t^N \in \mathbb{Q}(t)$.
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- Generalize the Kostka results to LR coefficients: finitely many $f_{\lambda, \mu}^{\nu}(N)$ with $c_{\lambda, \mu}^{\nu} = k$ and exact classification for $k = 3$.
- Show that the coefficients of the stretched Kostka (and more generally LR) polynomials are nonnegative.

Thank you