Select Geometry Qual Problems

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1 Preface

This is a compilation of solutions to many of the past UCLA Geometry/Topology Qual problems I have written up while preparing for the exam. The problems are sorted into two sections, focusing on the differential aspect of geometry and the algebraic aspect of topology respectively. The problems tend to be sorted by the year but there's no particular order I stuck to. You can find a problem by Ctrl+F and looking for the exam and problem in the format yyF.# (for Fall exams) and yyS.# (for Spring exams). Not all problems are solved here.

Many thanks to Josh Enwright for helpful discussions while compiling these.

2 Geometry

01S.2 On the compact connected n-manifold M, suppose α is a p-form and β is a $(n-p-1)$-form. Suppose ∂M has two components: $\partial_0 M$ and $\partial_1 M$. Let i_0 and i_1 be the inclusion of $\partial_0 M$ and $\partial_1 M$ into M. Given that $i_0^*\alpha = i_1^*\beta = 0$, prove that

$$\int\limits_M d\alpha \wedge \beta = (-1)^{p+1} \int\limits_M \alpha \wedge d\beta$$

Solution By Stokes' Theorem, we have that

$$\int\limits_M d(\alpha \wedge \beta) = \int\limits_{\partial M} \alpha \wedge \beta = \int\limits_M i_0^*(\alpha \wedge \beta) + \int\limits_M i_1^*(\alpha \wedge \beta)$$

But then

$$i_0^*(\alpha \wedge \beta) = i_0^*\alpha \wedge i_0^*\beta = 0 \text{ and } i_1^*(\alpha \wedge \beta) = i_1^*\alpha \wedge i_1^*\beta = 0$$

so that $d(\alpha \wedge \beta)$ must integrate to zero. Now we have that

$$0 = \int\limits_M d(\alpha \wedge \beta) = \int\limits_M [d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta] \Rightarrow \int\limits_M d\alpha \wedge \beta = (-1)^{p+1} \int\limits_M \alpha \wedge d\beta$$

01S.3 Suppose $f, g : S^1 \to \mathbb{R}$ are smooth embeddings and let

$$M = \{(a, b, \overrightarrow{v}) \in S^1 \times S^1 \times \mathbb{R} \mid f(a) - g(b) = \overrightarrow{v}\}$$

Show that M is a compact submanifold of $S^1 \times S^1 \times \mathbb{R}^2$. Let $\pi : M \to \mathbb{R}^2$ be the projection $\pi(a, b\overrightarrow{v}) = \overrightarrow{v}$. Apply Sard's Theorem to π and deduce that for almost every $\overrightarrow{v} \in \mathbb{R}^2$, $f(S^1)$ is transverse to $g(S^1) + \overrightarrow{v}$.

Solution M is a smooth submanifold of $S^1 \times S^1 \times \mathbb{R}^2$ because it is a level set of the smooth function $(a, b, \overrightarrow{v}) \mapsto f(a) - g(b) - \overrightarrow{v}$. To show that M is compact, it will suffice by Heine-Borel to show that M is closed and bounded. If $(a, b, \overrightarrow{v}) \in M$ then by the triangle inequality,

$$|\overrightarrow{v}| \leq \sup_{a \in S^1} |f(a)| + \sup_{b \in S^1} |g(b)| =: R < \infty$$

since f, g are smooth and S^1 is compact, so that $|(a, b, \overrightarrow{v})| \leq \sqrt{R^2 + 2} < \infty$, so M is bounded. Next, suppose that $(a_n, b_n, \overrightarrow{v_n})$ is a sequence in M converging to some $(a, b, \overrightarrow{v}) \in S^1 \times S^1 \times \mathbb{R}^n$. Then $a \in S^1$, $b \in S^1$, and

$$\overrightarrow{v} = \lim_{n\to\infty} \overrightarrow{v}_n = \lim_{n\to\infty} [f(a_n) - g(b_n)] = \lim_{n\to\infty} f(a_n) - \lim_{n\to\infty} g(b_n) = f(a) - g(b)$$

since f, g are smooth, so that $(a, b, \overrightarrow{v}) \in M$. Hence M is closed, so M is compact. Now let π be as given; then π is smooth so by Sard's Theorem, almost every $\overrightarrow{v} \in \mathbb{R}^2$ is a regular value of π. Now suppose $\overrightarrow{v} \in \mathbb{R}^2$ is a regular value of π, and let $(a, b, \overrightarrow{v})$ be any element in $\pi^{-1}(\overrightarrow{v})$. Then

$$\begin{aligned} &T_{(a,b,\overrightarrow{v})}M = \{(v, w, df_a(v) - dg_b(w)) \mid v \in T_aS^1, w \in T_bS^1\} \text{ so that} \\ &d\pi_{(a,b,\overrightarrow{v})}(T_{(a,b,\overrightarrow{v})}M) = df_a(T_aS^1) + dg_b(T_bS^1) \end{aligned}$$

Since $\overrightarrow{v}$ is a regular value of π, the left-hand side has dimension 2 so that the two subspaces $df_a(T_aS^1), dg_b(T_bS^1)$ must be transverse subspaces of $\mathbb{R}^2$, for all such a, b. But these are precisely the a and b for which $f(a) = g(b) + \overrightarrow{v}$, corresponding to all intersections of $f(S^1)$ and $g(S^1) + \overrightarrow{v}$, so that these two curves must be transverse.

10F.2, 16S.2 Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

Solution Let $f : X \times Y \to \mathbb{R}^n$ be defined by $f(x, y) = y - x$. Then f is smooth, so by Sard's Theorem almost every $a \in \mathbb{R}^n$ is a regular value of f. Take such an a, and let (x, y) be any element in $f^{-1}(a)$. Then

$$\begin{aligned} &T_{(x,y)}X \times Y = T_xX \oplus T_yY \text{ so that} \\ &df_{(x,y)}(T_{(x,y)}X \times Y) = T_xX + T_yY \end{aligned}$$

Since a is a regular value of f, the left-hand side is n-dimensional so that T_xX and T_yY are transverse subspaces of $\mathbb{R}^n$, for all such x, y. But these are precisely the x and y for which $x + a = y$, corresponding to all intersections of $X + a$ and Y, so that X and Y intersect transversely.

19F.1 State the classical Divergence Theorem for a compact 3-dimensional submanifold of $\mathbb{R}^3$ with smooth boundary. Derive it from Stokes' Theorem for differential forms.

Solution The classical divergence theorem states that for any compact 3-dimensional submanifold M of $\mathbb{R}^3$ with smooth boundary and any smooth vector field $\mathbf{F}$ in M,

$$\int_{\partial M} \mathbf{F} \cdot d\mathbf{S} = \int_{M} \text{div}\mathbf{F} dV$$

where $\mathbf{S}$ denotes the outward normal vector to ∂M and dV is the standard volume form. To obtain this, we first write $\mathbf{F}(x, y, z) := (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$; then each of F_1, F_2, F_3 is a smooth function because $\mathbf{F}$ is smooth. Now,

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{S} &= F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy \\ \Rightarrow d(\mathbf{F} \cdot d\mathbf{S}) &= (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}) dx \wedge dy \wedge dz = \text{div}\mathbf{F} dV \end{aligned}$$

The outward normal $\mathbf{S}$ gives an orientation of M, so that by Stokes' Theorem

$$\int_{\partial M} \mathbf{F} \cdot d\mathbf{S} = \int_{M} d(\mathbf{F} \cdot d\mathbf{S}) = \int_{M} \text{div}\mathbf{F} dV$$

19F.3 For which $n > 0$ does the real projective space $\mathbb{RP}^n$ admit a nowhere-vanishing vector field? If it exists, give an explicit one.

Solution If n is even, then consider $\pi : S^n \to \mathbb{RP}^n$ defined by $\pi(x_0, ..., x_n) = [x_0 : ... : x_n]$. π is a covering map which identifies antipodal points of S^n (and therefore has degree 2), so that the Euler characteristic is $\chi(\mathbb{RP}^n) = \frac{1}{2}\chi(S^n) = \frac{1}{2}2 = 1$. Thus by Poincaré-Hopf no even-dimensional $\mathbb{RP}^n$ can admit a nowhere-vanishing vector field.

If n is odd, then S^n admits the nowhere-vanishing vector field $(x_0, x_1, ..., x_{n-1}, x_n)$ $\mapsto (-x_1, x_0, -x_3, x_2, ..., -x_n, x_{n-1})$, which gives (after applying the same covering map π as above) the nowhere-vanishing vector field $[x_0, x_1, ..., x_{n-1}, x_n] \mapsto [-x_1, x_0, -x_3, x_2, ..., -x_n, x_{n-1}]$ on $\mathbb{RP}^n$ for odd n. Thus the n for which $\mathbb{RP}^n$ admits a nowhere-vanishing vector field are precisely the odd n.

19F.5 A vector field X on a Lie group G is *left-invariant* $(L_g)_*X = X$ for all $g \in G$, where $L_g : G \to G$ denotes left-multiplication by g. Show that if X, Y are left-invariant, so is $[X, Y]$. You must prove any fact about Lie brackets that you use.

Solution This will follow immediately from the fact that for any smooth $f : G \to G$, $f_*[X, Y] = [f_*X, f_*Y]$, as then we will have that

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y]$$

In order to show this fact, take any smooth $h : G \to \mathbb{R}$. Then

$$\begin{aligned} f_*[X, Y](h) &= [X, Y](h \circ f) \\ &= X(Y(h \circ f)) - Y(X(h \circ f)) \\ &= X(f_*Y(h) \circ f) - Y(f_*X(h) \circ f) \\ &= f_*X(f_*Y(h)) - f_*Y(f_*X(h)) = [f_*X, f_*Y](h) \end{aligned}$$

17F.1 Let M be a smooth manifold. Verify the following identities for vector fields X, Y and a smooth 1-form on M:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

Solution Fix Y and let X_1, X_2 be two vector fields and $f : M \to \mathbb{R}$ be any smooth function. Then

$$\begin{aligned}
&(fX_1 + X_2)\omega(Y) - Y\omega(fX_1 + X_2) - \omega([fX_1 + X_2, Y]) \\
&= (fX_1 + X_2)\omega(Y) - Y\omega(fX_1 + X_2) - \omega(-Y(f)X_1 + f[X_1, Y] + [X_2, Y]) \\
&= fX_1\omega(Y) + X_2\omega(Y) - fY\omega(X_1) - Y\omega(X_2) - Y(f)\omega(X_1) + Y(f)\omega(X_1) - f\omega([X_1, Y]) - \omega([X_2, Y]) \\
&= f(X_1\omega(Y) - Y\omega(X_1) - \omega([X_1, Y])) + (X_2\omega(Y) - Y\omega(X_2) - \omega([X_2, Y]))
\end{aligned}$$

Similarly fix X and let Y_1, Y_2 be two vector fields, $f : M \to \mathbb{R}$ be any smooth function. Then

$$\begin{aligned}
&X\omega(fY_1 + Y_2) - (fY_1 + Y_2)\omega(X) - \omega([X, fY_1 + Y_2]) \\
&= X\omega(fY_1 + Y_2) - (fY_1 + Y_2)\omega(X) - \omega(X(f)Y_1 + f[X, Y_1] + [X, Y_2]) \\
&= fX\omega(Y_1) + X(f)\omega(Y_1) + X\omega(Y_2) - fY_1\omega(X) - Y_2\omega(X) - X(f)\omega(Y_1) - f\omega([X, Y_1]) - \omega([X, Y_2]) \\
&= f(X\omega(Y_1) - Y_1\omega(X) - \omega([X, Y_1])) + (X\omega(Y_2) - Y_2\omega(X) - \omega([X, Y_2]))
\end{aligned}$$

The above computations show that the right-hand side of the desired identity is tensorial, so it will suffice to check the desired result on $X = \frac{\partial}{\partial x_j}, Y = \frac{\partial}{\partial x_k}$, where $[X, Y] = 0$ by Clairaut's Theorem. Write $\omega = \sum \omega_i dx_i$, so that

$$d\omega(X,Y) = \frac{\partial \omega_k}{\partial x_j} - \frac{\partial \omega_j}{\partial x_k} = X(\omega(Y)) - Y(\omega(X))$$

17F.2 Let $M_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real coefficients.
a) Show that $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = \mathrm{Id}\}$ is a smooth submanifold of $M_n(\mathbb{R})$.
b) Show that $O(n)$ has a trivial tangent bundle.

Solution a) $O(n)$ is smooth as it is the level set at the identity of the smooth function $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $f(A) = AA^T$.

b) Let $\pi : TO(n) \to O(n)$ be the canonical projection. For each $A \in O(n)$, let $L_A : O(n) \to O(n)$ where $L_A(B) = AB$. Then L_A is smooth and its inverse function is $L_{A^{-1}}$ which is smooth, so L_A is a diffeomorphism for each $A \in O(n)$. Now let $F : O(n) \times T_{\mathrm{Id}}O(n) \to TO(n)$ be defined by $F(A, v) = (L_A)_* v$. Then F is a bijection since L_A is a diffeomorphism, so it will suffice to show that F is smooth.

Since $TO(n)$ is locally trivial, let U be a trivializing neighborhood of Id. Then any chart of U gives a chart of each $V := L_A(U)$ via L_A, which together gives the atlas of $O(n)$. This then gives charts $\{V \times T_{\mathrm{Id}}O(n), \phi\}$ in $O(n) \times T_{\mathrm{Id}}O(n)$ and $\{\pi^{-1}(V), \psi\}$ in $TO(n)$, where by construction $F = \psi^{-1} \circ \phi$(because this is certainly true for $V = U$ and the others follow since L_A is a diffeomorphism), so in particular F is smooth. Thus F is a bundle isomorphism, so $O(n)$ has a trivial tangent bundle.

17F.4, 20S.3 Consider the differential 1-form $\omega = xdy - ydx + dz$ in $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

Solution We have that

$$d(f\omega) = df \wedge \omega + f d\omega = (2f - x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y})dx \wedge dy + \frac{\partial f}{\partial x} dx \wedge dz + \frac{\partial f}{\partial y} dy \wedge dz$$

If this is zero, then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, but then $2f = 0$ as well which contradicts the fact that f is nowhere zero. Thus $f\omega$ is not closed for any such f.

17F.5 Let x, y, z denote the standard Euclidean coordinates on $\mathbb{R}^3$ and let dA denote the standard area form on $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Determine the values of $n = 0, 1, 2, ...$ for which $\omega = z^n dA$ is exact.

Solution Let $i : S^2 \to \mathbb{R}^3$ be the standard inclusion. We have that

$$\omega = z^n i^*(zdx \wedge dy + ydz \wedge dx + xdy \wedge dz) =: z^n i^* \eta$$

Let $B \subseteq \mathbb{R}^3$ denote the closed ball of unit 1. Then B is a compact orientable manifold with boundary $\partial B = S^2$, so that by Stokes' Theorem

$$\int_{S^2} \omega = \int_B d(z^n \eta) = \int_B (n+3) z^n dx \wedge dy \wedge dz$$

ω is exact if and only if $[\omega] \in H^2_{deRham}(S^2)$ is trivial, if and only if this integral is zero. When n is odd, z^n is an odd function so since B is symmetric about the xy-plane, this integral is zero. When n is even, $z^n \geq 0$ on B with a positive volume set $(B \cap \{1/2 \leq z \leq 1/2\}\})$ on which it is bounded from below by 2^{-n}, so that the integral is nonzero. Thus ω is exact if and only if n is odd.

17F.6, 21F.7 a) Define what it means for a manifold M to be orientable.
b) Show that every connected nonorientable manifold M admits a connected, oriented double cover.

Solution a) The set of ordered bases on T_pM can be divided into two equivalence classes by orientation-preserving isomorphisms. An orientation of T_pM is an assigment of each equivalence class to $\{\pm 1\}$. M is orientable if there exists a smooth (in p) choice of orientations for each T_pM.

b) Let $\tilde{M} = \{(p, o) \,|\, p \in M, o \text{ is an orientation on } T_pM\}$. Let $\pi : \tilde{M} \to M$ be defined by $\pi(p, o) = p$. In a sufficiently small neighborhood of each (p, o), every o corresponds to the standard basis $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ for T_pM ordered in the same way, so that π is a diffeomorphism on this neighborhood and is thus the desired double cover, and furthermore, $(d\pi)^{-1}$ induces a local orientation on $\tilde{M}$.

Now suppose $\tilde{M}$ is not connected. Then it can be written as $\tilde{M} = U \cup V$ where U, V are disjoint open sets, and since π is a double cover, this means that $\pi_U : U \to M$ is a diffeomorphism. But then since U is orientable, so is M, which is a contradiction, so $\tilde{M}$ must be our desired connected, oriented double cover.

16S.1 Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Solution Let $[a : b : c] \in \mathbb{RP}^2$. If a, b are not both zero, then the equation $ax + by + c = 0$ defines a line in $\mathbb{R}^2$. If $[d : e : f] = [a : b : c]$ then (d, e, f) is a scalar multiple (call this scalar λ) of (a, b, c) so that $dx + ey + f = \lambda(ax + by + c) = 0 \Rightarrow ax + by + c = 0$ so it gives the same line. Thus $[a : b : c] \mapsto ax + by + c = 0$ is a well-defined injective function from $\mathbb{RP}^2 \setminus \{[0 : 0 : 1]\}$ into the set of lines in $\mathbb{R}^2$, and moreover every line can be represented this way so we can give the space of all straight lines a smooth structure such that this is a diffeomorphism.

Let M be the orientation cover (see 17F.6) of $\mathbb{RP}^2$. Then since M is a degree 2 cover, the orientation cover of $\mathbb{RP}^2 \setminus \{[0 : 0 : 1]\}$ is M with two points removed. Since $\mathbb{RP}^2$ is nonorientable, M is connected, so M with two points removed is still connected, so $\mathbb{RP}^2 \setminus \{[0 : 0 : 1]\}$ is nonorientable.

16S.3 Consider the vector field $X(z) = z^{2016} + 2016 z^{2015} + 2016$ on $\mathbb{C} = \mathbb{R}^2$. (By this we mean the following: take a complex coordinate z on $\mathbb{C}$, identify $T_z\mathbb{C} = \mathbb{C}$, and let $X(z) \in T_z\mathbb{C}$.) Compute the sum of indices of X over all the zeroes of X.

Solution All the zeroes of X (which must all be isolated) are contained in some compact disk D. Then by Poincaré-Hopf the sum of indices of X over all of its zeroes is equal to the Euler characteristic of D, which is 1.

16S.10 Consider the 3-form on $\mathbb{R}^4$ given by

$$\alpha = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3$$

Let S^3 be the unit sphere in $\mathbb{R}^4$ and $\iota : S^3 \to \mathbb{R}^4$ be its inclusion map.
a) Evaluate $\int_{S^3} \iota^* \alpha$.
b) Let γ be the 3-form on $\mathbb{R}^4 \setminus \{0\}$ given by

$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k}$$

for $k \in \mathbb{R}$. Determine the values of k for which γ is closed and those for which it is exact.

Solution a) Let $B \subseteq \mathbb{R}^4$ denote the closed ball of unit 1. Then B is a compact orientable manifold with boundary $\partial B = S^3$, so that by Stokes' Theorem

$$\int_{S^3} \iota^* \alpha = \int_B d\alpha = \int_B 4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = 2\pi^2$$

b) We have that

$$\begin{aligned} d\gamma &= \frac{-2k(x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4)}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{k+1}} \wedge \alpha + \frac{d\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k} \\ &= \left[\frac{-2k(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{k+1}} + \frac{4}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k} \right] dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ &= \frac{4 - 2k}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

so that γ is closed if and only if $k = 2$. Now,

$$\int_{S^3} \iota^* \gamma = \int_{S^3} \iota^* \alpha \neq 0$$

by part (a), so that $H^3_{deRham}(S^3) \ni [\iota^* \gamma] \neq 0$. But since $\pi : \mathbb{R}^4 \setminus \{0\} \to S^3$ defined by $\pi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)/(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$ is a deformation retraction, it induces an isomorphism of de Rham cohomologies. Since $\pi^*(\iota^* \gamma)) = \gamma$, we thus have that

$$H^3_{deRham}(\mathbb{R}^4 \setminus \{0\}) \ni [\gamma] \neq 0 \text{ so that } \gamma \text{ is not exact.}$$

Therefore, there does not exist a k for which γ is exact.

14S.2 Let M be a smooth manifold with boundary, and $f : \partial M \to \mathbb{R}^n$ a smooth map for some $n \geq 1$. Show that there is a smooth map $F : M \to \mathbb{R}^n$ such that $F|_{\partial M} = f$.

Solution Let U be a tubular neighborhood of ∂M in M, with the function $\pi : U \to \partial M$ projecting along the normal vector. Let ϕ be a smooth bump function which is identically 1 on ∂M and identically 0 outside U. Let $F : M \to \mathbb{R}^n$ by $F(x) = \phi(x) f(\pi(x))$ (which is well-defined since ϕ is zero outside of U where π is defined). F is smooth, and for all $x \in \partial M$ we have that $\pi(x) = x$ so that $F(x) = 1 \cdot f(x)$ so that $F|_{\partial M} = f$ as desired.

14S.4 Let $\omega_1, ..., \omega_n$ be 1-forms on a smooth manifold M. Show that $\{\omega_i\}$ is linearly independent if and only if $\omega_1 \wedge ... \wedge \omega_n \neq 0$.

Solution Suppose that the ω_i are linearly dependent. Then write $\omega_n = \sum_{i=1}^{n-1} a_i\omega_i$, so that

$$\omega_1 \wedge ... \wedge \omega_n = \omega_1 \wedge ... \wedge \omega_{n-1} \wedge (\sum_{i=1}^{n-1} a_i\omega_i) = \sum_{i=1}^{n-1}(-1)^{n-1-i}\omega_1 \wedge ... \wedge \omega_i \wedge \omega_i \wedge ... \wedge \omega_{n-1} = 0$$

Conversely, suppose the ω_i are linearly independent. Then at a given point p, they form a basis for T_p^*M, so take $v_1, ..., v_n$ to be their dual basis. Then

$$(\omega_1 \wedge ... \wedge \omega_n)_p(v_1, ..., v_n) = 1 \text{ so that } \omega_1 \wedge ... \wedge \omega_n \neq 0$$

19S.1 Let M be a smooth manifold. Show that there is a proper map $f : M \to \mathbb{R}$.

Solution M has a countable chart (since it is second-countable) where every point lies in finitely many of the sets (since it is paracompact), so let $U_1, U_2, ...$ be such a chart and take a partition of unity $\{\phi_n\}$ subordinate to this cover. Define $f : M \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} n\phi_n(x)$. Since $\phi_n(x)$ is only nonzero for finitely many n, $0 < f(x) < \infty$ so that f is a well-defined smooth function. Now for any real number R, if $f(x) \leq R$, then $x \in \bigcup_{n=1}^{\lfloor R \rfloor} \overline{U}_n$, since otherwise

$$\phi_n(x) = 0 \text{ for all } n = 1, ..., \lfloor R \rfloor \Rightarrow f(x) = \sum_{n=\lfloor R \rfloor+1}^{\infty} n\phi_n(x) \geq \lfloor R \rfloor + 1 > R \text{ which is a contradiction.}$$

Therefore $f^{-1}([-R, R]) \subseteq \bigcup_{n=1}^{\lfloor R \rfloor} \overline{U}_n$. Since the former set is closed and the latter is compact, the former must be closed. Now any compact $K \subseteq \mathbb{R}$ is bounded, so it is contained in some $[-R, R]$, and therefore $f^{-1}(K)$ is a closed subset of $f^{-1}([-R, R])$ which is compact, so it too is compact. Hence f is proper.

19S.2 Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be smooth and suppose 0 is a regular value of f. Let $M = f^{-1}(\{0\})$. Show that $M \times S^1$ is parallelizable.

Solution S^1 is parallelizable, since it admits a nowhere vanishing vector field $(x, y) \mapsto (y, -x)$. Then

$$T(M \times S^1) \simeq TM \times TS^1 \simeq TM \times S^1 \times \mathbb{R}$$

M is a codimension 1 submanifold of $\mathbb{R}^{n+1}$, so it has a 1-dimensional normal bundle. ∇f is a normal vector field which cannot be zero in M because 0 is a regular value of f, so it spans each normal space and so the normal bundle NM is trivial. Hence

$$TM \times \mathbb{R} \simeq \{(p, v, w) \mid p \in M, v \in T_pM, w \in N_pM\} \simeq TM \times \mathbb{R} \simeq M \times \mathbb{R}^{n+1}$$

since $T_pM \oplus N_pM = \mathbb{R}^{n+1}$ at each $p \in M$. Therefore

$$T(M \times S^1) \simeq TM \times S^1 \times \mathbb{R} \simeq M \times S^1 \times \mathbb{R}^{n+1}$$

so that the tangent bundle of $M \times S^1$ is trivial and hence $M \times S^1$ is parallelizable.

19F.4 Prove that for any vector fields X, Y with Lie derivatives acting on k-forms,

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$$

Solution

$$\begin{aligned}
[\mathcal{L}_X, \mathcal{L}_Y] &= [\mathcal{L}_X, d \circ i_Y + i_Y \circ d] \text{ by Cartan's formula} \\
&= \mathcal{L}_X \circ d \circ i_Y - d \circ i_Y \circ \mathcal{L}_X + \mathcal{L}_X \circ i_Y \circ d - i_Y \circ d \circ \mathcal{L}_X \\
&= d \circ [\mathcal{L}_X, i_Y] + [\mathcal{L}_X, i_Y] \circ d \\
&= d \circ i_{[X,Y]} + i_{[X,Y]} \circ d = \mathcal{L}_{[X,Y]} \text{ by Cartan's formula again.}
\end{aligned}$$

19S.5 Show that a closed 1-form on a manifold M is exact if and only if $\int_{S^1} f^*\omega = 0$ for every smooth $f : S^1 \to M$.

Solution Suppose ω is exact and let $f : S^1 \to M$ be any smooth function. Write $\omega = dg$ Then $f(S^1)$ is a compact orientable manifold with empty boundary, so by Stokes' Theorem

$$\int\limits_{S^1} f^*\omega = \int\limits_{f(S^1)} \omega = \int\limits_{\emptyset} dg = 0$$

Conversely, suppose $\int_{S^1} f^*\omega = 0$ for every smooth $f : S^1 \to M$. Without loss of generality assume M is path-connected (otherwise, do this for each path-component), and fix a base point x_0. For all $x \in M$ let $g(x) = \int_\gamma \omega$ where γ is a path from x_0 to x. g is well-defined because if γ, γ' are two paths from x_0 to x then attaching them gives a loop Γ, which is smooth up to homotopy and therefore satisfies $\int_\Gamma \omega = 0$, so that

$$\int_{\gamma'} \omega = \int_\Gamma \omega + \int_\gamma \omega = \int_\gamma \omega$$

Now by the Fundamental Theorem of Calculus we have for every $x \in M$ that

$$dg_x = \frac{d}{dt}|_{t=1} \int\limits_\gamma \omega = \omega_x$$

so that ω is exact.

19S.6 Let $f : X \to Y$ be a smooth, finite covering map between smooth manifolds. Show that the induced map on de Rham cohomology

$$f^* : H^n_{deRham}(Y;\mathbb{R}) \to H^n_{deRham}(X;\mathbb{R})$$

is injective.

Solution For each $p \in Y$, there exists a neighborhood U of p such that $f^{-1}(U) = \bigcup_{j=1}^k U_j$. Define the form $g(\omega)$ by

$$g(\omega)_p := \frac{1}{k}\sum_{j=1}^k ((f|U_j)^{-1})^*(\omega|U)_p$$

Now consider $\omega' = \omega + d\alpha$. We have that

$$\begin{aligned} g(\omega')_p &= \frac{1}{k}\sum_{j=1}^k ((f|U_j)^{-1})^*(\omega'|U)_p = \frac{1}{k}\sum_{j=1}^k ((f|U_j)^{-1})^*(\omega|U)_p + \frac{1}{k}\sum_{j=1}^k ((f|U_j)^{-1})^*(d\alpha|U)_p \\ &= g(\omega)_p + d\left[\frac{1}{k}\sum_{j=1}^k ((f|U_j)^{-1})^*(\alpha|U)_p\right] =: g(\omega)_p + d\beta_p \end{aligned}$$

so that g is a well-defined map $g : H^n_{deRham}(X;\mathbb{R}) \to H^n_{deRham}(Y;\mathbb{R})$ on homology. Finally, for any $\omega \in H^n_{deRham}(Y;\mathbb{R})$,

$$(g \circ f^*)(\omega)_p = \frac{1}{k}\sum_{j=1}^k ((f|U_j)^{-1})^*(f^*\omega|U)_p = (\omega|U)_p = \omega_p$$

so that $g \circ f^*$ is the identity on $H^n_{deRham}(Y;\mathbb{R})$, and so f^* must be injective.

16F.1 (Smooth Urysohn Lemma) Let M be a smooth manifold. Prove that for any two disjoint closed subsets $A, B \subseteq M$ there is a smooth function $f : M \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

Solution Let $B \subseteq \mathbb{R}^n$ be an open ball, and let x_0, r be its center and radius respectively. Then the function

$$\phi_{x_0,r}(x) := \begin{cases} \exp\left(\frac{1}{|x-x_0|^2-r^2}\right) & |x-x_0| < r \\ 0 & |x-x_0| \geq r \end{cases}$$

is a smooth function $B \to \mathbb{R}$ where $\phi^{-1}_{x_0,r}(0) = \mathbb{R}^n \setminus B$. Now for any closed set A, write $\mathbb{R}^n \setminus A$ as a union of open balls. Since M is paracompact, this can be done in a locally finite way, so take such a union and for each ball, take its corresponding $\phi_{y,r}$ and let $\phi(x) = \sum \phi_{y,r}(x)$. Then $\phi^{-1}(0) = A$.

Now let $A \subseteq M$ be any closed set. By paracompactness, there exists a locally finite open cover of M by $\{U_\alpha\}$ such that each $\overline{U_\alpha}$ is contained in a chart (V_α, ψ). By the above there exists a $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi^{-1}(0) = \psi(A \cap \overline{U_\alpha})$. Now extend $\psi \circ \phi$ to $f_\alpha : M \to \mathbb{R}$ by declaring it zero outside of V, and let $f_A := \sum f_\alpha$. Then $f_A^{-1}(0) = A$. Finally, given any two disjoint closed subsets A, B, f_A and f_B cannot both be zero at once, so we have a well-defined smooth function

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$$

When $x \in A$, $f_A(x) = 0$ so that $f(x) = 0$, and when $x \in B$, $f_B(x) = 0$ so that $f_A(x) = 1$, so f is our desired function.

16F.2 (Whitney's Immersion Theorem) Let $M \subseteq \mathbb{R}^N$ be a smooth k-dimensional submanifold. Prove that M can be immersed into $\mathbb{R}^{2k}$.

Proof Proceed by induction. Suppose there is an immersion $f : M \to \mathbb{R}^L$, which is certainly true for $L = N$ since M is embedded. If $L \leq 2k$ then we are done, so assume that $L > 2k$. Define

$$g : TM \to \mathbb{R}^L \text{ by } g(p, v) = Df_p(v)$$

By Sard's Theorem, take a regular value a of g. Since $L > 2k$, $\mathbb{R}^L$ has a higher dimension than TM (which is $2k$-dimensional), so a cannot be in the image of g. Let $\pi : \mathbb{R}^L \to \mathbb{R}^{L-1}$ be the projection onto the orthogonal complement of $\operatorname{span}\{a\}$. Now suppose $D(\pi \circ f)_p(v) = 0$. Then

$$0 = D(\pi \circ f)_p(v) = (\pi \circ Df_p)(v) \Rightarrow Df_p(v) = g(p, v) \in \operatorname{span}\{a\}$$

Since a is not in the image of g, this implies that v must be zero. Therefore $D(\pi \circ f)_p$ is injective at each $p \in M$, so that $\pi \circ f : M \to \mathbb{R}^{L-1}$ is an immersion as well, which completes the induction.

16F.4 Show that

$$D = \ker(dx_3 - x_1 dx_2) \cap \ker(dx_1 - x_4 dx_2) \subseteq T\mathbb{R}^4$$

is a smooth distribution of rank 2, and determine whether D is integrable.

Solution The differential forms $\alpha := dx_3 - x_1 dx_2$ and $\beta := dx_1 - x_4 dx_2$ are (pointwise) linear maps $T_p\mathbb{R}^4 \to \mathbb{R}$, so with the standard basis $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_4}\}$ for $T_p\mathbb{R}^4$ we can represent α and β by the matrices $\begin{pmatrix} 0 & -x_1 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -x_4 & 0 & 0 \end{pmatrix}$ respectively. We see that $X = \sum_{i=1}^4 f_i \frac{\partial}{\partial x_i}$ lies in D if and only if

$$\begin{pmatrix} 0 & -x_1 & 1 & 0 \\ 1 & -x_4 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = 0$$

For all $(x_1, ..., x_4)$, the leftmost matrix has rank 2 (because of the 1's), so its kernel is 2-dimensional. Hence D is a smooth distribution of rank 2. For all $(x_1, ..., x_4)$, the vector fields

$$X_1 = \frac{\partial}{\partial x_4} \text{ and } X_2 = x_4 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$$

are linearly independent, so they form a global basis for D. But

$$[X_1, X_2] = \frac{\partial}{\partial x_4}(x_4 \frac{\partial}{\partial x_1}) + \text{several terms which are } 0 = \frac{\partial}{\partial x_1} \notin D$$

so that by the Frobenius Integrability Theorem, D is not integrable.

17S.1 Let M be a connected smooth manifold of dimension at least 2. Prove that for any $2n$ distinct points $x_1, ..., x_n, y_1, ..., y_n \in M$ there is a diffeomorphism $f : M \to M$ such that $f(x_i) = y_i$ for all i.

Solution By induction on n. First let $\sim$ be the equivalence relation defined on M by $x \sim y$ if there exists a diffeomorphism $f : M \to M$ with $f(x) = f(y)$. For any $x \in M$, fix a chart (U, ϕ) such that $\phi(x) = 0$. Then $\phi(y) = a \neq 0$ for any $x \neq y \in U$, so let $X = a \cdot (\sum_{i=1}^n \frac{\partial}{\partial x_i})$. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a bump function which is identically 1 on $\overline{B(0, |a|)}$ and identically 0 outside of $B(0, 2|a|)$. Then ψX is a compactly supported vector field on $\phi(U) \subseteq \mathbb{R}^n$, so that $(\phi^{-1})_*(\psi X)$ is a compactly supported vector field on U, and can therefore be extended to a vector field Y on M by declaring it zero outside U. Y therefore has a global flow Φ_t for all $t \in \mathbb{R}$ which is a diffeomorphism. In particular, by definition $\Phi_1(x) = y$, so Φ_1 is our desired diffeomorphism in the case that $n = 1$. Additionally, note that Φ_1 is the identity outside of a compact subset of M.

Now suppose that for any connected smooth manifold N of dimension at least 2 and any $2n - 2$ distinct points $x_1, ..., x_{n-1}, y_1, ..., y_{n-1} \in N$ there is a diffeomorphism $g : N \to N$ such that $g(x_i) = y_i$ for all i which is the identity outside of a compact subset of N. Given any $2n$ distinct points $x_1, ..., x_n, y_1, ..., y_n \in M$, let $N = M \setminus \{x_n, y_n\}$, which is also a connected smooth manifold of dimension at least 2 if M is. Let g be the diffeomorphism given by the above inductive hypothesis. Since g is the identity outside a of a compact subset of N, there exist neighborhoods of x_n, y_n where g is the identity. Therefore g extends to a diffeomorphism $\tilde{g} : M \to M$ by $\tilde{(g)}(x_n) = x_n$ and $\tilde{(g)}(y_n) = y_n$. Similarly, returning to the $n = 1$ case we consider $N' = M \setminus \{x_1, ..., x_{n-1}, y_1, ..., y_{n-1}\}$, and let $h : N' \to N'$ be a diffeomorphism with $h(x_n) = y_n$ which is the identity outside a compact subset of N'. Then again there exist neighborhoods of $x_1, ..., x_{n-1}, y_1, ..., y_{n-1}$ where h is the identity, so h extends to a diffeomorphism $\tilde{h} : M \to M$ by $\tilde{h}(x_i) = x_i$ and $\tilde{h}(y_i) = y_i$ for $1 \leq i \leq n - 1$. Then $f := \tilde{g} \circ \tilde{h} : M \to M$ is a diffeomorphism which is the identity outside a compact subset of M which satisfies $f(x_i) = y_i$ for each $1 \leq i \leq n$, which completes the induction.

17S.2 Let $M_{2n\times 2n}(\mathbb{R}) = \mathbb{R}^{4n^2}$ be the space of $2n \times 2n$ real matrices. Consider the following matrix in block form

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in M_{2n\times 2n}(\mathbb{R})$$

where I_n is the $n \times n$ identity matrix. Show that the subspace

$$S = \{A \in M_{2n\times 2n}(\mathbb{R}) \mid A^T \Omega A = \Omega\}$$

is a smooth submanifold of $M_{2n\times 2n}(\mathbb{R})$, and compute its dimension.

Solution Let $Skew_{2n}(\mathbb{R})$ be the subspace of skew-symmetric $2n \times 2n$ matrices. Since a skew-symmetric matrix is uniquely determined by its $(i,j)^{th}$ entries for $i < j$, this is a submanifold of $M_{2n\times 2n}(\mathbb{R})$ of dimension $\binom{2n}{2}$. For every $A \in M_{2n\times 2n}(\mathbb{R})$, the matrix $A^T \Omega A$ is skew-symmetric since Ω is, so that $f : M_{2n\times 2n}(\mathbb{R}) \to Skew_{2n}(\mathbb{R}$ can be defined by $f(A) = A^T \Omega A$. Now,

$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \frac{f(A+hB) - f(A)}{h} = \frac{(A+hB)^T \Omega (A+hB) - A^T \Omega A}{h} \\ &= \lim_{h=0} \frac{A^T \Omega A - A^T \Omega A + h(B^T \Omega A + A^T \Omega B) + h^2 B^T \Omega B}{h} = B^T \Omega A + A^T \Omega B \end{aligned}$$

For every $A \in S = f^{-1}(\{\Omega\})$, we have that $\det(A^T \Omega A) = \det(A^2) = 1 = \det(\Omega)$, and in particular that A is invertible, so that for any $C \in Skew_{2n}(\mathbb{R})$, if $B = \frac{1}{2}\Omega^{-1}(A^{-1})^T C$, then by the above formula $df_A(B) = C$. Hence

$$df : M_{2n\times 2n}(\mathbb{R}) = T_A(M_{2n\times 2n}(\mathbb{R})) \to T_\Omega(Skew_{2n}(\mathbb{R}) = Skew_{2n}(\mathbb{R})$$

is surjective for every $A \in S$, so that Ω is a regular value of f and hence S is a smooth submanifold of $M_{2n\times 2n}(\mathbb{R})$. Its codimension is $\dim(Skew_{2n}(\mathbb{R})) = \binom{n}{2}$, so its dimension is $4n^2 - n(2n-1) = 2n^2 + n$.

17S.3 Use the Poincaré-Hopf index theorem to calculate the Euler characteristic of the n-sphere.

Solution For odd n, let $S^n \subseteq \mathbb{R}^{n+1} = \mathbb{C}^{(n+1)/2}$. Then the vector field $p \mapsto ip$ is nonvanishing, so by Poincaré-Hopf the Euler characteristic of S^n is an empty sum which is zero. For even n, let $S^n \subseteq \mathbb{R}^{n+1} = \mathbb{C}^{n/2} \times \mathbb{R}$. Then the vector field $X(p,r) = (ip, 0)$ has only two zeroes, at $(0, \pm 1)$, so by Poincaré-Hopf the Euler characteristic of S^n is the sum of the indices of X at these two zeroes. Let B be the neighborhood of $(0,1)$ consisting of all (p,r) where $r \geq 0$. Then B does not contain $(0,-1)$, so the index of X at $(0,1)$ is the degree of the map $X|_{\partial B} : \partial B = S^{n-1} \to S^{n-1}$, but this map is just $p \mapsto ip$ which has degree 1. Therefore X has index 1 at $(0,1)$, and a similar argument using B' the lower hemisphere (all (p,r) where $r \leq 0$) shows that X has index 1 at $(0,-1)$ as well. Hence the Euler characteristic of S^n is 2.

17S.4, 11S.2 a) State (11S.2: and also prove) the Cartan formula (also known as Cartan's magic formula) for the Lie derivative of a differential form with respect to a vector field.
b) Use this formula to show that a vector field X on $\mathbb{R}^3$ has a flow (defined locally and for a short time) that preserves volume if and only if the divergence of X is everywhre zero.

Solution a) The Cartan formula states that for each n-form ω,

$$\mathcal{L}_X \omega = d(i_X \omega) + i_X(d\omega)$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X and $i_X \omega$ is the $(n-1)$-form defined by
$i_X(\omega)(X_1, ..., x_{n-1}) = \omega(X, X_1, ..., X_{n-1})$. The proof can be found in any differential geometry text.

b) Let $\omega = dx \wedge dy \wedge dz$. Then if the flow Φ of X preserves volume, $\Phi_t^*\omega = \omega$ for sufficiently small t, so that in particular $\mathcal{L}_X\omega = 0$. Conversely, if $\mathcal{L}_X\omega = 0$, then $\mathcal{L}_X(\Phi_t^*\omega) = \Phi_t^*\mathcal{L}_X\omega = 0$ for all t, so that $\Phi_t^*\omega$ is constant in t so it must equal $\Phi_0^*\omega = \omega$. Thus the flow of X preserves volume if and only if $\mathcal{L}_X\omega = 0$. Since $d\omega = 0$, by Cartan's formula we have that

$$\mathcal{L}_X\omega = d(i_X\omega) + i_X(d\omega) = d(i_X\omega)$$

Write $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$. Then, evaluating on basis vectors gives

$$i_X(\omega) = fdy \wedge dz + gdz \wedge dx + hdx \wedge dy$$
$$\Rightarrow d(i_X\omega) = (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z})\omega = \mathrm{div}(F)\omega$$

We see that the flow of X preserves volume if and only if this is zero, if and only if $\mathrm{div}(F) = 0$.

17S.5 Let

$$\omega = \frac{-ydx + xdy}{(x^2+y^2)^\alpha}$$

be a 1-form on $\mathbb{R}^2 \setminus \{0\}$ with the usual coordinates (x, y), and for some $\alpha \in \mathbb{R}$. Consider $\int_\gamma \omega$, where $\gamma : S^1 \to \mathbb{R}^2 \setminus \{0\}$ is smooth.
a) For which $\alpha \in \mathbb{R}$ do we have $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ whenever ω_0 and ω_1 are smoothly homotopic?
b) What are the possible values of $\int_\gamma \omega$ when α is chosen as in part a?

Solution a) Each circle $S^1(R) \subseteq \mathbb{R}^2 \setminus \{0\}$ is smoothly homotopic to each other, so the integral of ω cannot depend on R. Putting polar coordinates $x = r\cos\theta, y = r\sin\theta$ gives

$$\int\limits_{S^1(R)} \omega = \int\limits_0^{2\pi} R^{2-2\alpha} d\theta = 2\pi R^{2-2\alpha}$$

Since this cannot depend on R, we must have that $\alpha = 1$. Conversely, if $\alpha = 1$ then $d\omega = 0$, so that ω is closed and therefore integrates to the same value on smoothly homotopic curves.

b) We have that $\pi_1(\mathbb{R}^2 \setminus \{0\} \simeq \pi_1(S^1) \simeq \mathbb{Z}$ by the usual deformation retraction, so the homotopy class of any loop corresponds to the homotopy class of the curve γ_n which wraps around S^1 n times. Therefore all the possible values are (where we use the integral from part a)

$$\int\limits_\gamma \omega = \int\limits_{\gamma_n} \omega = n\int\limits_{S^1} \omega = 2\pi n, n \in \mathbb{Z}$$

17S.10, 12S.9 Let G be a finite group and X be a smooth manifold on which G acts smoothly. If the action of G on X is free, then show that the natural quotient map $X \to X/G$ is a covering map (of smooth manifolds).

Solution Let $G = \{g_1, ..., g_n\}$. Under the natural quotient $\pi : X \to X/G$, write $\pi^{-1}(y) = \{g_1x, ..., g_nx\}$. By the freeness of the action, these are all distinct points. Choose charts $(W_i, \phi_i) \subseteq V_i$. Now $\pi(W_i) = \pi(g_iW_1) = \pi(W_1) =: W$ for each i, so that $\pi_{-1}(W)$ is the disjoint union of the W_i (and so W is open).

Suppose $\pi(x) = \pi(y)$ for $x, y \in W_i$. Then there exists $g_j \in G$ with $g_jx = y$, so $y \in g_jW_i = W_k$ where $g_jg_i = g_k$. But then $y \in W_k \cap W_i$, so since these sets are disjoint for $k \neq i$ we must have that $k = i$. But then $g_jg_i = g_i$ so $g_j = e$ so $y = g_jx = x$. Therefore $\pi|_{W_i}$ is injective for each i, and since it is surjective by construction, it is a bijection. $\pi|_{W_i}$ is an open map since for each open set $U \subseteq W_i$, $\pi^{-1}\pi(U)$ is the disjoint union of $g_jU \subseteq W_j$ for each j, which is certainly open. Therefore $\pi|_{W_i}$ is a homeomorphism for each i. (Cont'd on next page)

Finally, note that $W \xrightarrow{(\pi|_{W_i})^{-1}} W_i \xrightarrow{\phi_i} \mathbb{R}^n$ gives a chart on W the open neighborhood of $y \in X/G$, and repeating this construction gives a chart about each point so that X/G is a smooth manifold. Therefore, π is a smooth covering map of manifolds.

18F.1 Let M be a compact smooth n-manifold and $f : M \to \mathbb{R}^n$ be a smooth map. Let

$$S = \{p \in M \mid \operatorname{rank}(df_p) < n\}$$

a) Show that $S \neq \emptyset$.
b) Show that $f(S) \in \mathbb{R}^n$ has empty interior.

Solution a) If $S = \emptyset$, then f is a local diffeomorphism, so in particular it is an open map, so that $f(M)$ is an open and compact subset of $\mathbb{R}^n$, contradiction.

b) By definition, if $y \in \mathbb{R}^n \setminus f(S)$ then y is a regular value of f, so by Sard's Theorem $f(S)$ has empty interior.

Recurring Let M_n be the space of $n \times n$ matrices, viewed as the smooth manifold $\mathbb{R}^{n^2}$. Let M_n^k be the subset of matrices of rank k. Prove that M_n^k is a smooth submanifold of M_n. (18F.2, 15S.1, 13S.1)

Solution Each rank k matrix has an invertible $k \times k$ minor, so without loss of generality assume it's in the top left (otherwise just permute the rows and columns until this is the case). Let

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

where B is the invertible $k \times k$ minor. Then we see that

$$\operatorname{rank}(A) = k \iff \operatorname{rank}\left[A \begin{pmatrix} I_k & -B^{-1}C \\ 0 & I_{n-k} = k \end{pmatrix}\right] \iff \operatorname{rank}\begin{pmatrix} B & 0 \\ D & E - DB^{-1}C \end{pmatrix} \iff E - DB^{-1}C = 0$$

where I_k, I_{n-k} are the respective square identities, since the matrix we multiplied A by is clearly invertible. Now define $f : M_n \to M_{n-k}$ by $f(A) = f\begin{pmatrix} B & C \\ D & E \end{pmatrix} = E - DB^{-1}C$. Then for each $B \in M_{n-k}$,

$$df_A \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} = \lim_{h \to 0} \frac{f(A + h\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}) - f(A)}{h} = \lim_{h \to 0} \frac{hB + E - DB^{-1}C - E + DB^{-1}C}{h} = B$$

so that df_A is surjective for any A, and in particular that 0 is a regular value of f. Therefore $M_n^k = f^{-1}(0)$ is a smooth submanifold of M_n.

18F.3 Let θ be the restriction of

$$(x^2dx^1 - x^1dx^2) + (x^4dx^3 - x^3dx^4) + ... + (x^{2n}dx^{2n-1} - x^{2n-1}dx^{2n})$$

to the unit sphere $S^{2n-1} \subseteq \mathbb{R}^{2n}$. Prove $\ker(\theta)$ is a distribution on S^{2n-1}. Is it integrable?

Solution Let X be the vector field $X(x^1, x^2, ..., x^{2n-1}, x^{2n}) = (-x^2, x^1, ..., -x^{2n}, x^{2n-1})$; then

$$\theta(X) = \sum_{i=1}^{2n} (x^i)^2 = 1$$

so that θ is nonvanishing on S^{2n-1}, and therefore $\ker(\theta)$ has the same dimension everywhere so it is a distriution. Now,

$$d\theta = -2\sum_{j=1}^{n} dx^{2j-1} \wedge dx^{2j} \text{ so that}$$
$$\theta \wedge d\theta = 2\sum_{j=1}^{n}\sum_{k\neq j} x^{2j-1}dx^{2j} \wedge dx^{2k-1} \wedge dx^{2k} - x^{2j}dx^{2j-1} \wedge dx^{2k-1} \wedge dx^{2k}$$

If this equals zero, then $x^1 = ... = x^{2n} = 0$ which is a contradiction since $(x^1, ..., x^{2n}) \in S^{n-1}$. Therefore $\theta \wedge d\theta \neq 0$ so that $\ker(\theta)$ is not integrable by Frobenius.

18F.4 Let M be a compact smooth 3-manifold and $\omega \in \Omega^1(M)$ a nowhere zero 1-form, so that $\ker(\omega)$ is an integrable distribution. Prove the following.
(i) $\omega \wedge d\omega = 0$
(ii) There exists a 1-form α such that $d\omega = \alpha \wedge \omega$.
(iii) $d\alpha \wedge \omega = 0$.

Solution For all $p \in M$, $\ker(\omega_p)$ is 2-dimensional, so pick a basis $\{X, Y\}$ for it and extend it to a basis $\{X, Y, Z\}$ for T_pM. Then

$$(\omega \wedge d\omega)_p(X, Y, Z) = \omega_p(Z)d\omega_p(X, Y) - \omega_p(Z)d\omega_p(Y, X) = 2\omega_p(Z)d\omega_p(X, Y)$$

since all other terms vanish since they are in the kernel of ω_p. But if $X, Y \in \ker(\omega)$, then $[X, Y] \in \ker(\omega)$ by integrability, so that

$$0 = \omega([X, Y]) = X(\omega(Y)) - Y(\omega(X)) - d\omega(X, Y) = d\omega(X, Y)$$
$$\Rightarrow (\omega \wedge d\omega)(X, Y, Z) = 2\omega(Z)d\omega(X, Y) = 0$$

which proves (i). Now find local coordinates (x, y, z) such that $\omega = f dx$ locally. Then f is never zero since ω is nonvanishing, so

$$d\omega = \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial f}{\partial z}dz \wedge dx = \frac{1}{f}\left[\frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right] \wedge \omega =: \alpha \wedge \omega$$

locally. Taking enough neighborhoods where this is true to make a locally finite cover of M (which exists by paracompactness) and extending α globally over M via a partition of unity on this cover proves (ii). Finally,

$$0 = d(d\omega) = d\alpha \wedge \omega - \alpha \wedge d\omega = d\alpha \wedge \omega - \alpha \wedge \alpha \wedge \omega = d\alpha \wedge \omega$$

by (ii), so this gives (iii).

18F.5 Let $M \subseteq \mathbb{R}^n$ be a compact $(n-1)$-dimensional manifold, let $\iota : M \to \mathbb{R}^n$ be the inclusion map, and let $D \subseteq \mathbb{R}^n$ be the n-dimensional compact region with $\partial D = M$. Let $dV = dx^1 \wedge ... \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ be the standard volume form.

a) Define $dA \in \Omega^{n-1}(M)$, the standard volume form on M induced by ι.

b) Prove that $\iota^*(i_X(dV)) = \langle X, N\rangle dA$ for any smooth vector field X on $\mathbb{R}^n$, where N is the outward unit normal to M.

c) Prove that

$$\int_D \mathcal{L}_X dV = \int_M \langle X, N\rangle dA$$

d) Derive Gauss's Divergence Theorem in the case $n = 3$.

Solution a) $dA = \iota^*(i_N(dV))$.

b) Let $T = X - \langle X, N\rangle N$. Then T is tangent to M because we have subtracted off the projection onto the normal space, so $\iota^*(i_T(dV))(X_1, ..., X_{n-1}) = dV(T, d\iota X_1, ..., d\iota X_{n-1}) = 0$ since these vector fields are all tangent to M and therefore are linearly dependent since there are n of them and M is $(n-1)$-dimensional. Therefore

$$\begin{aligned} 0 &= \iota^*(i_T(dV)) = \iota^*(i_X(dV) - i_{\langle X,N\rangle N}(dV)) = \iota^*(i_X(dV)) - \langle X, N\rangle \iota^*(i_N(dV)) \\ &= \iota^*(i_X(dV)) - \langle X, N\rangle dA \Rightarrow \iota^*(i_X(dV)) = \langle X, N\rangle dA \end{aligned}$$

c) We have

$$\begin{aligned} \int_M \langle X, N\rangle dA &= \int_M \iota^*(i_X(dV)) \\ &= \int_D d(i_X(dV)) \text{ by Stokes' Theorem} \\ &= \int_D (\mathcal{L}_X - i_X \circ d)(dV) \text{ by Cartan's Formula} \\ &= \int_D \mathcal{L}_X(dV) \text{ since } d(dV) = 0 \end{aligned}$$

d) Write $X = f\partial_x + g\partial y + h\partial z$. Then

$$\begin{aligned} \mathcal{L}_X(dV) &= \mathcal{L}_X(dx) \wedge dy \wedge dz + dx \wedge \mathcal{L}_X(dy) \wedge dz + dz \wedge dy \wedge \mathcal{L}_X(dz) \\ &= d(\mathcal{L}_X(x)) \wedge dy \wedge dz + dx \wedge d(\mathcal{L}_X(y)) \wedge dz + dx \wedge dy \wedge d(\mathcal{L}_X(z)) \\ &= df \wedge dy \wedge dz + dx \wedge dg \wedge dz + dx \wedge dy \wedge dh \\ &= \frac{\partial f}{\partial x} dx \wedge dy \wedge dz + \frac{\partial g}{\partial y} dx \wedge dy \wedge dz + \frac{\partial h}{\partial z} dx \wedge dy \wedge dz = \operatorname{div}(X) dV \end{aligned}$$

so that we have the desired

$$\int_D \operatorname{div}(X) dV = \int_M \langle X, N\rangle dA$$

18F.7 Prove that the covering map $\pi : S^n \to \mathbb{RP}^n$ induces an isomorphism on de Rham cohomology if and only if n is odd. What is the orientable double cover of $\mathbb{RP}^n$?

Solution By the universal coefficient theorem and de Rham's Theorem we see that

$$H^k_{deRham}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases} \text{ and } H^k_{deRham}(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = n \text{ and } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Since π is a finite-sheeted covering map, $\pi^* : H^k_{deRham}(\mathbb{RP}^n) \to H^k_{deRham}(S^n)$ is injective, so by dimensionality we see that it is an isomorphism for all k except n, where it is an isomorphism if and only if n is odd. For n even, $x \mapsto -x$ is a nontrivial deck transformation for π which is orientation-reversing, so S^n is the orientable double cover of $\mathbb{RP}^n$ if n is even. If n is odd, $\mathbb{RP}^n$ is already orientable, so its orientable double cover is the disjoint union of two copies of itself.

15S.3 Consider two collections of 1-forms $\omega_1, ..., \omega_k$ and $\phi_1, ..., \phi_k$ on an n-dimensional manifold M. Assume that

$$\omega_1 \wedge ... \wedge \omega_k = \phi_1 \wedge ... \wedge \phi_k$$

never vanishes on M. Show that there are smooth functions $f_{ij} : M \to \mathbb{R}$ such that

$$\omega_i = \sum_{j=1}^{k} f_{ij}\phi_j$$

Solution Since $\phi_1 \wedge ... \wedge \phi_k \neq 0$ they are all linearly independent, but then $\phi_1 \wedge ... \wedge \phi_k \wedge \omega_i = \omega^1 \wedge ... \wedge \omega_i \wedge ... \wedge \omega_k \wedge \omega_i = 0$, so adding ω_i makes them linearly dependent, and therefore ω_i must be a linear combination of $\{\phi_1, ..., \phi_k\}$.

15S.6 Let

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

be a 2-form defined on $\mathbb{R}^3 \setminus \{0\}$. If $i : S^2 = \{x^2 + y^2 + z^2 = 1\} \to \mathbb{R}^3$ is the inclusion, compute $\int_{S^2} i^*\omega$. Also compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbb{R}^3$ maps $(x, y, z) \to (3x, 2y, 8z)$.

Solution Let $\alpha = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Then $\alpha = i^*\omega$ on S^2 and α is globally defined on $\mathbb{R}^3$. Let $B \subseteq \mathbb{R}^3$ be the closed ball of radius 1. Then B is compact and orientable and $\partial B = S^2$, so that by Stokes' Theorem,

$$\int_{S^2} i^*\omega = \int_B d\alpha = \int_B 3dx \wedge dy \wedge dz = 4\pi$$

Note that j is a diffeomorphism onto its image because its inverse is clearly $(x, y, z) \mapsto (x/3, y/2, z/8)$, so j factors as $j : S^2 \xrightarrow{\phi} j(S^2) \xrightarrow{k} \mathbb{R}^3 \setminus \{0\}$ where k is the inclusion map. Let A be the region outside S^2 inside $j(S^2)$, which is a well-defined annular region since S^2 lies inside $j(S^2)$, and we have that $\partial A = j(S^2) \cup S^2$, but with these two components oriented oppositely. Now by Stokes' Theorem, we have that

$$\int_{j(S^2)} k^*\omega - \int_{S^2} i^*\omega = \int_A d\omega$$

But computing $d\omega$ gives that

$$d\omega = (\frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 = z^2)^{5/2}} + \frac{x^2 + y^2 + z^2 - 3y^2}{(x^2 + y^2 = z^2)^{5/2}} + \frac{x^2 + y^2 + z^2 - 3z^2}{(x^2 + y^2 = z^2)^{5/2}})dx \wedge dy \wedge dz = 0$$

so that

$$\int_{S^2} j^*\omega = \int_{j(S^2)} \phi^*(k^*\omega) = \int_{j(S^2)} k^*\omega = \int_{iS^2} i^*\omega = 4\pi$$

15S.7 Define the de Rham cohomology groups $H^i_{dR}(M)$ of a manifold M and compute $H^i_{dR}(S^1)$ for $i = 0, 1, ...$

Solution $H^i_{dR}(M) = \{\text{closed } i - \text{forms on } M\}/\{\text{exact } i - \text{forms on } M\}$, so in particular $H^i_{dR}(S^1) = 0$ for all $i \geq 2$ since every i-form for $i \geq 2$ is zero, so it remains to consider $i = 0$ and 1. No nonzero 0-form is exact, so a 0-form $f \in H^0_{dR}(S^1)$ if and only if $df = 0$, if and only if f is a constant function, so that we must have $H^0_{dR}(S^1) = \{f : M \to \mathbb{R} \text{ constant}\} = \mathbb{R}$.

Let $I : H^1_{dR}(S^1) \to \mathbb{R}$ be defined by $I([\omega]) = \int_{S^1} \omega$. This is well-defined since by Stokes' Theorem

$$\int_{S^1} d\alpha = \int_{\partial S^1} \alpha = \int_{\emptyset} \alpha = 0$$

Furthermore, I is $\mathbb{R}$-linear and injective, since if $\int_{S^1} \omega = 0$ then ω is exact. Let $i : S^1 \to \mathbb{R}^2$ be the inclusion. Then let $B \subseteq \mathbb{R}^2$ be the closed ball of radius 1, which is compact, orientable, and safisties $\partial B = S^1$. Then

$$I(i^*(-ydx + xdy)) = \int_{S^1} i^*(-ydx + xdy) = \int_B d(-ydx + xdy) = \int_B 2dx \wedge dy = 2\pi \neq 0$$

so that I is surjective since any multiple of this form gives any real number. Therefore I is an isomorphism, so $H^1_{dR}(S^1) = \mathbb{R}$.

15S.10, 12F.7, 21F.5 Let M be a compact orientable smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M;\mathbb{R})$ is even.

Solution Suppose $\dim H^{2n+1}(M;\mathbb{R}) \simeq \mathbb{R}^k$. By de Rham's Theorem it suffices to consider the de Rham cohomology groups, where we have the natural map

$$\wedge : H^{2n+1}(M) \times H^{2n+1}(M) \to H^{4n+2}(M)$$

where we know that $H^{4n+2}(M) \simeq \mathbb{R}$ by Poincaré duality. Since $\wedge$ is alternating and bilinear, we can represent it by a skew-symmetric $k \times k$ matrix A where $\mathbb{R}^k \times \mathbb{R}^k \ni (v,w) \mapsto v^T Aw$. SUppose that $Aw = 0$; then $v^T Aw = 0$ for all v, which corresponds to a $(2n+1)$-form ω such that for every $(2n+1)$-form α, $\alpha \wedge \omega = 0$. Taking local coordinates, let $\alpha = dx_{i_1} \wedge ... \wedge dx_{i_{2n+1}}$ so that the corresponding term $a_{j_1 ... j_{2n+1}} dx_{j_1} \wedge ... \wedge dx_{j_{2n+1}}$ in ω (where $\{i_1, ..., i_{2n+1}, j_1, ..., j_{2n+1}\} = \{1, ..., 4n+2\}$ is zero. Therefore ω is zero since every term is zero, so $w = 0$ so that A must be invertible as well. Now $\det(A) = \det(A^T) = (-1)^k \det(A)$, so since A is invertible we must have that k is even as desired.

15F.1, 10F.3 Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real coefficients.
a) Show that $SL(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$ is a smooth submanifold of $M_n(\mathbb{R})$.
b) Show that $SL(n,\mathbb{R})$ has trivial Euler characteristic.

Solution a) It will suffice to show that 1 is a regular value of the smooth function $\det : M_n(\mathbb{R}) \to \mathbb{R}$. We have, for any $A \in SL(n,\mathbb{R})$ and any $r \in \mathbb{R}$,

$$\begin{aligned} d(\det)_A(\frac{r}{n}A) &= \lim_{h\to 0} \frac{\det(A + h(r/n)A) - \det(A)}{h} = \lim_{h\to 0} \frac{\det(A)(\det(I + h(r/n)A^{-1}A) - 1)}{h} \\ &= \lim_{h\to 0} \frac{(1 + h(r/n))^n - 1}{h} = \lim_{h\to 0} r(1 + h\frac{r}{n})^{n-1} = r \end{aligned}$$

so that $d(\det)_A$ is surjective. Therefore 1 is a regular value of det and therefore $SL(n,\mathbb{R}) = \det^{-1}(1)$ is a smooth submanifold of $M_n(\mathbb{R})$.

b) Let $A = UP$ be the polar decomposition (where U is unitary and P is positive definite since A is invertible) of any $A \in SL(2,\mathbb{R})$. Then $A \mapsto U = AP^{-1}$ is a smooth function $SL(2,\mathbb{R}) \to SO(2,\mathbb{R})$ such that if A is already orthogonal then $U = A$. In fact this is a deformation retract since A is homotopic to AP^{-1} via $A(tP^{-1} + (1-t)I)$, so it suffices to compute the Euler characteristic of $SO(2,\mathbb{R})$. But this is compact since $O(2,\mathbb{R})$ is, so let B_t be the matrix with 1's along the diagonal and t in the top right corner, and $f_t(A) = B_t A$. Then $f_0 = \mathrm{id}$ and f_1 has no fixed points since A is invertible, so the identity map is homotopic (via f_t) to a map with no fixed points, so by the Lefschetz fixed point theorem $\chi(SL(2,\mathbb{R})) = \chi(SO(2,\mathbb{R})) = 0$.

15F.3 For two smooth vector fields X, Y on a smooth manifold M, prove

$$[\mathcal{L}_X, i_X]\omega = i_{[X,Y]}\omega$$

where ω is a k-form for $k \geq 1$.

Solution For any $k-1$ vector fields $V_1, ..., V_{k-1}$ we have that

$$\begin{aligned}\mathcal{L}_X(i_Y(\omega))(V_1,...,V_{k-1}) &= X(i_Y(\omega)(V_1,...,V_k)) - \sum_{i=1}^{k-1} i_Y(\omega)(V_1,...,V_{i-1},[X,V_i],V_{i+1},...,V_{k-1}) \\ &= X(\omega(Y,V_1,...,V_k)) - \sum_{i=1}^{k-1} \omega(Y,V_1,...,V_{i-1},[X,V_i],V_{i+1},...,V_{k-1})\end{aligned}$$

and

$$\begin{aligned}i_Y(\mathcal{L}_X(\omega))(V_1,...,V_{k-1}) &= \mathcal{L}_X(\omega)(Y,V_1,...,V_{k-1}) \\ &= X(\omega(Y,V_1,...,V_k)) - -\omega([X,Y],V_1,...,V_{k-1}) \\ &- \sum_{i=1}^{k-1} \omega(Y,V_1,...,V_{i-1},[X,V_i],V_{i+1},...,V_{k-1})\end{aligned}$$

so that subtracting cancels most terms and gives

$$\begin{aligned}[\mathcal{L}_X, i_X]\omega(V_1,...,V_{k-1}) &= \mathcal{L}_X(i_Y(\omega))(V_1,...,V_{k-1}) - i_Y(\mathcal{L}_X(\omega))(V_1,...,V_{k-1}) \\ &= \omega([X,Y],V_1,...,V_{k-1}) = i_{[X,Y]}\omega(V_1,...,V_{k-1})\end{aligned}$$

15F.4 Let $M = \mathbb{R}^3/\mathbb{Z}^3$ be the 3-dimensional torus and $C = \pi(L)$ where $L \subseteq \mathbb{R}^3$ is the oriented line segment from $(0,1,1)$ to $(1,3,5)$ and $\pi : \mathbb{R}^3 \to M$ is the quotient map. Find a differential form on M that represents the Poincaré dual of C.

Solution Write $\pi = (\pi_1, \pi_2, \pi_3)$ and θ be the 1-form on S^1 such that $\int_{S_1} \theta = 1$. Then let $dx = \pi_1^*\theta, dy = \pi_2^*\theta, dz = \pi_3^*\theta$ be 1-forms on $\mathbb{R}^3$. Then $dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$ are linearly independent in $H^2_{deRham}(M)$, so they form a basis since the latter is 3-dimensional, so it suffices to write the Poincaré dual of C as $\omega = adx \wedge dy + bdy \wedge dz + cdz \wedge dx$. Letting $i : C \to M$ be the inclusion we have, for example, that

$$a = \int_M adx \wedge dy \wedge dz = \int_M dz \wedge \omega = \int_C i^*dz = \int_C i^*\pi_3^*\theta = \deg(\pi_3 \circ i) \int_{S^1} \theta = \deg(\pi_3 \circ i) = 4$$

Similarly, π_1 is a degree 1 cover of C while π_2 is a degree 2 cover, so that $b = 1$ and $c = 2$ and

$$\omega = 4dx \wedge dy + dy \wedge dz + 2dz \wedge dx$$

15F.6, 12F.3 Let $M^m \subset \mathbb{R}^n$ be a smooth manifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

Solution Let $p, q \in \mathbb{R}^n \setminus M$ and select a path $\gamma : [0,1] \to \mathbb{R}^n$. $\{0,1\}$ is a closed submanifold of $[0,1]$ and since $p, q \notin M$, $\gamma|_{\{0,1\}} \pitchfork M$ trivially, so by the Extension Theorem there exists a path $\gamma' : [0,1] \to \mathbb{R}^n$ such that $\gamma' \pitchfork M$. If there exists $x \in [0,1]$ such that $\gamma'(x) \in M$, then by transversality

$$d\gamma'_x(T_x[0,1]) \oplus T_{\gamma'(x)}M = T_{\gamma'(x)}\mathbb{R}^n$$

But the left-hand side is $m+1 < n$ dimensional, which is a contradiction. Therefore γ' is a path from p to q in $\mathbb{R}^n \setminus M$, so that $\mathbb{R}^n \setminus M$ is path-connected.

Let $\gamma : [0,1] \to \mathbb{R}^n \setminus M$ be a loop with $\gamma(0) = \gamma(1) =: p$. Since $\mathbb{R}^n$ is simply connected, γ is homotopic (say, via $H : [0,1]^2 \to \mathbb{R}^n$) to the constant map at p. Since γ does not intersect M and $p \notin M$, $H|_{\partial[0,1]^2}$ does not intersect M, and $\partial[0,1]^2$ is a closed submanifold of $[0,1]^2$ so that by the Extension Theorem again there exists

$$G : [0,1]^2 \to \mathbb{R}^n \text{ such that } G|_{\partial[0,1]^2} = H|_{\partial[0,1]^2} \text{ and } G \pitchfork M$$
$$\Rightarrow \begin{cases} G(0,x) = H(0,x) = \gamma(x) \\ G(1,x) = H(1,x) = p \\ G(t,0) = H(t,0) = p \\ G(t,1) = H(t,1) = p \end{cases}$$

so that G is also a path homotopy between γ and the constant loop at p. If there exists $(t,x) \in [0,1]^2$ such that $G(t,x) \in M$, then by transversality

$$dG_{(t,x)}(T_{(t,x)}[0,1]^2) \oplus T_{G(t,x)}M = T_{G(t,x)}\mathbb{R}^n$$

But the left-hand side is $m+2 < n$ dimensional, which is a contradiction. Therefore G also does not intersect M, so γ is homotopic to the constant loop in $\mathbb{R}^n \setminus M$ and therefore it is simply connected.

21F.1 Let $V_k(\mathbb{R}^n)$ denote the space of k-tuples of orthonormal vectors in $\mathbb{R}^n$. Show that $V_k(\mathbb{R}^n)$ is a manifold of dimension $k(n - \frac{k+1}{2}$

Solution Let $F : M_{n\times k}(\mathbb{R}) \to Sym_{k\times k}(\mathbb{R}) \simeq \mathbb{R}^{k(k+1)/2}$ (where $Sym_{k\times k}(\mathbb{R})$ is the space of symmetric $k \times k$ real matrices) defined by $F(A) = A^TA$. The columns of A form an orthonormal k-tuple in $\mathbb{R}^n$ if and only if $A^TA = I_k$, so that $V_k(\mathbb{R}^n) = F^{-1}(I)$ and it remains to prove that I is a regular value of F. For every $A \in V_k(\mathbb{R}^n)$,

$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \frac{f(A+hB) - f(A)}{h} = \frac{(A+hB)^T(A+hB) - A^TA}{h} \\ &= \lim_{h=0} \frac{A^TA - A^TA + h(B^TA + A^TB) + h^2B^TB}{h} = B^TA + A^TB \end{aligned}$$

For every $A \in V_k(\mathbb{R}^n)$, rank$(A) = k$ since $A^TA = I$. Let C be any symmetric $k \times k$ matrix. Then the matrix C' constructed by taking the upper triangular entries and half the diagonal entries can be written as B^TA for some $B \in M_{n\times k}(\mathbb{R})$ since A has sufficient rank. But then $C = C' + (C')^T = B^TA + A^TB$. Therefore df_A is surjective for every A, so that I is indeed a regular value of F. By the Preimage Theorem, $V_k(\mathbb{R}^n)$ is a smooth manifold of codimension $k(k+1)/2$, so it has dimension

$$\dim(V_k(\mathbb{R}^n)) = nk - \frac{k(k+1)}{2} = k(n - \frac{k+1}{2})$$

21F.2 (Kevaire's Theorem) Show that $S^p \times S^q$ is parallelizable if p or q is odd.

Solution Consider both S^p and S^q embedded in $\mathbb{R}^{p+1}$ and $\mathbb{R}^{q+1}$ respectively. Without loss of generality suppose p is odd (otherwise just relabel p, q). Then the normal bundle NS^q is trivial because S^q is orientable. Additionally, S^p admits a nonvanishing vector field because p is odd so it has Euler characteristic zero, so its tangent bundle can be written as $\alpha \oplus \xi$ where ξ is a trivial line bundle. Now let π_{S^p}, π_{S_q} denote the projections onto the respective factors, so that

$$\begin{aligned} T(S^p \times S^q) &= \pi_{S^p}^*(TS^p) \oplus \pi_{S^q}^*(TS^q) = \pi_{S^p}^*(\alpha \oplus \xi) \oplus \pi_{S^q}^*(TS^q) \\ &= \pi_{S^p}^*(\alpha) \oplus (\xi \oplus \pi_{S^q}^*(TS^q)) \end{aligned}$$

But since NS^q is trivial, $\xi = \pi_{S^q}^*(NS^q)$, so we obtain

$$\begin{aligned} T(S^p \times S^q) &= \pi_{S^p}^*(\alpha) \oplus (\xi \oplus \pi_{S^q}^*(TS^q)) = \pi_{S^p}^*(\alpha) \oplus (\xi \oplus \pi_{S^q}^*(TS^q)) = \pi_{S^p}(\alpha) \oplus (\pi_{S^q}^*(TS^q) \oplus \pi_{S^q}^*(NS^q)) \\ &= \pi_{S^p}^*(\alpha) \oplus \pi_{S^q}^*(T\mathbb{R}^{q+1}) = \pi_{S^p}^*(\alpha) \oplus \xi^{q+1} \end{aligned}$$

since $\mathbb{R}^{q+1}$ is parallelizable. Now we have that

$$\begin{aligned} T(S^p \times S^q) &= \pi_{S^p}^*(\alpha) \oplus \xi^{q+1} = \pi_{S^p}^*(\alpha \oplus \xi) \oplus \xi^q \\ &= \pi_{S^p}^*(TS^p) \oplus \xi^q = \pi_{S^p}^*(TS^p \oplus \xi) \oplus \xi^{q-1} \end{aligned}$$

But NS^p is also trivial (again because S^p is orientable) so just as before $\xi = \pi_{S^q}^*(NS^q)$ so that

$$\begin{aligned} T(S^p \times S^q) &= \pi_{S^p}^*(TS^p \oplus \xi) \oplus \xi^{q-1} = \pi_{S^p}^*(TS^p \oplus NS^p) \oplus \xi^{q-1} \\ &= \pi_{S^p}^*(T\mathbb{R}^{p+1}) \oplus \xi^{q-1} = \xi^{p+1} \oplus \xi^{q-1} = \xi^{p+q} \end{aligned}$$

so that $T(S^p \times S^q)$ is trivial, and therefore $S^p \times S^q$ is parallelizable.

21F.4 Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ be a compactly supported n-form. Show that $\omega = d\eta$ for some compactly supported $(n-1)$-form $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} \omega = 0$.

Solution By Poincaré duality, the compactly supported cohomology group $H_c^n(\mathbb{R}^n)$ is isomorphic to the de Rham cohomology group $H^0(\mathbb{R}^n) \simeq \mathbb{R}$. First, if $\omega = d\eta$ then for any compact subset C of $\mathbb{R}^n$ whose interior contains the support of η, by Stokes' Theorem

$$\int_C \omega = \int_{\partial C} d\eta = 0$$

But if η is supported inside C, so is ω, so that $\int_{\mathbb{R}^n} \omega = \int_C \omega = 0$.

For the converse direction, consider $f : H_c^n(\mathbb{R}^n) \to \mathbb{R}$ defined by $f(\omega) = \int_{\mathbb{R}^n} \omega$. This is well-defined by the previous direction since exact forms integrate to zero. f is linear, and if $\varphi : \mathbb{R}^n \to [0,1]$ is a smooth function supported inside the cube $[0,1]^n$ which is identically 1 on the smaller cube $[1/3, 2/3]^n$ (which exists by the smooth Urysohn Lemma), then

$$0 < 3^{-n} \leq \int_{\mathbb{R}^n} \varphi dx_1 \wedge dx_2 \wedge ... \wedge dx_n \leq 1$$

so that f is nonzero, and hence (because the dimension of its domain and codomain are both 1) is a linear isomorphism. Therefore $\omega \in \Omega_c^n(\mathbb{R}^n)$ is exact if and only if $f(\omega) = 0$, if and only if $\int_{\mathbb{R}^n} \omega = 0$.

3 Topology

10F.8 Let G be a connected topological group. Show that $\pi_1(G)$ is an abelian group.

Solution Let $f, g \in \pi_1(G)$ be parametrized by t such that $f(0) = f(1) = g(0) = g(1) = 1$ the identity element of G. Let e be the loop $e(t) = 1$ for all $t \in [0,1]$. Then for all $t \in [0,1]$

$$(fg)(t) = \begin{cases} f(2t) & t \leq \frac{1}{2} \\ g(2t-1) & t > \frac{1}{2} \end{cases} = \begin{cases} f(2t) \cdot e(2t) & t \leq \frac{1}{2} \\ e(2t) \cdot g(2t-1) & t > \frac{1}{2} \end{cases} = (fe)(t) \cdot (eg)(t)$$

where $\cdot$ denotes multiplication of elements of G. Similarly, we have that

$$(gf)(t) = \begin{cases} g(2t) & t \leq \frac{1}{2} \\ f(2t-1) & t > \frac{1}{2} \end{cases} = \begin{cases} e(2t) \cdot g(2t) & t \leq \frac{1}{2} \\ f(2t-1) \cdot e(2t-1) & t > \frac{1}{2} \end{cases} = (ef)(t) \cdot (ge)(t)$$

Since e is the identity loop, we have that $ef = fe$ and $eg = ge$ up to path homotopy, so that the above gives $fg = gf$ as desired. Hence $\pi_1(G)$ is abelian.

10F.9 Show that if $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, then $m = n$.

Solution Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. Then f extends to a homeomorphism $\tilde{f} : S^m \to S^n$ viewing each as the one-point compactification of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and thus $\tilde{f}_* : H_k(S^m) \to H_k(S^n)$ is an isomorphism of integer homology groups for each k. But we have that

$$H_k(S^m) = \begin{cases} \mathbb{Z} & k = 0, m \\ 0 & \text{otherwise} \end{cases} \text{ and } H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

so m must be equal to n.

17F.8 Let $M = [0,1] \times [0,1]/\sim_0$ where $(x,1) \sim_0 (1-x, 0)$ for all $x \in [0,1]$, and let $X = (M \times \{0,1\})/\sim_1$ where $(y,1) \sim_1 (y,0)$ for all $y \in \partial M$.

Solution M deformation retracts onto a circle by $(x,y) \mapsto (\frac{1}{2}, y)$, so that $\pi_1(M) = \mathbb{Z}$. ∂M is the union of $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$, which maps onto two copies of the circle by the deformation retraction. Now consider X, which contains two copies of M joined along their boundary. Call the two copies U_1 and U_2, which by above both have infinite cyclic fundamental group, so let their generators be a, b respectively. Let i_1, i_2 denote the inclusion of each U_1, U_2 and the path homotopy class of the boundary curve be c. Then $(i_1)_*(c) = a^2, (i_2)_*(c) = b^2$ by above, so by Van Kampen's Theorem,

$$\pi_1(X) = \pi_1(U_1 \cup U_2) = \langle a, b \mid a^2 b^{-2} \rangle$$

16S.4 Let M be a compact odd-dimensional manifold with nonempty boundary ∂M. Show that the Euler characteristics of M and ∂M are related by

$$\chi(M) = \frac{1}{2}\chi(\partial M)$$

Solution Let $\tilde{M} = M_1 \cup M_2$ be the manifold given by gluing two copies of the interior of M along ∂M. Mayer-Vietoris gives the following long exact sequence,

$$... \to H_{k+1}(\tilde{M}; \mathbb{F}_2) \to H_k(\partial M; \mathbb{F}_2) \to H_k(M_1; \mathbb{F}_2) \oplus H_k(M_2; \mathbb{F}_2) \to H_k(\tilde{M}; \mathbb{F}_2) \to ...$$

(where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is the field of order 2) which terminates at $H_0(M_1; \mathbb{F}_2) \oplus H_0(M_2; \mathbb{F}_2) = H_0(M; \mathbb{F}_2) \oplus H_0(M; \mathbb{F}_2) = \mathbb{F}_2^2$ and $H_0(\tilde{M}; \mathbb{F}_2) = \mathbb{F}_2$, so that for each k, we have by induction that

$$\dim(H_k(\tilde{M}; \mathbb{F}_2)) = 2\dim(H_k(M; \mathbb{F}_2)) - \dim(H_k(\partial M; \mathbb{F}_2))$$

so that summing over all k gives that

$$\begin{aligned}\chi(\tilde{M}) &= \sum_{k=0}^{n}(-1)^k \dim(H_k(\tilde{M};\mathbb{F}_2)) \\ &= 2\sum_{k=0}^{n}(-1)^k \dim(H_k(M;\mathbb{F}_2)) - \sum_{k=1}^{n}(-1)^k \dim(H_k(\partial M;\mathbb{F}_2)) \\ &= 2\chi(M) - \chi(\partial M)\end{aligned}$$

$\tilde{M}$ is an odd-dimensional manifold with no boundary and every manifold is always orientable mod 2, so by Poincaré duality,

$$\begin{aligned}\chi(\tilde{M}) &= \sum_{k=0}^{n}(-1)^k \dim(H_k(\tilde{M};\mathbb{F}_2)) = \sum_{k=0}^{n}(-1)^k \dim(H^{n-k}(\tilde{M};\mathbb{F}_2)) \\ &= -\sum_{k=0}^{n}(-1)^{n-k} \dim(H^{n-k}(\tilde{M};\mathbb{F}_2)) = -\chi(\tilde{M})\end{aligned}$$

(since $n = \dim(\tilde{M})$ is odd) so that $\chi(\tilde{M}) = 0$ and we have $\chi(\partial M) = 2\chi(M)$ as desired.

16S.6 Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus with coordinates $(x, y) \in \mathbb{R}^2$ and let $p \in T^2$.
a) Compute the de Rham cohomology of the punctured torus $T^2 \setminus \{p\}$.
b) Is the volume form $dx \wedge dy$ exact on $T^2 \setminus \{p\}$?

Solution a) The punctured torus deformation retracts onto $S_1 \vee S_1$ (for instance, by taking any longitude and meridian circles not passing through p), so it has the same singular homology with any coefficients. In particular, using real coefficients, by de Rham's Theorem we have that

$$H^k_{deRham}(T^2 \setminus \{p\}) = H_k(T^2 \setminus \{p\};\mathbb{R}) = H_k(S_1 \vee S_1) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R}^2 & k = 1 \\ 0 & k \geq 2 \end{cases}$$

b) ω (in fact any volume form) is exact because it is closed and by part a $H^2_{deRham}(T^2 \setminus \{p\}) = 0$.

16S.7 Exhibit a space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$. Exhibit another space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

Solution For each positive integer $k \geq 2$, let M_k be the union of S^1 with a closed disk B such that ∂B coincides with k loops around S^1. Then $\pi_1(M_k)$ is generated by the homotopy class of the boundary loop (call it c) since any loop in the interior of B can be contracted since B is contractible. By construction c^p corresponds to a loop in B (which is then contractible) if and only if k divides p, so that $[c]$ has order k in $\pi_1(M_k)$, and thus $\pi_1(M_k) \simeq \mathbb{Z}/k\mathbb{Z}$. Now letting $X = M_m \vee M_n$ gives a space X with fundamental group isomorphic to $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$ by Van Kampen's Theorem.

Now denote the generators of $\pi_1(M_m)$ and $\pi_1(M_n)$ be a, b respectively. Then let $Y = M_m \cup M_n$ where the two spaces are glued along a loop corresponding to $aba^{-1}b^{-1}$. Then by Van Kampen's Theorem,

$$\pi_1(Y) \simeq \langle a, b \mid a^m, b^n, aba^{-1}b^{-1} = (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$$

16S.8 Let L_x be the x-axis, L_y be the y-axis, and L_z be the z-axis of $\mathbb{R}^3$. Compute

$$\pi_1(\mathbb{R}^3 \setminus L_x \setminus L_y \setminus L_z)$$

Solution Since $\mathbb{R}^3 \setminus L_x \setminus L_y \setminus L_z$ does not contain 0, it deformation retracts onto a subset of S^2 via the map $x \mapsto x/||x||$. The image of each axis is $\{\pm(1,0,0)\}, \{\pm(0,1,0)\}, \{\pm(0,0,1)\}$ respectively, so the image of $\mathbb{R}^3 \setminus L_x \setminus L_y \setminus L_z$ is S^2 with these six points removed. This punctured sphere is then homotopic to $\mathbb{R}^2$ with five points removed by stereographic projection about one of the removed points. Thus $\pi_1(\mathbb{R}^3 \setminus L_x \setminus L_y \setminus L_z)$ is isomorphic to $\pi_1(\mathbb{R}^2 \setminus \{\text{five distinct points}\})$ which is isomorphic to $\mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z}$, the free group on five generators.

14S.3, 19F.3 Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere. Determine the values of $n \geq 0$ for which the antipodal map $S^n \to S^n$ is isotopic to the identity.

Solution If n is even, then the identity map has degree 1 while the antipodal map has degree -1 so they cannot be isotopic. If n is odd, consider $\mathbb{R}^{n+1} \simeq \mathbb{C}^{(n+1)/2}$. Then $H : [0,1] \times S^n \to S^n$ defined by $H(t,x) = e^{i\pi t}x$ is a homotopy between the identity $H(0,x)$ and the antipodal map $H(1,x)$. Thus the identity and antipodal maps on S^n are isotopic if and only if n is odd.

19F.8 a) Show that any continuous map $\mathbb{RP}^2 \to S^1 \times S^1$ is nullhomotopic.
a) Find, with proof, a continuous map $S^1 \times S^1 \to \mathbb{RP}^2$ that is not nullhomotopic.

Solution a) Since $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$, we see that $f_*\pi_1(\mathbb{RP}^2) \subseteq \pi_1(S^1 \times S^1) = \mathbb{Z}^2$ must map to the trivial subgroup (the only finite subgroup of $\mathbb{Z}^2$). Therefore, by the lifting criterion f lifts to $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}^2$ since $\mathbb{R}^2$ is the universal cover of $S^1 \times S^1$. Let $p : \mathbb{R}^2 \to S^1 \times S^1$ be the universal covering map and let $f_t := p \circ (t\tilde{f})$. Then $f_0 = 0$ and $f_1 = f$ so f is nullhomotopic.

b) View as $\mathbb{RP}^2$ as S^2 with antipodal points identified, and consider $f : S^1 \times S^1 \to \mathbb{RP}^2$ by $f(e^{i\theta}, e^{i\phi}) = [\cos(\theta/2) : \sin(\theta/2) : 0]$. Considering the latter as a point of S^2, we see that f covers half of the equator, so that if γ is a loop in $S^1 \times S^1$ winding once around the first coordinate, $f^*\gamma$ is a loop in $\mathbb{RP}^2$ which corresponds to a curve in S^2 which is half of the equator, and so $f^*\gamma$ is a nontrivial element of $\pi_1(\mathbb{RP}^2)$, and so f cannot be nullhomotopic.

16F.3 (Ham Sandwich Theorem) Let $U_1, ..., U_n$ be n bounded, connected, open subsets of $\mathbb{R}^n$. Prove that there exists an $(n-1)$-dimensional hyperplane $H \subseteq \mathbb{R}^n$ that bisects every U_i; i.e. if A and B are the two components of $\mathbb{R}^n \setminus H$, then

$$\text{volume}(U_i \cap A) = \text{volume}(U_i \cap B) \text{ for each } i$$

Solution Let $S^{n-1} \subseteq \mathbb{R}^n$ be the unit sphere. Then for each $p \in S^{n-1}$ and each $r \in \mathbb{R}$, there exists a unique hyperplane perpendicular to the vector rp oriented in the direction of p. Fix $p \in S^{n-1}$, and for each $r \in \mathbb{R}$ let $f(r)$ denote the area of the part of U_n on the positive side of the hyperplane defined by rp. Then $f : \mathbb{R} \to \mathbb{R}$ is continuous, and since U_n is bounded, for sufficiently large negative r $f(r) = 0$ and for sufficiently large positive r $f(r) = \text{volume}(U_n)$, so by the Intermediate Value Theorem there exists $r^* \in \mathbb{R}$ such that $f(r) = \frac{1}{2}\text{volume}(U_n)$. Let the hyperplane defined by r^*p be $H(p)$; this is well-defined since f is increasing in r so that r^* is unique. Now define $g : S^{n-1} \to \mathbb{R}^{n-1}$ by

$$g(p) = (\text{volume of } U_1 \text{ on the positive side of } H(p), ..., \text{volume of } U_{n-1} \text{ on the positive side of } H(p))$$

g is continuous, so by the Borsuk-Ulam theorem there is $p \in S^{n-1}$ such that $g(p) = g(-p)$. But the hyperplane $H(p)$ determined by r^*p is the same as that determined by $(-r^*)(-p)$, (which then must be $H(-p)$), just oriented oppositely. Therefore each side of the hyperplane $H = H(p)$ is the positive side of either $H(p)$ or $H(-p)$. Then, since $g(p) = g(-p)$, we see that each of $U_1, ..., U_{n-1}$ has the same volume on both sides of H, and so does U_n by definition, so H is the desired hyperplane.

16F.7 Let X be a connected CW-complex with $\pi_1(X,x)$ finite. Show that any continuous map $X \to (S^1)^n$ is nullhomotopic.

Solution Let $f : X \to (S^1)^n$ be continuous. Since $\pi_1(X)$ is finite, we see that $f_*\pi_1(X) \subseteq \pi_1((S^1)^n) = \mathbb{Z}^2$ must map to the trivial subgroup (the only finite subgroup of $\mathbb{Z}^n$). Therefore, by the lifting criterion f lifts to $\tilde{f} : X \to \mathbb{R}^n$ since $\mathbb{R}^n$ is the universal cover of $(S^1)^n$. Let $p : \mathbb{R}^n \to (S^1)^n$ be the universal covering map and let $f_t := p \circ (t\tilde{f})$. Then $f_0 = 0$ and $f_1 = f$ so f is nullhomotopic.

16F.9 Let $S^2 \xleftarrow{q_1} S^2 \vee S^2 \xrightarrow{q_2} S^2$ be the maps that crush out one of the two summands. Let $f : S^2 \to S^2 \vee S^2$ be a map such that $q_i \circ f : S^2 \to S^2$ is a map of degree d_i. Compute the homology groups of $X = (S^2 \vee S^2) \cup_f D^3$.

Solution X has one 3-cell, two 2-cells, and one 0-cell, which gives the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

Let the 2-cells be e_1, e_2 and the 3-cell be F. Then the only nontrivial map is $\partial_3(F) = d_1e_1 + d_2e_2$ by the cellular boundary formula. If d_1 and d_2 are not both zero, ∂_3 is injective, and $H_2(X) = \mathbb{X}^2/\mathrm{Im}(\partial_3)$. The Smith normal form of the matrix $\begin{pmatrix} d_1 & d_2 \end{pmatrix}$ is $\begin{pmatrix} gcd(d_1,d_2) & 0 \end{pmatrix}$ so that $H_2(X) = \mathbb{X}^2/\mathrm{Im}(\partial_3) = (\mathbb{Z}/gcd(d_1,d_2)\mathbb{Z}) \times \mathbb{Z}$. Therefore, when d_1, d_2 are not both zero,

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ (\mathbb{Z}/gcd(d_1,d_2)\mathbb{Z}) \times \mathbb{Z} & n = 2 \\ 0 & n > 2 \end{cases}$$

When $d_1 = d_2 = 0$, $\partial_3 = 0$ so all chain maps are trivial, so we have that

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ \mathbb{Z}^2 & n = 2 \\ \mathbb{Z} & n = 3 \\ 0 & n > 3 \end{cases}$$

17S.7 Let $X = S^1 \times D^2$ with boundary $\partial X = S^1 \times S^1$. Compute $H_k(X, \partial X; \mathbb{Z})$.

Solution Since (S^1, D^2) is a good pair, so is $(\partial X, X)$. Using the long exact sequence of reduced homology groups

$$... \to \tilde{H}_k(\partial X) \to \tilde{H}_k(X) \to H_k(X, \partial X) \to \tilde{H}_{k-1}(\partial X) \to ...$$

Since D^2 is contractible to a point, $H_k(X) = H_k(S^1)$ for all k, and so we have that

$$\tilde{H}_k(\partial X) = \begin{cases} 0 & k = 0, k \geq 3 \\ \mathbb{Z}^2 & k = 1 \\ \mathbb{Z} & k = 2 \end{cases} \quad \text{and } \tilde{H}_k(X) = \begin{cases} 0 & k \neq 1 \\ \mathbb{Z} & k = 1 \end{cases}$$

For $k > 3$, every term of the long exact sequence is zero, so our long exact sequence becomes

$$0 \to H_3(X, \partial X) \to \mathbb{Z} \to 0 \to H_2(X, \partial X) \to \mathbb{Z}^2 \to \mathbb{Z} \to H_1(X, \partial X) \to 0 \to 0 \to H_0(X, \partial X) \to 0$$

Clearly, $H_0(X, \partial X) = 0$ and $H_3(X, \partial X) = \mathbb{Z}$. The map $\mathbb{Z}^2 \to \mathbb{Z}$ corresponds to the map $H_1(S^1 \times S^1) \to H_1(S^1 \times D^2)$ induced by the inclusion map. Under the inclusion map, one generator of the homology of $S^1 \times S^1$ (the meridian) is contractible while the other generator (the longitude) maps to the generator of the homology of $S^1 \times D^2$, so that this map is a surjection with kernel $\mathbb{Z}$. Hence $H_1(X, \partial X) = 0$ and $H_2(X, \partial X) = \mathbb{Z}$, so that we have

$$H_k(X, \partial X) = \begin{cases} 0 & k = 0, 1, k \geq 4 \\ \mathbb{Z} & k = 2, 3 \end{cases}$$

17S.8 Let X be a CW complex and let $\tilde{X} \to X$ be a covering space. Let G be the group of deck transformations on $\tilde{X} \to X$.
a) Show that for any k and for any abelian group M, the group G acts naturally on $H_k(\tilde{X}; M)$.
b) Show that the map $p_* : H_k(\tilde{X}; M) \to H_k(X; M)$ through the quotient of $H_k(\tilde{X}; M)$ by the subgroup S generated by $m - g \cdot m$ for all $m \in H_k(\tilde{X}; M)$ and $g \in G$.
c) Give an example for which the induced map $H_k(\tilde{X}; M)/S \to H_k(X; M)$ in (b) is not surjective.

Solution a) Each $g \in G$ gives a homeomorphism (deck transformation) $g : \tilde{X} \to \tilde{X}$, which naturally induces a map $g_* : H_k(\tilde{X}; M) \to H_k(\tilde{X}; M)$.

b) $pg = p$ for all $g \in G$, so that $p_*(m - mg_*) = 0$ for all $m \in M, g \in G$ so p_* factors through the quotient by $(m - g \cdot m)$.

c) Take $p : \mathbb{R} \to S^1$ by $p(t) = e^{it}$. Then $H_1(\mathbb{R}; M) = 0$ but $H_1(S^1; M) = \mathbb{Z}$, so the morphism from (b) cannot be surjective.

17S.9, 11S.8 a) Find $H_k(\mathbb{RP}^n;\mathbb{Z})$ for all k.

b) Describe a cell decomposition for $\mathbb{RP}^2 \times \mathbb{RP}^2$. Use it to show (without appealing to the Künneth theorem) that $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2;\mathbb{Z})$ is nontrivial.

Solution a) Give $\mathbb{RP}^n$ the following CW structure. Let $p : S^{n-1} \to \mathbb{RP}^{n-1}$ be the standard double cover; then $\mathbb{RP}^{n-1} \cup_p D^n \simeq \mathbb{RP}^n$. Therefore $\mathbb{RP}^n$ has one cell in each dimension. Now consider the map

$$S^{n-1} \xrightarrow{p} \mathbb{RP}^{n-1} \xrightarrow{\pi} \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} = S^{n-1}$$

Preimages in π (of points not in $\mathbb{RP}^{n-2}$) are single points, so preimages of the total map are antipodal points of S^{n-1}. If n is even, these points have the same orientation, and if n is odd, they have opposite orientations, so this total map has degree 2 if n is even and 0 if n is odd. Therefore by the cellular boundary formula, we have the following chain complex

$$0 \to \mathbb{Z} \to \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

(where the map k denotes multiplication by k) If n is even, the top map is 2 which is injective so $H_n(\mathbb{RP}^n) = 0$. If n is odd, the top map is 0 so that $H_n(\mathbb{RP}^n) = \mathbb{Z}$. Hence,

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n \text{ even} \\ 0 & 0 < k < n \text{ odd} \\ \mathbb{Z} & k = n \text{ even} \\ 0 & k = n \text{ odd} \end{cases}$$

b) Write $\mathbb{RP}^2 = e_0 \cup e_1 \cup e_2$ and $\mathbb{RP}^2 = f_0 \cup f_1 \cup f_2$ as the cell decompositions, where by part (a) we have that $\partial e_2 = 2e_1, \partial e_1 = 0$ and similarly for f_0, f_1, f_2. Then $\mathbb{RP}^2 \times \mathbb{RP}^2$ has $(i+j)$-cells $e_i \times f_j$ with $\partial(e_i \times f_j) = \partial e_i \times f_j + (-1)^{\dim(e_i)} e_i \times \partial f_j$. In particular,

$$\partial_3(e_1 \times f_2) = -2e_1 \times f_1 \text{ and } \partial_3(e_2 \times f_1) = 2e_1 \times f_1$$

so that $\ker(\partial_3)$ is generated by $e_1 \times f_2 + e_2 \times f_1$. We also have

$$\partial_4(e_2 \times f_2) = 2e_1 \times f_2 + 2e_2 \times f_1$$

so that $\mathrm{im}(\partial_4)$ is generated by $2(e_1 \times f_2 + e_2 \times f_1)$, and therefore $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z} \neq 0$.

18F.6 Can a finite rank free group have a finite index subgroup of a smaller rank?

Solution Let X be a wedge sum of n circles; then $\pi_1(X)$ is free on n generators. Every subgroup of $\pi_1(X)$ of index k corresponds to a covering space Y of X of index k, which must be a graph with k vertices and one outgoing and incoming edge per vertex, making kn edges. Then Y has a spanning tree which is $k-1$ edges long, so after contracting this to a point we are left with a wedge sum of $kn-k+1$ circles. Therefore $\pi_1(Y)$ is free on $kn-k+1$ generators. But $(k-1)n = kn - n \geq k-1 \Rightarrow kn-k+1 \geq n$ so that a finite rank free group cannot have a finite index subgroup of a smaller rank.

18F.9 Let X be a connected CW complex. Show that there is a natural isomorphism

$$\tilde{H}_k(\Sigma X; M) \simeq \tilde{H}_{k-1}(X; M)$$

for all k and all abelian groups M.

(Related: 15F.9)

Solution Let p_0, p_1 be the points where we have collapsed $X \times \{0\}$ and $X \times \{1\}$ respectively in ΣX. Then let U, V be neighborhoods of p_0, p_1 respectively avoiding the other point, such that $U \cup V = \Sigma X$. Then $U \cap V = X \times (a, b)$ where $0 < a < b < 1$ which deformation retracts onto a copy of X, while U and V are both contractible to their respective points p_0, p_1. By Mayer-Vietoris, we have the long exact sequence

$$... \to \tilde{H}_k(U) \oplus \tilde{H}_k(V) \to \tilde{H}_k(\Sigma X) \to \tilde{H}_{k-1}(U \cap V) \to \tilde{H}_{k-1}(U) \oplus \tilde{H}_{k-1}(U) \to ...$$

But since U, V are contractible, their respective terms are zero, giving the exact sequence $0 \to \tilde{H}_k(\Sigma X) \to \tilde{H}_{k-1}(X) \to\to 0$ for each k, so these are isomorphic for each k. Finally, applying the universal coefficient theorem gives the desired result for all abelian groups M.

18F.10 Let Y be a connected and simply connected CW complex.
a) Compute the fundamental group of $Y \vee S^1$.
b) Describe the universal cover of $Y \vee S^1$, together with the action of deck transformations.
c) Describe all finite covers of $Y \vee S^1$, again with the action of deck transformations.
d) Describe what changes in the first two parts for $Y = \mathbb{RP}^2$

Solution a) By Van Kampen's Theorem, $\pi_1(Y \vee S^1) = \pi_1(Y) * \pi_1(S^1) = 0 * \mathbb{Z} = \mathbb{Z}$.

b) Y is its own universal cover and $\mathbb{R}$ is the universal cover of S^1, so the universal cover of $Y \vee S^1$ is $\mathbb{R}$ with a copy of Y glued at the base point to every integer. The deck transformations are given by the deck transformations of $\mathbb{R} \to S^1$, so they are all given by translation by an integer.

c) All finite covers are given by finite quotients of subgroups of $\mathbb{Z}$, so they all correspond to $\mathbb{Z}/k\mathbb{Z}$ for some integer k. In this case we obtain a circle with k integer points and a copy of Y glued at its base point to each integer point.

d) Let $Y = \mathbb{RP}^1$. Then by Van Kampen's Theorem, $\pi_1(Y \vee S^1) = \pi_1(Y) * \pi_1(S^1) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z} = \langle a, b | a^2 \rangle$. Now the universal cover of Y is S^2, so the universal cover of $Y \vee S^1$ is S^2 glued with $\mathbb{R}$ at the lifts of the base points. In particular, we have two copies $\mathbb{R}$ and with their integers aligned, and copies of S^2 glued between them such that the north pole of each S^2 intersects one real line at an integer and the south pole intersects the other real line at an integer.

15S.4 Consider a smooth map $F : \mathbb{RP}^2 \to \mathbb{RP}^2$.
a) When n is even show that F has a fixed point.
b) When n is odd give an example where F does not have a fixed point.

Solution a) Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) = \mathbb{Q}$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) = 0$, we have, for n even, that

$$H^k(\mathbb{RP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since F^* is a cohomology ring homomorphism, $F^*(1) = 1$ in $H^0(\mathbb{RP}^n; \mathbb{Q}$ so that $L(F) = \mathrm{tr}(F^* | H^0(\mathbb{RP}^n; \mathbb{Q}) = 1$. Now by the Lefschetz fixed point theorem, F has a fixed point.

b) Write $n = 2k - 1$. Then identify $S^n \subseteq \mathbb{R}^n = \mathbb{C}^k$ and let $f(x) = ix$ on S^n. Then $S^n \xrightarrow{f} S^n \xrightarrow{\pi} \mathbb{RP}^n$ gives a well-defined function $F : \mathbb{RP}^n \to \mathbb{RP}^n$ by $F([x]) = [f(x)]$, since $\pi(f(-x)) = \pi(-ix) = \pi(ix) = \pi(f(x))$. Suppose $F([x]) = [x]$. Then $ix = \pm x$, which is a contradiction, so F has no fixed points.

15S.8 Let X be a CW complex consisting of one vertex p, 2 edges a and b, and two 2-cells f_1 and f_2 where the boundaries of a and b map to p, the boundary of f_1 mapsto ab^2, and the boundary of f_2 is mapped to ba^2.
a) Compute $\pi_1(X)$. Is it finite?
b) Compute the homology $H_i(X)$.

Solution a) Without the 2-cells X is just a wedge sum of circles, so after adding them we obtain $\pi_1(X) = \langle a, b | ab^2, ba^2 \rangle$. But now $a = b^{-2}$ so that $b = a^{-2} = b^4$, so $b^3 = e$ so that $a = b^{-2} = b$. Therefore $\pi_1(X) = \langle b | b^3 \rangle = \mathbb{Z}/3\mathbb{Z}$ is indeed finite.

b) There are no cells in dimension higher than 2 so $H_i(X) = 0$ for $i \geq 3$. $H_1(X)$ is the abelianization of $\pi_1(X)$, so it is just $\mathbb{Z}/3\mathbb{Z}$. Since $\partial a = \partial b = p - p = 0$, we must have that $H_0(X) = \mathbb{Z}$ since the previous boundary map into $\mathbb{Z}$ has zero image. Finally, suppose $\partial(nf_1 + mf_2) = 0$. Then $(n+2m)a + (2n+m)b = 0$, so that $n + 2m = 2n + m = 0 \Rightarrow n = m = 0$, so that the first boundary map has trivial kernel and so $H_2(X) = 0$. Hence

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/3\mathbb{Z} & i = 1 \\ 0 & i \geq 2 \end{cases}$$

15S.5 Recall that the Hopf fibration $\pi : S^3 \to S^2$ is defined as follows: If we identify

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

and $S^2 = \mathbb{CP}^1$ with homogeneous coordinates $[z_1, z_2]$, then $\pi(z_1, z_2) = [z_1, z_2]$. Show that π does not admit a section.

Solution Suppose s were a section. Then $\pi_* \circ s_* = \mathrm{id}_*$ so that in particular s_* is an injection on homology. But then $S_* : H_2(S^2) \to H_2(S^3)$ is an injection from $\mathbb{Z}$ to 0, which is a contradiction, so there can be no such section s.

21F.6 Let $f : \mathbb{C} \to \mathbb{C}$ be a nowhere zero continuous function. Prove that there exists a continuous function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$.

Solution Consider $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. Identify $\mathbb{C} \setminus \{0\} \simeq \mathbb{R}^+ \times S^1$ using polar coordinates. Then the imaginary axis covers $S^1 \subseteq \mathbb{C}$ via $ix \mapsto e^{ix}$, and the real axis covers the positive reals via $x \mapsto e^x$ (which is a homeomorphism, so in particular it's a covering map). Therefore $z \mapsto e^z : \mathbb{C} \simeq \mathbb{R}^2 \to \mathbb{R}^+ \times S^1 \simeq \mathbb{C} \setminus \{0\}$ is a covering map. But $f_* : \pi_1(\mathbb{C}) = 0 \to \pi_1(\mathbb{C} \setminus \{0\}$ must be the zero map, so f lifts to some $g : \mathbb{C} \to \mathbb{C}$ under the cover $z \mapsto e^z$. But then by definition $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$.

21F.8 Let M be a connected non-orientable manifold whose fundamental group G is simple. Prove that G must be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Solution Let $\pi : \tilde{M} \to M$ be its orientable double cover (see 17F.6). Suppose $\tilde{M}$ is not connected. Then it can be written as $\tilde{M} = U \cup V$ where U, V are disjoint open sets, and since π is a double cover, this means that $\pi|_U : U \to M$ is a diffeomorphism. But then since U is orientable, so is M, which is a contradiction, so $\tilde{M}$ must be connected. The function $f : \tilde{M} \to \tilde{M}$ defined by $f(p, o) = (p, -o)$ is continuous by Problem 7 and satisfies $\pi \circ f = \pi$ so it gives a deck transformation which acts transitively on the fibers of the cover, so that π is a normal covering of M since $\tilde{M}$ is connected, which defines an index 2 normal subgroup of $G = \pi_1(M)$. But since G is simple, this normal subgroup must therefore be trivial, so we must have $G \simeq \mathbb{Z}/2\mathbb{Z}$.

21F.10 (17F.10 is a similar problem) Consider the following subsets of $\mathbb{R}^3$:

$$\begin{aligned} Z &= \{(0,0,z) \mid z \in \mathbb{R}\} \\ C_1 &= \{(\cos(\theta), \sin(\theta), 0) \mid \theta \in \mathbb{R}\} \\ C_2 &= \{(2+\cos(\theta), \sin(\theta), 0) \mid \theta \in \mathbb{R}\} \end{aligned}$$

Prove that there is no self-homeomorphism of $\mathbb{R}^3$ that takes $Z \cup C_1$ to $Z \cup C_2$.

Solution Take the one-point compactification S^3 of $\mathbb{R}^3$, in which Z becomes a circle (and C_1, C_2 remain circles). If there were a self-homeomorphism of $\mathbb{R}^3$ taking $Z \cup C_1$ to $Z \cup C_2$, it would extend to a self-homeomorphism of S^3 taking $Z \cup C_1$ to $Z \cup C_2$ by taking the point at infinity to itself (since $\infty \in Z$), so it suffices to show that no such self-homeomorphism of S^3 exists. $S_3 \setminus Z = \mathbb{R}^3 \setminus Z$ since $\infty \in Z$, so $\pi_1(S^3 \setminus Z) = \pi_1(\mathbb{R}^3 \setminus Z)$. But the latter deformation retracts onto $S^2 \setminus \{\pm(0,0,1)\}$ via $x \mapsto x/|x|$, and this is homeomorphic to $\mathbb{R}^2 \setminus \{(0,0)\}$ via stereographic projection, which again deformation retracts onto S^1, so that $\pi_1(S^3 \setminus Z) = \pi_1(S^1) = \mathbb{Z}$, and in fact this is true of any circle with the same argument if we just move the location of ∞.

Now to compute $\pi_1(S^3 \setminus (Z \cup C_2))$, note that Z and C_2 are path homotopic to circles on opposite hemispheres of S^3, so without loss of generality let this be the case and let U, V be each open hemisphere with the respective circle removed. Then $\pi_1(U) = \pi_1(V) \simeq \pi_1(\mathbb{R}^3 \setminus Z) = \mathbb{Z}$, and $U \cap V$ retracts onto $\mathbb{R}^2$ (the equator), which is simply connected, so that by Van Kampen's Theorem $\pi_1(S^3 \setminus (Z \cup C_2)) = \pi_1(U \cup V) = \mathbb{Z} * \mathbb{Z}$ the free group on two generators.

To compute $\pi_1(S^3 \setminus (Z \cup C_1))$, note that $S^3 \setminus (Z \cup C_1) = \mathbb{R}^3 \setminus (Z \cup C_1)$ (because $\infty \in Z$) so that this deformation retracts onto a torus. To see this, consider the vertical half-plane at each angle (so, $\{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 \mid r > 0, \theta = \theta_0\}$). The half-planes already contain every point except those of Z, and removing C_1 removes one point $(\cos\theta_0, \sin\theta_0, 0)$, so each vertical half-plane is a homeomorphic copy of $\mathbb{R}^2$ minus one point, which deformation retracts onto a circle, which gives a torus. Hence $\pi_1(S^3 \setminus (Z \cup C_1)) = \pi_1(S^1 \times S^1) = \mathbb{Z}^2$.

Finally, since any self-homeomorphism of S^3 which sends $Z \cup C_1$ to $Z \cup C_2$ also sends their complements to their complements, we must have that $\mathbb{Z}^2 = \pi_1(S^3 \setminus (Z \cup C_1)) \simeq \pi_1(S^3 \setminus (Z \cup C_2)) = \mathbb{Z} * \mathbb{Z}$. But this is a contradiction (in particular, the left-hand group is abelian and the right-hand group is not), so no such self-homeomorphism exists.