SAAMARTH JUNEJA

#2

2) Must show: 1) $\leq_\alpha$ is a partial ordering
2) all elements are comparable
3) every non empty subset has a minimum

1) Reflexivity: $\forall a \in \alpha$ $a = a$ i.e $a \leq_\alpha a$

Anti-symmetry: let $a, b \in \alpha$ s.t $a \leq_\alpha b$ and $b \leq_\alpha a$

if $a \neq b$ then $a \in b$ and $b \in a$ ↯ since α is grounded

Transitivity: Let $a \leq_\alpha b$ and $b \leq_\alpha c$

if $a = b$ then $a = b \leq_\alpha c$
if $b = c$ then $a \leq_\alpha b = c$
if $a \neq b, b \neq c$ then $a \in b \in c$ so $a \in c$

2) Let $a, b \in \alpha$. Then be connectedness,
$a \in b \vee b \in a \vee a = b$
so $a \leq_\alpha b$ or $b \leq_\alpha a$ i.e all elements are comparable.

3) Let $X \subseteq \alpha$. To show: $\exists x \in X$ s.t $\forall a \in X, x \leq_\alpha a$

Assume otherwise. Choose any $x_1 \in X$
Then $\exists$ $x_2 \in X$ s.t $x_1 \nleq_\alpha x_2$ i.e $x_2 <_\alpha x_1$
recursively define x_{n+1} as $x_{n+1} \in X$ and $x_{n+1} <_\alpha x_n$
This is defined since for each x_n, x_n is not minimal.
So $x_1 \ni x_2 \ni x_3 \cdots$ ↯ since α is grounded

$\square$

SAAMARTH JUNEJA

#3

3. Will prove $seg_\alpha(X) = X$ by double set containment

($\subseteq$) Let $a \in seg_\alpha(X)$

then $a \leq_\alpha X$ and $a \neq X$

i.e. $a \in X$

($\supseteq$) Let $a \in X$

then $a \neq X$ otherwise α is not grounded

and $a \leq_\alpha X$ i.e. $a <_\alpha X$ i.e. $a \in seg_\alpha(X)$

$\square$

4) We prove existence and uniqueness of an ordinal similar to an arbitrary well ordered set $(U, \leq_U)$. We identify U with $(U, \leq_U)$.

~~#3~~

#4

Existence: Define by recursion on U, $V_U(x) := \{V_U(y) \mid y <_U x\}$.

Define $ord(U) := V_U[U]$ (which exists by replacement).

We first claim that $ord(U) \in ON$. We will show that it is transitive, pure, grounded, and connected.

Transitive: Let $x \in y \in ord(U)$. $ord(U) = V_U[U]$ so $\exists y' \in U$ s.t. $y = V_U(y')$.

Then $x \in V_U(y')$ so $\exists x' \in U$, $x' <_U y'$ s.t. $x = V_U(x')$.

Then $x \in V_U[U] = ord(U)$, so $ord(U)$ is transitive.

Grounded: Suppose $x_0 \ni x_1 \ni \ldots$, $x_n \in ord(U)$ $\forall n \in \mathbb{N}$.

As $ord(U) = V_U[U]$, $\exists y_n \in U$ s.t. $x_n = V_U(y_n)$.

As $x_{n+1} \in x_n$, $V_U(y_{n+1}) \in V_U(y_n) = \{V_U(y) \mid y <_U y_n\}$, so $y_{n+1} <_U y$.

Thus, we have $y_0 >_U y_1 >_U y_2 >_U \cdots$ in U, so $\emptyset \neq \{y_n \mid n \in \mathbb{N}\} \subseteq U$ has no least element which is a contradiction. Thus, such an $\in$-chain cannot exist so $ord(U)$ is grounded.

Pure: $ord(U)$ is, as above, transitive, so it suffices to show that $ord(U)$ contains no atoms. $ord(U) = V_U[U] = \{V_U(y) \mid y \in U\}$. Each $V_U(y)$ was, by definition, a set. Thus, $ord(U)$ contains no atoms so $ord(U)$ is pure.

Connected: Let $x, y \in ord(U)$. Then $\exists x', y' \in U$ s.t. $x = V_U(x')$, $y = V_U(y')$.

As $x', y' \in U$ a woset, $x' <_U y'$ or $x' = y'$ or $y' <_U x'$.

If $x' <_U y'$, $V_U(x') \in V_U(y')$ so $x \in y$. Symmetrically, $y' <_U x'$ yields $y \in x$.

$x' = y'$ yields $V_U(x') = V_U(y')$ so $x = y$. Thus, $ord(U)$ is connected.

Thus, $ord(U) \in ON$.

We claim $ord(U) =_o U$. We exhibit a similarity $U \longrightarrow ord(U)$.

$V_U : U \to ord(U)$ as above. $ord(U) = V_U[U]$ so V_U is a surjection.

Let $x <_U y$ in U. Then $V_U(x) \in V_U(y)$ so $V_U(x) < V_U(y)$ so V_U is order-preserving.

We now show V_U is an injection. Let $x \neq y$ in U. Then $x <_U y$ or $y <_U x$, so $V_U(x) \in V_U(y)$ or $V_U(y) \in V_U(x)$.

Thus, if $V_U(x) = V_U(y)$, $V_U(x) \in V_U(x)$, contradicting groundedness of $ord(U)$.

Thus, $V_U(x) \neq V_U(y)$ so V_U is an injection.

Putting the above together, we have that V_U is a similarity, so $ord(U) =_o U$.

This concludes the existence of $ord(U)$.

Uniqueness: We first claim that for ordinals $\alpha, \beta \in ON$, $\alpha \neq \beta$ then $\alpha \neq_o \beta$.

As $\alpha \neq \beta$, by comparability, $\alpha \in \beta$ or $\beta \in \alpha$. WLOG say $\alpha \in \beta$.

Then by 3), $\alpha = seg_\beta(x)$ some $x \in \beta$. As β a woset, $\beta \neq_o seg_\beta(x)$ so $\beta \neq_o \alpha$ as desired.

Thus, $\alpha =_o \beta \Rightarrow \alpha = \beta$.

Suppose $\alpha, \beta \in ON$ with $U =_o \alpha$, $U =_o \beta$. Then $\alpha =_o \beta$ so as above $\alpha = \beta$.

Thus, $ord(U)$ is the unique ordinal such that $U =_o ord(U)$.

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#5

5. Prop: $\alpha \leq \beta \iff (\exists \pi : \alpha \rightarrowtail \beta)[x \in y \in \alpha \Rightarrow \pi(x) \in \pi(y)]$

$(\Rightarrow)$ $\alpha \leq \beta \iff (\alpha, \leq_\alpha) \leq_o (\beta, \leq_\beta)$

i.e $\exists x \in \beta$ and $f : \alpha \rightarrowtail\!\!\!\!\twoheadrightarrow seg_\beta(x)$
s.t f is order preserving

i.e $f : \alpha \rightarrowtail \beta$ is order preserving

$(\Leftarrow)$ Let $\alpha \nleq \beta$ for contradiction

so $\alpha \nleq_o \beta$ i.e $\beta \leq_o \alpha$ and $\beta \neq_o \alpha$

so $\exists a \in \alpha$, $\exists g : \beta \rightarrowtail\!\!\!\!\twoheadrightarrow seg_\alpha(a)$, a proper initial segment

~~so~~ $g \circ \pi : \alpha \rightarrowtail \cancel{seg_\alpha}(a)$, an order preserving injection to a proper initial segment ↯

$\square$

Allison Wong

#7

#7: The class ON is well ordered by the condition $\leq$.

Proof: By #6, we can use that $\alpha \leq \beta \iff \alpha = \beta$ or $\alpha \in \beta$.

$\leq$ is reflexive: For any $\alpha \in ON$, $\alpha = \alpha$, so $\alpha \leq \alpha$.

$\leq$ is transitive: Let $\alpha, \beta, \gamma \in ON$, and suppose $\alpha \leq \beta$ and $\beta \leq \gamma$.

Then $\alpha = \beta$ or $\alpha \in \beta$, and $\beta = \gamma$ or $\beta \in \gamma$.

If $\alpha = \beta$, then $\alpha = \gamma$ or $\alpha \in \gamma$, so $\alpha \leq \gamma$.

If $\beta = \gamma$, then $\alpha = \gamma$ or $\alpha \in \gamma$, so $\alpha \leq \gamma$.

If $\alpha \neq \beta$ and $\beta \neq \gamma$, then $\alpha \in \beta$ and $\beta \in \gamma$, so since γ is transitive, $\alpha \in \gamma$, and thus, $\alpha \leq \gamma$.

$\leq$ is anti-symmetric: Suppose $\alpha, \beta \in ON$ and $\alpha \leq \beta$ and $\beta \leq \alpha$.

Suppose, for a contradiction, that $\alpha \neq \beta$. Then $\alpha \in \beta$ and $\beta \in \alpha$.

Since α, β are transitive, $\alpha \in \beta$ implies $\alpha \subseteq \beta$, and similarly, $\beta \in \alpha$ implies $\beta \subseteq \alpha$. Thus, $\alpha = \beta$.

$\leq$ is linear: Suppose $\alpha, \beta \in ON$. By #4, $(\alpha, \leq_\alpha)$ and $(\beta, \leq_\beta)$ are well ordered sets. By 7.31, either $(\alpha, \leq_\alpha) \leq_o (\beta, \leq_\beta)$ or $(\beta, \leq_\beta) \leq_o (\alpha, \leq_\alpha)$. In the first case, $\alpha \leq \beta$, and in the second, $\beta \leq \alpha$ (by definition 3), so for any $\alpha, \beta \in ON$, either $\alpha \leq \beta$ or $\beta \leq \alpha$.

"Every non-empty set of ordinals has a $\leq$-least element": It suffices to show that for every definite condition P on ordinals,

$(\exists \alpha) P(\alpha) \Rightarrow (\exists \alpha)[P(\alpha)$ and $(\forall \beta < \alpha)[\neg P(\beta)]]$.

Suppose, for a contradiction, that $(\exists \alpha) P(\alpha)$ but for all α such that $P(\alpha)$, $(\exists \beta < \alpha) P(\beta)$ for some definite condition P on ordinals.

Then we can pick an infinite descending $\in$-chain of ordinals $\alpha_1 \ni \alpha_2 \ni \alpha_3 \ni \cdots$ such that $P(\alpha_i)$ for all i.

Since α_1 is transitive, $\alpha_i \in \alpha_1$ for all i (by induction), so $\alpha_1 \ni \alpha_2 \ni \alpha_3 \cdots$ is an infinite, descending $\in$-chain in α_1.

This contradicts the fact that α_1 is grounded.

Allison Wong

#8

#8: If $\mathcal{E}$ is a non-empty set of ordinals, then

$\sup \mathcal{E}$ = the least $\beta\ (\forall \alpha \in \mathcal{E})[\alpha \leq \beta] = \bigcup \mathcal{E}$

$\min \mathcal{E} = \bigcap \mathcal{E}$.

Proof: We first show that $\sup \mathcal{E} = \bigcup \mathcal{E}$.

Claim: $\bigcup \mathcal{E}$ is an ordinal

$\bigcup \mathcal{E}$ is transitive: Suppose $\alpha \in \beta \in \bigcup \mathcal{E}$. Then there is some $\gamma \in \mathcal{E}$ such that $\beta \in \gamma$, meaning $\alpha \in \beta \in \gamma$. Since γ is transitive, $\alpha \in \gamma$, so $\alpha \in \bigcup \mathcal{E}$.

$\bigcup \mathcal{E}$ is pure: It suffices to show that $TC(\bigcup \mathcal{E})$ contains no atoms. Since $\bigcup \mathcal{E}$ is transitive, $TC(\bigcup \mathcal{E}) = \bigcup \mathcal{E} \cup \{\bigcup \mathcal{E}\}$, so it suffices to show that $\bigcup \mathcal{E}$ contains no atoms, and that $\bigcup \mathcal{E}$ is not an atom. Indeed, if $a \in \bigcup \mathcal{E}$, then there is some $\gamma \in \mathcal{E}$ such that $a \in \gamma$. Since γ is an ordinal, and hence pure, γ cannot contain any atoms, so a is not an atom. This shows that $\bigcup \mathcal{E}$ contains no atoms. Moreover, $\bigcup \mathcal{E}$ is not an atom because it is a set by the unionset axiom.

$\bigcup \mathcal{E}$ is grounded: Suppose not. Then by DC, there exists an infinite chain $\alpha_1 \ni \alpha_2 \ni \cdots$ in $\bigcup \mathcal{E}$.
Since $\alpha_1 \in \bigcup \mathcal{E}$, there must be some ordinal $\gamma \in \mathcal{E}$ such that $\alpha_1 \in \gamma$. By the transitivity of γ, $\alpha_i \in \gamma$ for all i, and hence, $\alpha_1 \ni \alpha_2 \ni \cdots$ is an infinite descending $\in$-chain in γ.
By DC, this contradicts the fact that γ is grounded.

$\bigcup \mathcal{E}$ is connected: Suppose $x, y \in \bigcup \mathcal{E}$. We want to show that $x \in y$ or $y \in x$ or $x = y$.
By the definition of $\bigcup \mathcal{E}$, there exist $\gamma_1, \gamma_2 \in \mathcal{E}$ such that $x \in \gamma_1$ and $y \in \gamma_2$.
Since $\gamma_1, \gamma_2 \in ON$, by #7, $\gamma_1 < \gamma_2$ or $\gamma_1 = \gamma_2$ or $\gamma_2 < \gamma_1$, and by #6, equivalently we have $\gamma_1 \in \gamma_2$ or $\gamma_1 = \gamma_2$ or $\gamma_2 \in \gamma_1$.
If $\gamma_1 = \gamma_2$, then $x, y \in \gamma_1$, and since γ_1 is connected, $x = y$ or $x \in y$ or $y \in x$.
If $\gamma_1 \neq \gamma_2$, we can assume without loss of generality that $\gamma_1 \in \gamma_2$. Thus, $x \in \gamma_1 \in \gamma_2$, so since γ_2 is transitive, $x, y \in \gamma_2$, and so $x = y$ or $y \in x$ or $x \in y$ because γ_2 is connected.

Claim: $\sup \mathcal{E} = \bigcup \mathcal{E}$

<u>$\bigcup \mathcal{E}$ is an upper bound for $\mathcal{E}$</u>: We want to show that for all $\alpha \in \mathcal{E}$, $\alpha \leq \bigcup \mathcal{E}$, or equivalently, by #6, $\alpha = \bigcup \mathcal{E}$ or $\alpha \in \bigcup \mathcal{E}$. ~~[illegible]~~

~~[illegible]~~

Take some $\gamma \in \mathcal{E}$. Since γ and $\bigcup \mathcal{E}$ are both ordinals, by #7 either $\gamma \leq \bigcup \mathcal{E}$ or $\bigcup \mathcal{E} < \gamma$.

If $\gamma \leq \bigcup \mathcal{E}$, we are done.

If not, then $\bigcup \mathcal{E} < \gamma$, so by #6, $\bigcup \mathcal{E} \in \gamma$. Since γ is transitive, this means that $\bigcup \mathcal{E} \subseteq \gamma$. Since $\gamma \in \mathcal{E}$, we also know that $\gamma \subseteq \bigcup \mathcal{E}$. Thus, $\gamma = \bigcup \mathcal{E}$, meaning $\gamma \leq \bigcup \mathcal{E}$.

<u>$\bigcup \mathcal{E}$ is a least upper bound for $\mathcal{E}$</u>: Suppose γ is an upper bound for $\mathcal{E}$. Then for all $\beta \in \mathcal{E}$, $\beta \leq \gamma$. We want to show that $\gamma \geq \bigcup \mathcal{E}$.

Suppose not. Then by #7, $\gamma < \bigcup \mathcal{E}$, or equivalently, $\gamma \in \bigcup \mathcal{E}$. This means that there is a $\beta \in \mathcal{E}$ such that $\gamma \in \beta$. But by #6, this means that $\gamma < \beta$, a contradiction.

Thus, $\gamma \geq \bigcup \mathcal{E}$.

We now show that $\min \mathcal{E} = \bigcap \mathcal{E}$.

Claim: $\bigcap \mathcal{E}$ is an ordinal.

<u>$\bigcap \mathcal{E}$ is transitive</u>: Suppose $x \in \alpha \in \bigcap \mathcal{E}$. Then $\alpha \in \gamma$ for all $\gamma \in \mathcal{E}$, so $x \in \alpha \in \gamma$ for all $\gamma \in \mathcal{E}$. Since each γ is transitive, this implies that $x \in \gamma$ for all $\gamma \in \mathcal{E}$, or $x \in \bigcap \mathcal{E}$.

<u>$\bigcap \mathcal{E}$ is pure</u>: It suffices to show that $TC(\bigcap \mathcal{E})$ contains no atoms. Since $\bigcap \mathcal{E}$ is transitive, $TC(\bigcap \mathcal{E}) = \bigcap \mathcal{E} \cup \{\bigcap \mathcal{E}\}$, so it suffices to show that $\bigcap \mathcal{E}$ contains no atoms, and that $\bigcap \mathcal{E}$ is not an atom. Indeed, if $a \in \bigcap \mathcal{E}$, $a \in \gamma$ for all $\gamma \in \mathcal{E}$. Since each γ is pure, it cannot contain any atoms, hence a is not an atom.

Moreover, $\bigcap \mathcal{E}$ is a set, so it cannot be an atom. Thus, $TC(\bigcap \mathcal{E})$ contains no atoms, and $\bigcap \mathcal{E}$ is pure.

<u>$\bigcap \mathcal{E}$ is grounded</u>: Suppose not. Then by DC, there exists an infinite descending $\in$-chain in $\bigcap \mathcal{E}$, say $\alpha_1 \ni \alpha_2 \ni \ldots$

Pick some $\gamma \in \mathcal{E}$. Then since $\alpha_i \in \bigcap \mathcal{E}$ for all i, $\alpha_i \in \gamma$ for all i, meaning that the above is an infinite descending $\in$-chain in γ. This contradicts the fact that γ is grounded.

Allison Wong

$\bigcap \mathcal{E}$ is connected: Suppose $x, y \in \bigcap \mathcal{E}$. Pick some $\gamma \in \mathcal{E}$. Then $x, y \in \gamma$, so since γ is connected, $x \in y$ or $y \in x$ or $x = y$.

claim: min $\mathcal{E} = \bigcap \mathcal{E}$

$\bigcap \mathcal{E}$ is a lower bound for $\mathcal{E}$: Take any $\gamma \in \mathcal{E}$. Since $\bigcap \mathcal{E}, \gamma$ are both ordinals, either $\bigcap \mathcal{E} \leq \gamma$ or $\gamma > \bigcap \mathcal{E}$ by #7.

If $\bigcap \mathcal{E} \leq \gamma$, we are done.

If not, then $\gamma < \bigcap \mathcal{E}$, or equivalently, $\gamma \in \bigcap \mathcal{E}$.

By the fact that $\bigcap \mathcal{E}$ is transitive, this implies that $\gamma \subseteq \bigcap \mathcal{E}$.

We have that $\bigcap \mathcal{E} \subseteq \gamma$ by definition, so $\bigcap \mathcal{E} = \gamma$, or $\bigcap \mathcal{E} \leq \gamma$.

$\bigcap \mathcal{E}$ is a greatest lower bound for $\mathcal{E}$: Suppose, for a contradiction, that γ is a lower bound for $\mathcal{E}$, but $\bigcap \mathcal{E} < \gamma$.

Since γ is a lower bound for $\mathcal{E}$, that means that for all $\beta \in \mathcal{E}$, $\gamma \leq \beta$. Thus, $\bigcap \mathcal{E} < \beta$ for all $\beta \in \mathcal{E}$, or $\bigcap \mathcal{E} \in \beta$ for all $\beta \in \mathcal{E}$.

This means that $\bigcap \mathcal{E} \cup \{\bigcap \mathcal{E}\} = S(\bigcap \mathcal{E}) \subseteq \bigcap \mathcal{E}$, a contradiction.

Thus, $\bigcap \mathcal{E} \geq \gamma$ for all lower bounds γ for $\mathcal{E}$.

Allison Wong

#9

#9: The class ON is not a set.

Proof: Suppose, for a contradiction, that ON is a set.

By #10, ON is not empty, as $\emptyset \in ON$.

Thus, by #8, $\sup ON = \bigcup ON$.

(In fact, $\bigcup ON$ is the maximum element of ON, since by the proof in #8, $\bigcup ON \in ON$).

Now consider $S(\bigcup ON)$.

By #10, $S(\bigcup ON)$ is the least ordinal $> \bigcup ON$.

This means that $\bigcup ON < S(\bigcup ON)$, and that $S(\bigcup ON) \in ON$.

But since $\bigcup ON$ is the supremum of ON, that means that for all $\beta \in ON$, $\beta \leq \bigcup ON$. In particular, $S(\bigcup ON) \leq \bigcup ON$.

This is a contradiction.

Note that if ON is a class, the same contradiction does not occur, as $\bigcup ON$ is not guaranteed to be a set.

JUNEJA

10. Prop: 1) $0 = \phi$ is the least ordinal
2) $S(\alpha) = \alpha \cup \{\alpha\}$ is the least ordinal $> \alpha$.
3) if $\alpha \neq 0$, α not a successor then $\alpha = \sup\{\gamma \mid \gamma < \alpha\}$

1) if $a \leq 0$ then $a = 0$ or $a \in \phi$ i.e $a = 0$

2) if $a > \alpha$ then $\alpha \in a$ i.e $\alpha \subseteq a$ and $\alpha \in a$.
so $S(\alpha) = \alpha \cup \{\alpha\} \subseteq a$
i.e $S(\alpha) \leq a$.

3) Let $\alpha \neq 0$, not a successor. will show
$\alpha = \sup\{\gamma \mid \gamma < \alpha\}$ by set containment
($\supseteq$) Let $x \in \sup\{\gamma \mid \gamma < \alpha\}$

Then $x \in \bigcup_{\gamma < \alpha} \gamma$ i.e $\exists \beta < \alpha$ s.t $x \in \beta$

so $x \in \beta \in \alpha$ so $x \in \alpha$.

($\subseteq$) Let $x \in \alpha$ then $x < \alpha$ i.e $x \subseteq \bigcup_{\gamma < \alpha} \gamma$
i.e $x \in \bigcup_{\gamma < \alpha} \gamma$

$\square$

SAAMARTH JUNEJA #11

1P Let $(\forall \beta \in \alpha) P(\beta) \Rightarrow P(\alpha)$

Assume, for contradiction, that $\exists x \in ON$ s.t $\neg P(x)$

Since ON are well ordered by $\leq$, there is a least x_0 s.t $\neg P(x_0)$ since there exists x s.t $\neg P(x)$

Then $\forall a < x_0$, $P(a)$
i.e $\forall a \in x_0$, $P(a)$

So $P(x_0)$ $\lightning$ since $\neg P(x_0)$

$\square$

12) Let $H(w, \alpha, x)$ be a definite operation. We claim that $\exists! \ F(\alpha, x)$ s.t. $F(\alpha,x) = H(F_x|_\alpha, \alpha, x)$, where $F_x|_\alpha = \{(\beta, F(\beta,x)) \mid \beta \in \alpha\}$. For $\alpha \in ON$, x, define $G_{\alpha,x} : \alpha \to G_{\alpha,x}[\alpha]$ via $G_{\alpha,x}(\beta) = H(G_{\alpha,x}|_\beta, \beta, x)$ via recursion on α with parameters, as α is a woset.

#12

Lemma: Let $\beta < \alpha$ in ON. We claim that $\forall \gamma < \beta$, $G_{\alpha,x}(\gamma) = G_{\beta,x}(\gamma)$.

Proof: Suppose not. Then let $\gamma < \beta$ minimal such that $G_{\alpha,x}(\gamma) \neq G_{\beta,x}(\gamma)$.

By minimality, $\forall \delta < \gamma$ $G_{\alpha,x}(\delta) = G_{\beta,x}(\delta)$.

Thus, $G_{\alpha,x}|_\gamma = G_{\beta,x}|_\gamma$. Hence,

$$\begin{array}{ccc} H(G_{\alpha,x}|_\gamma, \gamma, x) & = & H(G_{\beta,x}|_\gamma, \gamma, x) \\ \| & & \| \\ G_{\alpha,x}(\gamma) & \neq & G_{\beta,x}(\gamma) \end{array}$$

So we have reached a contradiction, so $\forall \gamma < \beta$ $G_{\alpha,x}(\gamma) = G_{\beta,x}(\gamma)$.

Thus, for $\beta < \alpha$ in ON, $\forall \gamma < \beta$ $G_{\alpha,x}|_\delta = G_{\beta,x}|_\beta$.

Define $F(\alpha, x) = G_{S(\alpha),x}(\alpha)$.

We claim $F(\alpha,x) = H(F_x|_\alpha, \alpha, x)$.

Let $\beta \in \alpha$. $F_x|_\alpha(\beta) = F(\beta,x) = G_{S(\beta),x}(\beta) = G_{S(\alpha),x}(\beta)$ as $S(\beta) < S(\alpha)$, so the lemma applies.

Thus, $F_x|_\alpha = G_{S(\alpha),x}|_\alpha$

Hence, $H(F_x|_\alpha, \alpha, x) = H(G_{S(\alpha),x}|_\alpha, \alpha, x) = G_{S(\alpha),x}(\alpha) = F(\alpha,x)$ as desired.

Uniqueness: Let E and F both satisfy the condition that $E(\alpha,x) = H(E_x|_\alpha, \alpha, x)$
$F(\alpha,x) = H(F_x|_\alpha, \alpha, x)$.

Fix x, we claim $E(\alpha,x) = F(\alpha,x)$. If not, let $\alpha \in ON$ minimal s.t. $E(\alpha,x) \neq F(\alpha,x)$.

Then $\forall \beta < \alpha$ $E(\beta,x) = F(\beta,x)$, so $E_x|_\alpha = F_x|_\alpha$.

$$\begin{array}{ccc} \text{Thus, } H(F_x|_\alpha, \alpha, x) & = & H(E_x|_\alpha, \alpha, x) \\ \| & & \| \\ F(\alpha,x) & \neq & E(\alpha,x) \end{array}$$

which is a contradiction, so $E(\alpha,x) = F(\alpha,x)$. Thus $\forall x \ \forall \alpha \in ON$ $E(\alpha,x) = F(\alpha,x)$, so $E = F$, so we have uniqueness.

For the case without parameters, simply use the above with some fixed x, say $x = \emptyset$.

Jas Singh

#14

SAAMARTH JUNEJA

14. Consider $\alpha = \omega$ and $\beta = 1$

Then $1 + \omega = \omega + \omega + 1$

$\square$

#18. For $\forall \alpha \cdot \beta \cdot \gamma \in ON$ $\alpha\cdot(\beta\cdot\gamma)=(\alpha\cdot\beta)\cdot\gamma$

To prove #18 by induction we need first assume #22 (Right distributive law).

We use induction on γ

#18

Base Case: $\gamma=0$ $\alpha\cdot(\beta\cdot 0)=\alpha\cdot 0=0$ by definition

$(\alpha\cdot\beta)\cdot 0=0$ by definition

so $\alpha\cdot(\beta\cdot 0)=(\alpha\cdot\beta)\cdot 0$

Now Assume $\forall \delta<\gamma$ $\alpha\cdot(\beta\cdot\delta)=(\alpha\cdot\beta)\cdot\delta$.

Case 1: γ is not a limit point, i.e. $S^{-1}(\gamma)$ exists.

Then $\alpha\cdot(\beta\cdot\gamma)=\alpha\cdot(\beta\cdot S(S^{-1}(\gamma)))=\alpha\cdot(\beta\cdot S^{-1}(\gamma)+\beta)$ by definition

$=\alpha\cdot(\beta\cdot S^{-1}(\gamma))+\alpha\cdot\beta$ by right distributive law

$=(\alpha\cdot\beta)\cdot S^{-1}(\gamma)+\alpha\cdot\beta$ by induction hypothesis.

$=(\alpha\cdot\beta)\, S(S^{-1}(\gamma))=(\alpha\cdot\beta)\cdot\gamma$ by definition.

Case 2: γ is a limit point.

Then $\alpha\cdot(\beta\cdot\gamma)=\alpha\cdot\sup\{\beta\cdot\delta\mid\delta<\gamma\}$ $(\alpha\cdot\beta)\cdot\gamma=\sup\{(\alpha\cdot\beta)\cdot\delta\mid\delta<\gamma\}$.

Let $L=\sup\{\beta\cdot\delta\mid\delta<\gamma\}$

WTS $\alpha\cdot L$ is the least upper bound of the set $\{(\alpha\cdot\beta)\cdot\delta\mid\delta<\gamma\}$.

① since $(\alpha\cdot\beta)\cdot\delta=\alpha\cdot(\beta\cdot\delta)$ by induction hypothesis.

and $\beta\cdot\delta\leq\sup\{\beta\cdot\delta\mid\delta<\gamma\}=L$ $\Rightarrow\alpha\cdot(\beta\cdot\delta)\leq\alpha\cdot L$ $\Rightarrow\alpha\cdot L$ is an upper bound.

② suppose K is an upper bound for $\{(\alpha\cdot\beta)\cdot\delta\mid\delta<\gamma\}$

i.e. $\forall\delta<\gamma$ $(\alpha\cdot\beta)\cdot\delta=\alpha\cdot(\beta\cdot\delta)\leq K$. $\Rightarrow\alpha\cdot\sup\{\beta\cdot\delta\mid\delta<\gamma\}\leq K$.

i.e $\alpha\cdot L\leq K$ thus. $\alpha\cdot L$ is the least upper bound of $\{(\alpha\cdot\beta)\cdot\delta\mid\delta<\gamma\}$.

which means $\alpha\cdot(\beta\cdot\gamma)=(\alpha\cdot\beta)\cdot\gamma$

Thus concludes the proof.

Jas Singh

#19

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14) $\exists\, \alpha, \beta \in On$ s.t. $\alpha \cdot \beta \neq \beta \cdot \alpha$.

Proof: Let $\alpha = 2$, $\beta = \omega$.

$2 \cdot \omega = \sup\{2n \mid n < \omega\} = \sup\{n \mid n < \omega\} = \omega$.

$\omega \cdot 2 = \omega + \omega > \omega$ by problem 18.

Thus, $2 \cdot \omega = \omega \neq \omega \cdot 2$ as desired.

Jas Singh

#20

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20 Claim: For $\alpha \in ON$, $0 \cdot \alpha = 0$

Proof: We proceed by induction on α.

Base case: If $\alpha = 0$ then $0 \cdot \alpha = 0 \cdot 0 = 0$

Suppose $\forall \beta < \alpha$ that $0 \cdot \beta = 0$.

Successor: Suppose $\alpha = S(\gamma)$. Then $0 \cdot \alpha = 0 \cdot S(\gamma) = 0 \cdot \gamma + 0 = 0 + 0 = 0$, as $\gamma < \alpha$ so $0 \cdot \gamma = 0$.

Limit: Suppose α a limit ordinal. Then $0 \cdot \alpha = \sup\{0 \cdot \beta \mid \beta < \alpha\} = \sup\{0 \mid \beta < \alpha\} = 0$.

Thus, by ordinal induction, $\forall \alpha \in ON$ $0 \cdot \alpha = 0$.

Claim: For $\alpha, \beta \in ON$, $0 < \alpha$ and $1 < \beta \Rightarrow \alpha < \alpha \cdot \beta$.

Proof: We proceed by induction on β.

Base case: $\beta = 2 \Rightarrow \alpha \cdot \beta = \alpha \cdot S(1) = \alpha + \alpha > \alpha$ by (15) as $\alpha > 0$. Thus, $\alpha \cdot \beta > \alpha$.

Suppose $\forall\, 1 < \gamma < \beta$, $\forall\, 0 < \alpha \in ON$ $\alpha < \alpha \cdot \beta$

Successor: Suppose $\beta = S(\gamma)$. Then $\alpha \cdot \beta = \alpha \cdot S(\gamma) = \alpha\gamma + \alpha > \alpha + \alpha$ by the inductive hypothesis
$> \alpha$ as above.

Thus, $\alpha \cdot \beta > \alpha$

Limit: Suppose β a limit ordinal. Then $\alpha\beta = \sup\{\alpha\gamma \mid \gamma < \beta\} \geq \alpha \cdot 2 > \alpha$ by the inductive hypothesis.

Thus, $\alpha \cdot \beta > \alpha$.

Thus, by ordinal induction, $\forall\, 1 < \beta$, $0 < \alpha$ $\alpha \cdot \beta > \alpha$

Claim: $\alpha \leq \beta$, $\gamma \leq \delta \Rightarrow \alpha \cdot \gamma \leq \beta \cdot \delta$

Proof: We proceed by induction on δ.

Base case: If $\delta = 0$ then as $\gamma \leq \delta$, $\gamma = 0$. Then the desired inequality of $\alpha\gamma \leq \beta\delta$ becomes $0 \leq 0$, which is true.

Suppose that $\forall\, \varepsilon < \delta$ in ON, $\alpha \leq \beta$, $\gamma \leq \varepsilon \Rightarrow \alpha\gamma \leq \beta\varepsilon$

Successor: Suppose $\delta = S(\varepsilon)$. If $\gamma \leq \varepsilon$ then $\beta\delta = \beta S(\varepsilon) = \beta\varepsilon + \beta \geq \alpha\gamma + \beta$ by the inductive hypothesis
$\geq \alpha\gamma$,

so $\beta\delta \geq \alpha\delta$.

If on the other hand $\varepsilon < \gamma$, $S(\varepsilon) \leq \gamma \leq \delta = S(\varepsilon)$ so $\gamma = \delta$.

Then $\beta\delta = \beta S(\varepsilon) = \beta\varepsilon + \beta \geq \alpha\varepsilon + \beta$ by the inductive hypothesis
$\geq \alpha\varepsilon + \alpha$ by (15) as $\alpha \leq \beta$
$= \alpha S(\varepsilon) = \alpha\delta = \alpha\gamma$ so $\beta\delta \geq \alpha\gamma$.

Limit: Suppose δ a limit ordinal. Then $\beta\delta = \sup\{\beta\varepsilon \mid \varepsilon < \delta\}$

If $\gamma < \delta$ then $\beta\delta = \sup\{\beta\varepsilon \mid \varepsilon < \delta\} \geq \beta\delta \geq \alpha\delta$ by the inductive hypothesis.

If, on the other hand, $\gamma = \delta$, then for $\varepsilon < \delta$ $\beta\varepsilon \ge \alpha\varepsilon$ by the induction hypothesis.

Thus, $\sup\{\beta\varepsilon \mid \varepsilon < \delta\} \ge \sup\{\alpha\varepsilon \mid \varepsilon < \delta\}$

$$\underset{\beta\delta}{\shortparallel} \qquad \ge \qquad \underset{\alpha\delta}{\shortparallel} = \alpha\gamma, \text{ so } \beta\delta \ge \alpha\gamma.$$

Thus, by ordinal induction $\alpha \le \beta$, $\gamma \le \delta$ $\Rightarrow$ $\alpha\gamma \le \beta\delta$.

Claim: $0 < \alpha \le \beta$, $\gamma < \delta \Rightarrow \alpha\gamma < \beta\delta$.

Proof: We proceed by cases.

If $\delta = 1$ then $\gamma = 0$ so the desired inequality becomes $\alpha \cdot 0 < \beta \cdot 1$, so $0 < \beta$, which is true as $\beta \ge \alpha > 0$ by assumption.

If $\delta = S(\varepsilon)$ then if $\varepsilon < \gamma$, $S(\varepsilon) \le \gamma < \delta = S(\varepsilon)$ so $S(\varepsilon) < S(\varepsilon)$, contradiction.

Thus, $\gamma \le \varepsilon$ so $\beta\delta = \beta S(\varepsilon) = \beta\varepsilon + \beta \ge \alpha\gamma + \beta$ by the previous result.

$> \alpha\gamma$ by (15) as $\beta > 0$.

Thus, $\beta\delta > \alpha\gamma$

If δ is a limit ordinal then as $\gamma < \delta$, $S(\gamma) < \delta$.

$\beta\delta = \sup\{\beta\varepsilon \mid \varepsilon < \delta\} \ge \beta S(\gamma) = \beta\gamma + \beta \ge \alpha\gamma + \beta$ by the previous result

$> \alpha\gamma$ by (15) as $\beta > 0$.

Thus, in all cases, $\beta\delta > \alpha\gamma$.

Claim: $\alpha < \beta \not\Rightarrow \alpha\gamma < \beta\gamma$, even for $\gamma > 0$.

Proof: Let $\alpha = 1$, $\beta = 2$, $\gamma = \omega$. Then $\alpha < \beta$, $\gamma > 0$.

However, $\alpha\gamma = 1 \cdot \omega = \omega$

$\beta\gamma = 2 \cdot \omega = \omega$

Thus, $\alpha\gamma \not< \beta\gamma$ so the implication is not always true.

We use the fact from x12.12 in the textbook that for all $\alpha \geq \omega$, $(\alpha+1)\cdot\omega = \alpha\cdot\omega$.

Roughly, this fact is true because we can think of $(\alpha+1)\cdot\omega$ as ω copies of $\alpha+1$ placed in sequence. By associativity of ordinal addition, we have

$$
\begin{aligned}
&(\alpha+1)\cdot\omega \\
&= (\alpha+1)+(\alpha+1)+(\alpha+1)+(\alpha+1)+\cdots \\
&= \alpha+(1+\alpha)+(1+\alpha)+(1+\alpha)+(1+\alpha)+\cdots \\
&= \alpha+\alpha+\alpha+\alpha+\alpha+\cdots \\
&= \alpha\cdot\omega.
\end{aligned}
$$

Now, letting $\alpha=\omega$, $\beta=1$, and $\gamma=\omega$, we have

$$
\begin{aligned}
(\alpha+\beta)\cdot\gamma &= (\omega+1)\cdot\omega \\
&= \omega\cdot\omega
\end{aligned}
$$

by the fact above. We also have

$$\alpha\cdot\gamma+\beta\cdot\gamma = \omega\cdot\omega+\omega.$$

Since $\omega\cdot\omega \neq \omega\cdot\omega+\omega$,

$(\alpha+\beta)\cdot\gamma \neq \alpha\cdot\gamma+\beta\cdot\gamma$.

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#24

24. Let $\leqslant$ be a best well-ordering on A

Let $\alpha = ord(A, \leqslant)$

Prop: $|A| = \alpha$

Pf: $|A| = (\mu\beta \in ON)[A =_c \beta]$

Since $A =_c ord(A, \leqslant) =_c \alpha$ and $\alpha \in ON$, we have $|A| = \alpha$ or $|A| \in \alpha$

if $|A| = \alpha$, we are done, so let $|A| \in \alpha$

Then $|A| <_o \alpha$ i.e $|A| \sqsubsetneq ord(A, \leqslant)$

So $\exists x \in A$ s.t $|A| =_o seg_{\leqslant}(x)$, a proper initial segment.

So $A =_c |A| =_c seg_{\leqslant}(x)$ $\lightning$

since $\leqslant$ is a best-well ordering

$\square$

#25

SAAMARTH JUNEJA

25. Prop: For any well orderable sets A, B, $A =_c B \Leftrightarrow |A| = |B|$

$(\Rightarrow)$ Let $A =_c B$

$|A| = (\mu \beta \in ON)(A =_c \beta)$

since $A =_c B$, $A =_c \beta \Leftrightarrow B =_c \beta$

so $|A| = (\mu \beta \in ON)(B =_c \beta) = |B|$

$(\Leftarrow)$ Let $|A = |B|$

Then $A =_c |A| =_c |B| =_c B$

so $A =_c B$

$\square$

#26. Card(κ) $\iff$ $\kappa \in ON$. $(\forall \alpha \in \kappa)[\alpha <_c \kappa]$.

Shangjie Zhang

proof: "$\Rightarrow$" Assume $\kappa \in$ Card.

i.e. $\exists A$ s.t. $[\kappa = |A|]$ since $|A| =_c A$ $\Rightarrow \kappa =_c |A| =_c A$

For $\forall \alpha \in \kappa$. $\Rightarrow \alpha \subseteq \kappa$ $\Rightarrow \alpha \leq_c \kappa$.

Suppose, toward a contradiction that $\alpha =_c \kappa$.

$\Rightarrow$. $\alpha =_c \kappa =_c A$.

which means κ is not the least ordinal s.t. $\kappa =_c A$

$\Rightarrow |A| \neq \kappa$. "$\Rightarrow\Leftarrow$"

Thus $\alpha <_c \kappa$

"$\Leftarrow$" Assume For $\kappa \in ON$, $\forall \alpha \in \kappa$, $\alpha <_c \kappa$

since κ itself is a woset.

and. $|\kappa| =_c \kappa$

and $\forall \alpha < \kappa$. $\alpha <_c \kappa$.

which means κ is the smallest ordinal s.t $\kappa =_c \kappa$.

$\Rightarrow \kappa \in$ Card.

q.e.d.

SAAMARTH JUNEJA

#27

27. Prop: The class Card is not a set

Pf: Assume towards a contradiction that the class Card is a set

Let this set be K

Since all cardinals are ordinals, K is a set of ordinals so $\bigcup K = \sup K$ is an ordinal

Consider $\chi(\bigcup K)$ the hartogs set of $\bigcup K$ then $\chi(\bigcup K) \not\leq_c \bigcup K$

We know $\chi(\bigcup K)$ is well orderable set so $|\chi(\bigcup K)|$ exists

Since $|\chi(\bigcup K)|$ is a cardinal so $|\chi(\bigcup K)| \in K$

So $|\chi(\bigcup K)| \leq_o \bigcup K$ since $\bigcup K = \sup K$

So $|\chi(\bigcup K)| \leq_c \bigcup K$

So $\chi(\bigcup K) \leq_c \bigcup K$ $\lightning$

So the class Card is not a set.

$\square$

(38) Our goal is to show $V_\omega \notin Z$.

First, let's recall the definitions of V_ω and Z. They are defined recursively as follows:

$V_0 = \phi$
$V_{\alpha+1} = P(V_\alpha)$
$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ if λ is a limit point

$Z_0 = N_0 = \{\phi, \{\phi\}, \{\{\phi\}\}, \ldots\}$
$Z_{n+1} = P(Z_n)$
$Z = \bigcup_{n\in N} Z_n$

Now to show that $V_\omega \notin Z$, it suffices to show that $\forall n \in N$, $V_\omega \notin Z_n$.

Suppose $n=0$. Suppose $V_\omega \in Z_0$. Then V_ω is a singleton. But observe that:

$V_0 = \phi$
$V_1 = \{\phi\}$
$V_2 = \{\phi, \{\phi\}\}$

So $\phi, \{\phi\} \in V_\omega$. So V_ω is not a singleton, a contradiction. So $V_\omega \notin Z_0$.

Now suppose $n \neq 0$. Then $n = m+1$ for some $m \in N$. We claim that $\forall m \in N$, $V_\omega \nsubseteq Z_m$. For if this is true, then $V_\omega \notin P(Z_m) = Z_{m+1}$, as desired.

We show the claim by induction on m. Suppose $m=0$. Observe that:

$V_0 = \phi$
$V_1 = \{\phi\}$
$V_2 = \{\phi, \{\phi\}\}$
$V_3 = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Then $\{\phi, \{\phi\}\} \in V_\omega$. But $\{\phi, \{\phi\}\}$ is not a singleton, so $\{\phi, \{\phi\}\} \notin Z_0$. So $V_\omega \nsubseteq Z_0$.

38 Now suppose $V_\omega \not\subseteq Z_m$. We want to show that $V_\omega \not\subseteq Z_{m+1}$.

Since $V_\omega \not\subseteq Z_m$, we know $\exists A$ so that $A \in V_\omega$ but $A \notin Z_m$.
Now we claim:

1. $\{A\} \in V_\omega$
2. $\{A\} \notin Z_{m+1}$

For if this is true, then $V_\omega \not\subseteq Z_{m+1}$, as desired.

1. Since $A \in V_\omega = \bigcup_{\alpha<\omega} V_\alpha$, we know $A \in V_\alpha$ for some $\alpha < \omega$. Now by definition of successor, $\alpha+1 = S(\alpha) \leq \omega$. If $S(\alpha) = \omega$, then this contradicts the definition of ω as a limit point. So $\alpha+1 = S(\alpha) < \omega$. So $V_{S(\alpha)} \subseteq \bigcup_{\alpha<\omega} V_\alpha = V_\omega$. Now $\{A\} \in P(V_\alpha) = V_{S(\alpha)} \subseteq V_\omega$, as desired.

2. Since $A \notin Z_m$, $\{A\} \notin P(Z_m) = Z_{m+1}$, as desired.

#42. P is a perfect set. than $P =_c \mathcal{C}$.

#42

Proof: Since $P \subseteq \mathcal{N} =_c \mathcal{C} \Rightarrow P \leq_c \mathcal{C}$.

conversely since P is perfect. $\exists$ some tree T s.t $P = [T]$ which is splitting.

by finite choice. $\exists\, l : T \to T$. $r : T \to T$ s.t.

$\forall u \in T$ $\begin{matrix} u \sqsubseteq l(u) \\ u \sqsubseteq r(u) \end{matrix}$ & $l(u) | r(u)$. i.e. l & r has splitting property.

By string recursion on l & r respectively.

$\exists! \sigma : \{0,1\}^* \longrightarrow T$ s.t. $\begin{cases} \sigma(\phi) = \phi. \\ \sigma(u \star \langle 0 \rangle) = l(\sigma(u)) \\ \sigma(u \star \langle 1 \rangle) = r(\sigma(u)). \end{cases}$

Thus σ has the following property:

since $\sigma(u) \sqsubseteq l(\sigma(u)) = \sigma(u \star \langle 0 \rangle)$ also $\sigma(u) \sqsubseteq \sigma(u \star \langle 1 \rangle)$.

① Thus $\forall u, v \in T$ $u \sqsubseteq v \Rightarrow \sigma(u) \sqsubseteq \sigma(v)$

② If. $u | v$ then $\sigma(u) | \sigma(v)$

(This is because $\sigma(u)$ & $\sigma(v)$ just extend u & v.).

Now we define $f : \{0,1\}^{\mathbb{N}} \longrightarrow [T]$. with

$f(x) = \sup\{\sigma(u) \mid u \sqsubseteq x\}$.

$\forall x, y \in \{0,1\}^{\mathbb{N}}$ $x \neq y$

$\Rightarrow \exists\, u \sqsubseteq x$ $v \sqsubseteq y$ s.t $u | v$.

$\Rightarrow \sup\{\sigma(u)\} \neq \sup\{\sigma(v)\}$

$\Rightarrow f(x) \neq f(y)$

Thus f is an injection $\Rightarrow \{0,1\}^{\mathbb{N}} = \mathcal{C} \leq_c P$

By Schröder-Berstein. Thm, $P =_c \mathcal{C}$.

String Recursion: (without parameter) Shangjie Zhang

For two sets A, E. & a function $h: E \times A \to E$. for some $a \in E$.

$\exists!\ f: A^* \to E$ s.t. $\begin{cases} f(\langle\ \rangle) = a \\ f(u \star \langle x \rangle) = h(f(u), x). \end{cases}$

Proof: By Recursion Thm on $\mathbb{N}$.

$\exists!\ \varphi$ s.t. $\varphi: \mathbb{N} \times A^* \to E$ s.t. $\varphi(0, u) = a$
$\varphi(n+1, u) = h(\varphi(n, u), u(n))$

Let $f(u) = \varphi(lh(u), u)$.

$f(\langle\ \rangle) = \varphi(0, u) = a$ satisfies the first condition.

WTS. $\forall n \in \mathbb{N}$. if $\forall i < n$, $u(i) = v(i)$ then $\varphi(n, u) = \varphi(n, v)$

proof by induction: base case $n=0$ $\varphi(0, u) = \varphi(0, v) = a$.

suppose it's true for n.

then for $\forall i < n+1$. with $u(i) = v(i)$

$\varphi(n+1, u) = h(\varphi(n, v), u(n)) = h(\varphi(n, v), v(n))$ by induction hypothesis
$= \varphi(n+1, v)$

so it holds true.

Thus. $f(u \star \langle x \rangle) = \varphi(lh(u)+1, u \star \langle x \rangle) = h(\varphi(lh(u), u \star \langle x \rangle), x)$
$= h(\varphi(lh(u), u), x) = h(f(u), x)$

q.e.d.