DETAILED PROOF OF THEOREM 4.1 IN SENSE AND DENOTATION AS ALGORITHM AND VALUE

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§4. Sense identity and indirect reference. Van Heijenoort [30] quotes an extensive passage from a 1906 letter from Frege to Husserl which begins with the following sentence:

> "It seems to me that we must have an objective criterion for recognizing a thought as the same thought, since without such a criterion a logical analysis is not possible."

This could be read as asserting the existence of a decision procedure for sense identity, but unfortunately, the letter goes on to suggest that logically equivalent sentences have the same sense, a position which is contrary to the whole spirit of [12]. It is apparently not clear what Frege thought of this question or if he seriously considered it at all. Kreisel and Takeuti [17] raise explicitly the question of *synonymity* of sentences which may be the same as that of identity of sense. If we identify sense with referential intension, the matter is happily settled by a theorem.

THEOREM 4.1. *For each recursor structure* $\mathbf{A} = (U_1, \dots, U_k, f_1, \dots, f_n)$ *of finite signature, the relation* $\sim_{\mathbf{A}}$ *of intensional identity on the terms of* FLR *interpreted on* $\mathbf{A}$ *is decidable.*

For each structure $\mathbf{A}$ and arbitrary integers n, m, let

$$S_{\mathbf{A}}(n,m) \iff n,\ m \text{ are Gödel numbers of sentences or terms } \theta_n,\ \theta_m \text{ of FLR and } \theta_n \sim_{\mathbf{A}} \theta_m. \tag{58}$$

The rigorous meaning of **4.1** is that this relation $S_{\mathbf{A}}$ is decidable, i.e., computable by a Turing machine. By the usual coding methods then, we get immediately:

COROLLARY 4.2. *The relation* $S_{\mathbf{A}}$ *of intensional identity on Gödel numbers of expressions of* FLR *is elementary (definable in* LPC*), over each acceptable structure* $\mathbf{A}$.

The Corollary is useful because it makes it possible to talk indirectly about FLR intensions within FLR. In general, we cannot do this directly because the intensions of a structure $\mathbf{A}$ are higher type objects over $\mathbf{A}$ which are not ordinarily[15] members of any basic set of the universe of $\mathbf{A}$. One reason we might want to discuss FLR intensions within FLR is to express *indirect reference*, where Frege's treatment deviates from his general doctrine of separate compositionality

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It is a revised version of Section §4 of [22], which fills a gap in the proof of the main Theorem 4.1, the decidability of the synonymy relation. It also provides many more details and some examples which (I hope) make the argument easier to follow. I have preserved the numbering of displayed formulas of [22], assigning new, interpolated numbers to new displays.

[15] If the universe of $\mathbf{A}$ contains the powerset of every basic set in it and the Cartesian product of every two basic sets, then of course it contains all recursors over basic sets and with suitably rich primitives we can develop the theory of intensions of $\mathbf{A}$ within LPCR. These are typed structures, however, of infinite signature, which lack a natural universe, a largest basic set. More interesting would be the structure of the universe of sets, whose only basic "set" is the class of all sets. The intensions of this structure are certainly not sets.

principles for sense and denotation. Frege argued that "the indirect denotation of a word is ... its customary sense," so that in

$$\text{Othello believed that Cassio and Desdemona were lovers,} \tag{59}$$

the object of Othello's belief would be the sense of the sentence 'Cassio and Desdemona were lovers'. Since we cannot refer to sense directly in any fragment of natural language formalizable within FLR (if we identify it with intension), we might attempt to make belief an attribute of sentences, or (equivalently) their Gödel numbers. This means that (59) is expressed by

$$\text{Othello believed 'Cassio and Desdemona were lovers',} \tag{60}$$

where 'Cassio and Desdemona were lovers' is the Gödel number of the sentence within the quotes. But then we would certainly expect (60) to imply

$$\text{Othello believed 'Desdemona and Cassio were lovers',} \tag{61}$$

and we would like to express in the language the general assertion that belief depends only on the sense, not the syntactic form of a sentence, i.e.,

$$[\text{Othello believes } m \ \&\ S_{\mathbf{A}}(m,n)] \Longrightarrow \text{ Othello believes } n. \tag{62}$$

The Corollary says precisely that (62) is expressible already in LPC. The method is evidently quite general: if we view propositional attitudes as attributes of Gödel codes, we can express in the language that they respect sense identity, those indeed which should respect it.

Proof of the Main Theorem **4.1**. The intension of a term in a recursor structure is computed in the associated functional expansion by Def. **3.6** of [25], so we may assume that the interpretations $f_1, \dots, f_n$ of the function symbols of the language in $\mathbf{A}$ are functionals. We fix such a functional structure $\mathbf{A}$ then and we assume (for the time being) that *all basic sets in* $\mathbf{A}$ *are infinite*. We will discuss the interesting case of finite basic sets at the end.

Since intensions are preserved by passing to the normal form, the problem of intensional identity on $\mathbf{A}$ comes down to this: given two irreducible, recursive terms

$$\phi \equiv \phi_0 \text{ where } \{p_1(u_1) \simeq \phi_1, \dots,\ p_n(u_n) \simeq \phi_n\},$$
$$\psi \equiv \psi_0 \text{ where } \{q_1(v_1) \simeq \psi_1, \dots,\ q_m(v_m) \simeq \psi_m\},$$

is $n = m$ and can we match the terms so that they define the same functionals? By trying all possible ways to match the parts[16] and using the form of irreducible, explicit terms (2B.4 of [24]), we can further reduce the problem to that of deciding whether an arbitrary identity in one of the following three forms holds in $\mathbf{A}$:

$$f(z_1, \dots, z_m) \simeq g(z_{m+1}, \dots, z_l), \tag{63}$$
$$f(z_1, \dots, z_m) \simeq p(w_1, \dots, w_k). \tag{64}$$
$$q(w_1, \dots, w_m) \simeq p(w_{m+1}, \dots, w_l). \tag{65}$$

Here the following conditions hold:

1. The functionals f and g are among the finitely many givens of $\mathbf{A}$, or the constants $\mathbf{t}$, $\mathbf{f}$ or the conditional.
2. Each z_i is an *immediate expression* (in the full set of variables) by 2B.2 of [24], i.e., either a basic variable, or $p(\vec{x})$ where the x_i's are basic variables, or $\lambda(\vec{s})p(\vec{x})$ with $p(\vec{x})$ as above.

[16]This trivial part of the algorithm is (on the face of it) in NP (non-deterministic, polynomial time) in the length of the given terms and the rest will be seen to be no worse. I do not know a better upper bound for the complexity of intensional identity on a fixed structure and the best lower bound I know is that of Theorem **3.6**.

3. Each w_j is either a basic variable or $r(\vec{x})$, where the x_i's are basic variables and where $r \equiv p$ and $r \equiv q$ are allowed.

The decision question is trivial for identities in form (65) because of the following elementary result from logic:

4.3. **Lemma.** An identity (65) is valid in any fixed structure with infinite basic sets only if its two sides are identical.

We can view (64) as a special case of (63), with the following *evaluation functional* substituted for g on the right:

$$ev^k(p, x_1, \dots, x_k) \simeq p(x_1, \dots, x_k). \tag{66}$$

Notice, however, that there are infinitely many such evaluation functionals. There are also infinitely many possible identities in form (63), because although f and g are chosen from a finite set, there is an infinite number of immediate expressions from which to choose the z_i's. The proof splits into two parts. First we will show that if we expand the structure by a fixed, finite number of evaluation functionals, then every identity in form (64) is effectively equivalent to one in form (63). In the second part we will show how to decide the validity of equations in form (63).

4.4. A basic variable v is *placed* in an identity (63) or (64) if $v \equiv z_i$ for some i. For example, the placed variables of $f(v, p(x, y), u) = p(s, r(x, y))$ are v and u.

4.5. **Lemma.**[17] Suppose the identity

$$f(z_1, \dots, z_m) \simeq p(w_1, \dots, w_k). \tag{67}$$

is valid in the structure $\mathbf{A}$ with infinite basic sets and $w_i \equiv r(\vec{x})$ is one of the terms on the right. Then there exists some z_j on the left such that either $w_i \equiv z_j$, or $z_j \equiv \lambda(s_1, \dots, s_k)r(\vec{y})$ and $r(\vec{x})$ can be obtained from $r(\vec{y})$ by the replacement of each s_i which actually occurs in $r(\vec{y})$ by a placed variable.

Proof. To keep the notation simple we assume that there is only one basic set and we consider the special case where

$$w_2 \equiv r(x, u, x, y, v), \quad u,\ v \text{ placed},\ x,\ y \text{ not placed}. \tag{68}$$

Case 1. $r \not\equiv p$. Choose disjoint, non-empty sets $D_{\rm x}$, $D_{\rm y}$, $D_{\rm u}$, $D_{\rm v}$, W such that $D_{\rm x}$, $D_{\rm y}$ and W have at least two points, choose some $\bar{c}$ outside all of them, and first set all variables other than x, y, u, v to $\bar{c}$ and all partial function variables other than r, p to constant functions with value $\bar{c}$. Next set u and v to constant values $\bar{u}$, $\bar{v}$ in the corresponding sets $D_{\rm u}$, $D_{\rm v}$. For each arbitrary partial function

$$\omega : D_{\rm x} \times D_{\rm x} \times D_{\rm y} \rightharpoonup W,$$

set r by the conditions

$$[s_1 \notin D_{\rm x} \vee s_2 \notin D_{\rm u} \vee s_3 \notin D_{\rm x} \vee s_4 \notin D_{\rm y} \vee s_5 \notin D_{\rm v}] \implies r(s_1, s_2, s_3, s_4, s_5) \simeq \bar{c},$$
$$[s_1 \in D_{\rm x}\ \&\ s_2 \in D_{\rm u}\ \&\ s_3 \in D_{\rm x}\ \&\ s_4 \in D_{\rm y}\ \&\ s_5 \in D_{\rm v}] \implies r(s_1, s_2, s_3, s_4, s_5) \simeq \omega(s_1, s_3, s_4)$$

and finally set

$$\sigma(t) = \begin{cases} t, \text{ if } t \in W, \\ \bar{c}, \text{ otherwise}, \end{cases} \qquad p(s_1, s_2, \dots, s_k) \simeq \sigma(s_2). \tag{69}$$

[17] I am grateful to Joan Moschovakis for a counterexample which killed a plausible simplification of this proof, before I invested too much time in it.

Consider the result of further substituting in (67) arbitrary values $x \in D_{\mathrm{x}}$, $y \in D_{\mathrm{y}}$. Suppose $w_j \equiv q(\vec{t})$ is one of the terms within p on the right. If $q \not\equiv r$, then with these substitutions w_j is defined, set either to $\bar{c}$ or to $\sigma(t_2)$, if $q \equiv p$. If $q \equiv r$ and the sequence of variables $\vec{t}$ is not *exactly* x, u, x, y, v, then w_j again takes the value $\bar{c}$. Thus the only term among the w_j's whose value may possibly depend on ω, x and y is w_2 (which may of course occur more than once) and hence the right-hand-side of (67) is defined exactly when w_2 is defined and we have a valid identity:

$$f(z_1(\omega, x, y), \ldots, z_m(\omega, x, y)) \simeq \omega(x, x, y), \quad (x \in D_{\mathrm{x}},\ y \in D_{\mathrm{y}},\ \omega : D_{\mathrm{x}} \times D_{\mathrm{x}} \times D_{\mathrm{y}} \rightharpoonup W). \tag{70}$$

The typical expression $z_i(\omega, x, y)$ on the left evaluates to the constant $\bar{c}$ or some function with the constant value $\bar{c}$ if neither r nor p occurs in z_i. If $z_i \equiv \lambda(\vec{s})p(\vec{t})$, then again z_i has a value independent of ω, x, y, because of the definition of p and the fact we set no variable equal to a member of W. Finally, if

$$z_i \equiv \lambda(\bar{s})r(t_1, t_2, t_3, t_4, t_5), \tag{71}$$

but some t_i is free or a constant and is not the i'th variable or constant in the pattern $x, \bar{u}, x, y, \bar{v}$, then again the expression evaluates to $\bar{c}$, by the definition of r. Thus z_i depends on ω, x, y only when at most t_1, t_3 or t_4 are free, and those among them which are free are set to the corresponding value x, x or y. If all three are free in some such z_i, then the lemma clearly holds. In the opposite case the partial function ω satisfies an identity of the form

$$\omega(x, x, y) \simeq h(\omega, \omega(\cdot, x, y), \omega(x, \cdot, y), \omega(x, x, \cdot), \omega(\cdot, \cdot, y), \omega(\cdot, x, \cdot), \omega(x, x, \cdot)), \tag{72}$$

where h is a monotone operation on partial functions and $\cdot$ is the algebraic notation for λ-abstraction, e.g.,

$$\omega(\cdot, x, \cdot) = \lambda(s, t)\omega(s, x, t).$$

For example, suppose

$$z_i \equiv \lambda(st)r(x, \bar{u}, s, t, \bar{v}) = \beta;$$

then

$$\beta(s, t) \simeq \begin{cases} \omega(x, s, t), \text{ if } s \in D_{\mathrm{x}},\ t \in D_{\mathrm{y}}, \\ \bar{c}, \text{ otherwise,} \end{cases}$$

so that $z_i = h_i(\omega(x, \cdot, \cdot))$ with a monotone h_i. A similar evaluation of z_i in terms of some *section* of ω is involved in each of the cases and the substitution of all these monotone h_i's into f yields a monotone operation.

Finally, we obtain a contradiction from the alleged validity of (72). Choose distinct points x_0, x_1, y_0, y_1 in the respective sets D_{x}, D_{y} and define two partial functions with only the indicated values, where $0, 1$ are distinct points in W.

$$\alpha(x_0, x_0, y_0) \simeq 0, \qquad \gamma(x, x', y) \simeq \begin{cases} 0, \text{ if } x = x_0 \vee x' = x_0 \vee y = y_0, \\ 1, \text{ otherwise.} \end{cases}$$

From (72) applied to α,

$$h(\alpha, \alpha(\cdot, x_0, y_0),\ \ldots,\ ) \simeq \alpha(x_0, x_0, y_0) \simeq 0. \tag{73}$$

But obviously $\alpha \subseteq \gamma$ and an easy computation shows that every section of γ at (x_1, x_1, y_1) extends the corresponding section of α at (x_0, x_0, y_0), for example

$$\lambda(s)\alpha(x_0, s, y_0) \subseteq \lambda(s)\gamma(x_1, s, y_1),$$

simply because $\gamma(x_1, x_0, y_1) \simeq 0$. Thus by the monotonicity of h, (73) and (72) applied to γ, we should have

$$0 \simeq h(\gamma, \gamma(\cdot, x_1, y_1),\ \ldots,\ ) \simeq \gamma(x_1, x_1, y_1),$$

while by its definition $\gamma(x_1, x_1, y_1) \simeq 1$. This completes the proof of the Lemma in the first case.

CASE 2. $r \equiv p$. We consider again a typical, simple case

$$f(z_1, \dots, z_m) \simeq p(w_1, p(x, y, u), w_2), \ u \text{ placed}, \ x, y \text{ not placed}.$$

As before, we restrict the variables to disjoint sets D_{x}, D_{y}, D_{u}, W and a $\bar{c}$ outside all of them, such that D_{x}, D_{y} and W have at least two points, we let

$$\omega : D_{\mathrm{x}} \times D_{\mathrm{y}} \rightharpoonup W$$

be some arbitrary partial function, and we set:

$$p(s_1, s_2, s_3) \simeq \begin{cases} \omega(s_1, s_2), \text{ if } s_1 \in D_{\mathrm{x}}, \ s_2 \in D_{\mathrm{y}}, \ s_3 \in D_{\mathrm{u}}, \\ s_2, \text{ otherwise, if } s_2 \in W, \\ \bar{c}, \text{ otherwise.} \end{cases}$$

From this it follows that we get a valid identity of the form (72) for an arbitrary $\omega : D_{\mathrm{x}} \times D_{\mathrm{y}} \rightharpoonup W$, the main points being that all the terms on the right which are not identical with w_2 are defined and only the sections show up on the left, and then the proof is finished as before. ⊣

4.6. **Lemma.** An identity of the form

$$f(z_1, \dots, z_m) \simeq p(w_1, \dots, w_k) \tag{74}$$

cannot be valid in a structure $\mathbf{A}$ with infinite basic sets if the number n of distinct terms (not variables) on the right is greater than a fixed number d, which depends only on the type of f; if $n \leq d$, then we can compute from (74) an equivalent identity of the form

$$f(z_1, \dots, z_m) \simeq ev^n(W_0, W_1, \dots, W_n). \tag{75}$$

Proof. If (74) is valid, then by the preceding Lemma **4.5**, each w_i which is a term either is identical with some z_j or can be obtained by the substitution of placed variables in some z_j. If there are $q \leq m$ placed variables, and if z_j is a λ-term, it is of the form $\lambda(s_1, \dots, s_{l(j)})z_j^*$, where the number $l(j)$ can be computed from the type of f, so it can generate by substitution of placed variables into its bound variables at most $q^{l(j)}$ distinct terms; hence the total number of distinct terms on the right cannot exceed

$$d = \sum_{j=1}^{m} q^{l(j)}. \tag{76}$$

Suppose the right-hand-side of (74) is $p(x, A, u, B, A, z)$, where distinct caps indicate distinct terms and the lower case letters are variables. We then have

$$\begin{aligned} p(x, A, u, B, A, x) &\simeq (\lambda(a, b)p(x, a, u, b, a, x))(A, B) \\ &\simeq ev^2(\lambda(a, b)p(x, a, u, b, a, x), A, B). \end{aligned}$$

The general case is similar. ⊣

This last lemma reduces the decision problem of intensional identity to equations in form (63), where there is a finite choice of f's, the functionals in the signature, and a finite choice of g's, those in the structure and the ev^k's, for k no more than d computed by (76) for every functional in the structure.

4.7. **Extended sets and assignments.** Before describing the procedure which determines the validity of identities in form (63), we consider a simple example which illustrates one of the

annoying subtleties we will need to deal with. Suppose g is a total, unary function on some set A and we define the total, binary function f by

$$f(x,y) \simeq g(x). \tag{77}$$

Clearly (77) is a valid identity involving the old g and the new f we just defined. Suppose we substitute a term in this to get

$$f(x,p(x)) \simeq g(x); \tag{78}$$

now this is not valid, because for some values of the partial function p, $p(x)$ will not be defined, so neither will $f(x,p(x))$, while the right-hand-side is defined. In dealing with partial functions and functionals as we have, *validity of identities is not preserved by substitution of terms.* One way to deal with this problem is to add an element $\perp$ to each basic set and view partial functions as total functions, which take the value $\perp$ when they should be undefined. For each set A, we set

$$A^{\perp} = A \cup \{\perp\} = \text{ the extension of } A. \tag{79}$$

We can now try to interpret identities by allowing the basic variables to range over the extended sets, so that the validity of (77) implies the validity of (78); this is fine, except that now (77) *fails for the f we originally defined*, because we still have $f(x,\perp) = \perp \neq g(x)$, when $x \in A$. Of course, some might say we defined the wrong f, but in fact it is these "strict" identities we need to decide to settle the question of intensional identity. In practice we will need to work both with strict and with extended identities and we must keep the context clear.

We will use "=" to denote equality in the extended basic sets and set

$$x\!\downarrow \iff x \in A \iff x \neq \perp, \quad (x \in A^{\perp}). \tag{80}$$

A *strict assignment* π in a structure $\mathbf{A}$ assigns partial functions to pf variables and members of the basic sets to basic variables, as usual. An *extended assignment* behaves exactly like a strict assignment on pf variables, but assigns members of the extended basic sets to the basic variables, i.e., it can set $v := \perp$. An identity is *strictly valid* when it holds for all strict assignments, and *extendedly valid* if it holds for all extended assignments.

4.8. **Dictionary lines.** Choose once and for all fixed, *special variables* $x_1, \ldots, x_l$ of types such that

$$f(x_1, \ldots, x_m) \simeq g(x_{m+1}, \ldots, x_l) \tag{81}$$

is well formed. A *dictionary line* for f and g is an implication of the form

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n \Longrightarrow f(x_1, \ldots, x_m) \simeq g(x_{m+1}, \ldots, x_l) \tag{82}$$

where each formula ϕ_k in the *antecedent of the line* may involve additional *extra, variables* other than the $x_1, \ldots, x_l$ and satisfies one of the following conditions.

1. $\phi_k \equiv x_i = u$, where x_i is one of the special, basic variables and u is an extra basic variable. At most one formula of this type in the antecedent involves each x_i.
2. ϕ_k is $\lambda(u_1, \ldots, u_t)x_i(u_1, \ldots, u_t) = \lambda(u_1, \ldots, u_t)y(u_{s_1}, \ldots, u_{s_n})$, where $t = \text{arity}(x_i)$, y is an extra pf variable of arity n, and $u_{s_1}, \ldots, u_{s_n}$ is a subsequence of the sequence of variables $u_1, \ldots, u_t$. We allow $n = 0$, in which case y is an extra, nullary pf variable. At most one formula of this type in the antecedent involves the variable x_i.
3. ϕ_k is $\lambda(\vec{s})x_i(\vec{u}) = \lambda(\vec{s})x_j(\vec{v})$ or $x_i(\vec{u}) = x_j(\vec{v})$, where the length of the sequence of distinct variables $\vec{s}$ is no larger than $\max(\text{arity}(x_i), \text{arity}(x_j))$ and $\vec{u}, \vec{v}$ are sequences of extra, basic variables. At most one formula of this type in the antecedent involves each pair x_i and x_j.

4. ϕ_k is $u\!\downarrow$ or $u \neq v$, where u and v are basic, extra variables which occur free in the line in formulas of type 1 or 3.

4.9. **Dictionaries.** A line is valid (in the given structure) if every extended assignment which satisfies its hypothesis also satisfies its conclusion. This means that the choice of specific extra variables is irrelevant to the validity of a line, and then a simple counting argument shows that the types of f and g determine an upper bound on the number of distinct (up to alphabetic change and reordering of hypotheses) lines. We fix a sufficiently large set of extra variables and list once and for all, all the lines in these variables which are valid for f and g in the given structure; this is *the dictionary for f and g*.

The dictionary of the structure $\mathbf{A}$ is the union of the dictionaries for all the pairs of functionals in $\mathbf{A}$. It is a finite list of lines, perhaps not easy to construct for specific structures with non-constructive givens, but in principle it can be written down.

We will associate (effectively) with each identity (63) a specific set of lines L such that the *strict validity* of (63) is equivalent to the *extended validity* of all the lines in L. It will be convenient to express these lines using the variables which occur in (63). To decide a specific (63), we translate the lines of L into equivalent lines in the fixed, chosen variables by an alphabetic change, and then (63) will be equivalent to the presence of these lines in the dictionary.

For example, (77) will be expressed by the two lines

$$x_1 = u, x_2 = v, x_3 = u,\ u\!\downarrow,\ v\!\downarrow,\ u \neq v \Longrightarrow f(x_1, x_2) = g(x_3)$$
$$x_1 = u, x_2 = u, x_3 = u, u\!\downarrow \Longrightarrow f(x_1, x_2) = g(x_3)$$

which are valid, while for (78) we will get the single line

$$x_1 = u, x_2 = v, x_3 = u,\ u\!\downarrow \Longrightarrow f(x_1, x_2) = g(x_3),$$

which is not—because it fails when we set $x_2 := v := \bot$.

4.10. **Bound variable unifiers.** A (bounded variable) *unifier* is a triple

$$\boldsymbol{u} = (\tau, \sigma, (s_1, \dots, s_l)),$$

where $\tau, \sigma : \{\text{variables}\} \to \{\text{variables}\}$ are variable transformations which respect types and $(s_1, \dots, s_l)$ is a (possible empty) sequence of variables, the *bound variables* of $\boldsymbol{u}$. We set

$$S = S(\boldsymbol{u}) = \{s_1, \dots, s_l\},$$

and sometimes we write $S(\boldsymbol{u})$ for $(s_1, \dots, s_l)$. Two unifiers $(\tau, \sigma, (s_1, \dots, s_l))$ and $(\tau', \sigma, (s'_1, \dots, s'_l))$ are *isomorphic* if they have the same number of bound variables and there is a bijection

$$\pi : S_{\boldsymbol{u}} \rightarrowtail\!\!\!\!\!\rightarrow S_{\boldsymbol{u}'}$$

such that $\tau' x \equiv \pi(\tau x), \sigma' x \equiv \pi(\sigma x)$, where it is assumed that $\pi x \equiv x$ if $x \notin S(\boldsymbol{u})$.

A unifier $\boldsymbol{u}$ *unifies* a pair (E, F) of immediate λ-expressions

$$E \equiv \lambda(u_1, \dots, u_m)A, \quad F \equiv \lambda(v_1, \dots, v_n)B, \tag{83}$$

if the following conditions hold:

(1) If $\tau x \not\equiv x$, then $x \equiv u_i$ for some i such that u_i occurs in A; and if $\sigma y \not\equiv y$, then $y \equiv v_j$ for some j such that v_j occurs in B.
(2) $s_1, \dots, s_l$ are variables which do not occur in A or B but do occur in both $\tau[A]$ and $\sigma[B]$.
(3) The substitutions unify the terms, i.e.,

$$\lambda(s_1, \dots, s_l)\tau[A] \equiv \lambda(s_1, \dots, s_l)\sigma[B]. \tag{84}$$

A pair of immediate λ-terms (E, F) is *unifiable* there is a bounded variable unifier for it.

For example, we can unify

$$\lambda(u, u')r(u, u', a), \quad \lambda(v)r(v, b, a)$$

by setting

$$\tau u \equiv \tau u' \equiv b, \quad \sigma v \equiv b, \quad \vec{s} = \emptyset,$$

which identifies both expressions with $\lambda(\,)r(b, b, a)$. It is obvious that this is not the best we can do, though, since we can also set

$$\tau' u \equiv s, \ \tau' u' \equiv b, \quad \sigma' v \equiv s, \quad \vec{s} = (s),$$

which unifies the terms "further" to $\lambda(s)r(s, b, a)$. The next definition and lemma capture this simple idea of the existence of a unique such "maximal" unifier, when one exists at all.

4.11. **Reducibility among unifiers**. Suppose $\boldsymbol{u} = (\tau, \sigma, \vec{s})$ and $\boldsymbol{u}' = (\tau', \sigma', \vec{s}')$ both unify (E, F) as in (83). A *reduction* of $\boldsymbol{u}$ to $\boldsymbol{u}'$ is any variable transformation π such that the following three conditions hold:

(i) If $\pi(x) \not\equiv x$, then $x \in S(\boldsymbol{u}')$.
(ii) For every u_i which occurs in A,
  (iia) $\tau(u_i) \equiv \pi\tau'(u_i)$.
  (iib) If $\tau(u_i) \in S(\boldsymbol{u})$, then $\tau'(u_i) \in S(\boldsymbol{u}')$.
(iii) For every v_j which occurs in B,
  (iiia) $\sigma(v_j) \equiv \pi\sigma'(v_j)$.
  (iiib) If $\sigma(v_j) \in S(\boldsymbol{u})$, then $\sigma'(v_j) \in S(\boldsymbol{u}')$.

If such a reduction π exists, we say that $\boldsymbol{u}$ *reduces to* $\boldsymbol{u}'$ and we write $\boldsymbol{u} \leq_\pi \boldsymbol{u}'$ or (simpler) $\boldsymbol{u} \leq \boldsymbol{u}'$.

Notice that for unifiers of a pair (E, F):

$$\text{if } \boldsymbol{u} \leq_\pi \boldsymbol{u}', \text{then } S(\boldsymbol{u}) \subseteq \pi[S(\boldsymbol{u}')], \text{ and hence } |S(\boldsymbol{u})| \leq |S(\boldsymbol{u}')|; \tag{84.1}$$

this is because each $s \in S(\boldsymbol{u})$ does not occur in A but occurs in $\tau[A]$, and so $s \equiv \tau(u_i)$ for some u_i which occurs in A; but then $\tau'(u_i) \in S(\boldsymbol{u}')$ by (iib) and $s \equiv \tau(u_i) \equiv \pi(\tau'(u_i))$, so $s \in \pi[S(\boldsymbol{u}')]$.

Notice also that if $\boldsymbol{u} \leq_\pi \boldsymbol{u}'$, u_i occurs in A and $\tau'(u_i) \notin S(\boldsymbol{u}')$, then $\tau(u_i) \equiv \tau'(u_i)$; this is because of clauses (iia) and (i), by which $\tau(u_i) \equiv \pi(\tau'(u_i)) \equiv \tau'(u_i)$.

Finally, for any two unifiers $\boldsymbol{u}, \boldsymbol{u}'$ of (E, F),

$$\text{if } \boldsymbol{u} \leq \boldsymbol{u}' \text{ and } \boldsymbol{u}' \leq \boldsymbol{u}, \text{then } \boldsymbol{u} \text{ and } \boldsymbol{u}' \text{ are isomorphic.} \tag{84.2}$$

This is because if the given reductions are π and π', then from (84.1) we get that $|S_1| = |S_2|$, and so the restriction $\pi^* = \pi' \restriction S_1$ is a surjection of S_1 onto S_2, which has the same size—and hence π^* a bijection, by the Pigeonhole Theorem; and then the definition of reduction yields immediately that π' is an isomorphism of $\boldsymbol{u}$ with $\boldsymbol{u}'$.

Consider, for example the two unifiers $\boldsymbol{u} = (\tau, \sigma, \emptyset)$ and $\boldsymbol{u}' = (\tau', \sigma', s)$ for $\lambda(u, u')r(u, u', a)$ and $\lambda(v)r(v, b, a)$ defined above; we can check easily that $\boldsymbol{u} \leq_\pi \boldsymbol{u}'$, with

$$\pi(s) \equiv b, \quad x \not\equiv s \Longrightarrow \pi(x) \equiv x.$$

Or take two alphabetic variants of the same expression

$$E \equiv \lambda(u_1, u_2)r(u_1, u_2), \quad F \equiv \lambda(v_1, v_2)r(v_1, v_2)$$

and the two unifiers

$$\tau_1(u_1) \equiv \sigma_1(v_1) \equiv x_1, \quad \tau_1(u_2) \equiv \sigma_1(v_2) \equiv x_2 \tag{$\boldsymbol{u}_1$}$$

$$\tau_2(u_1) \equiv \sigma_2(u_1) = \tau_2(u_2) \equiv \sigma_2(u_2) \equiv s, \tag{$\boldsymbol{u}_2$}$$

which unify (E, F) to $\lambda(x_1, x_2)r(x_1, x_2)$ and $\lambda(s)r(s, s)$ respectively. It is easy to check that $\boldsymbol{u}_2 \leq \boldsymbol{u}_1$ but $\boldsymbol{u}_1 \not\leq \boldsymbol{u}_2$, reflecting the fact that $\boldsymbol{u}_1$ is "more general" (less restrictive) than $\boldsymbol{u}_2$.

A unifier of (E, F) is *maximal*, if there is no unifier of (E, F) with a longer sequence of bound variables.

4.12. **Lemma.** *If a pair of immediate λ-expressions*

$$E \equiv \lambda(u_1, \dots, u_m)A, \quad F \equiv \lambda(v_1, \dots, v_n)B \tag{84.3}$$

is unifiable, then it has a unique (up to isomorphism) maximal unifier $\boldsymbol{u}^* = (\tau^*, \sigma^*, S^*)$, and every unifier of (E, F) is reducible to $\boldsymbol{u}^*$.

Moreover, if u_i occurs in A, then either $\tau^*(u_i)$ also occurs in A or $\tau^*(u_i) \in S^*$; and if v_j occurs in B, then either $\sigma^*(v_j)$ also occurs in B or $\sigma^*(v_j) \in S^*$.

PROOF. Since (E, F) is unifiable, we know that

$$A \equiv r(a_1, \dots, a_k), \quad B \equiv r(b_1, \dots, b_k), \tag{84.4}$$

i.e., A and B are immediate terms involving the same pf variable. Let

$$O = \{a_1, \dots, a_k, b_1, \dots, b_k\},$$
$$U = \{u_i \mid u_i \text{ occurs in } A\},\ V = \{v_j \mid v_j \text{ occurs in } B\},\ P = O \setminus (U \cup V), \tag{84.5}$$

and assume (without loss of generality) that $U \cap V = \emptyset$. Let $\sim$ be the smallest equivalence relation on O such that

$$a_t \sim b_t \quad (t = 1, \dots, k).$$

Fact 1. *If $\boldsymbol{u} = (\tau, \sigma, S)$ unifies (E, F), then*

$$x \sim y \Longrightarrow \tau(x) \equiv \tau(y) \text{ and } \sigma(x) \equiv \sigma(y) \quad (x, y \in O).$$

This is true because the relation

$$\{(x, y) \in O \times O \mid \tau(x) \equiv \tau(y) \text{ and } \sigma(x) \equiv \sigma(y)\}$$

is an equivalence relation which includes all the pairs (a_t, b_t).

By applying Fact 1 to the assumed unifier of (E, F) and using the fact that $\tau(x) \equiv \sigma(x) \equiv x$ if $x \notin U \cup V$, we get

$$x, y \in P \Longrightarrow x \sim y \iff x \equiv y,$$

so that each equivalence class has at most one "parameter" $x \in P$ in it. Call a variable $x \in U \cup V$ *forced* if $x \sim y$ for some $y \in P$ and set (unambiguously) for forced $u_i \in U, v_j \in V$,

$$\tau^*(u_i) = \text{the unique } x \in P \text{ such that } u_i \sim x,$$
$$\sigma^*(v_j) = \text{the unique } x \in P \text{ such that } v_j \sim x. \tag{84.6}$$

Let $e_1, \dots, e_l$ be an enumeration of the remaining (not forced) equivalence classes (if there are any), let $s_1^*, \dots, s_l^*$ be distinct, fresh variables which do not occur in A or B, and set for not forced $u_i \in O, v_j \in V$,

$$u_i \in e_t \Longrightarrow \tau^*(u_i) \equiv s_t^*; \quad v_j \in e_t \Longrightarrow \sigma^*(v_j) \equiv s_t^*. \tag{84.7}$$

If $x \notin U \cup V$, we set $\tau^*(x) \equiv \sigma^*(x) \equiv x$.

Fact 2. *The triple* $\boldsymbol{u}^* = (\tau^*, \sigma^*, (s_1^*, \dots, s_l^*))$ *unifies* (E, F).

This is trivial: property (1) of unifiers is true by definition; (2) follows because each $s_m^* \equiv \tau^*(u_i)$ for some (not forced) $u_i \in e_m$, and then also $s_m^* \equiv \sigma(v_j)$ for a (necessarily not forced) v_j such that for some t, $a_t \equiv u_i, b_t \equiv v_j$; and (3) holds because for each t, $a_t \sim b_t$ and hence $\tau^*(a_t) \equiv \sigma^*(b_t)$ by the definitions.

Fact 3. *If* $\boldsymbol{u} = (\tau, \sigma, S)$ *unifies* (E, F)*, then* $\boldsymbol{u} \leq \boldsymbol{u}^*$*, and, in particular,* $\boldsymbol{u}^*$ *is a maximal unifier of* (E, F).

We define the required reduction π by

$$\pi(s) \equiv \tau(u_i) \text{ where } \tau^*(u_i) \equiv s \quad (s \in S^*),$$

and setting $\pi(x) \equiv x$ if $x \notin S^*$, as required for reductions. The definition is good: because if $\tau^*(u_j) \equiv s$ for some $j \neq i$, then $u_i \sim u_j$ and then $\tau(u_i) \equiv \tau(u_j)$ by Fact 1. Now (iia) follows immediately since $\pi(\tau^*(u_i)) \equiv \tau(u_i)$ by the definition, and also (iib): if $\tau^*(u_i) \notin S^*$, then u_i is forced, and so $\tau^*(u_i) \equiv x$ for some variable which occurs in A, and so $\tau(u_i) \equiv \tau^*(u_i) \equiv x$ also, while no variable in S occurs in A. The argument is similar for (iiia) and (iiib).

The reduction $\boldsymbol{u} \leq \boldsymbol{u}^*$ implies $|S(\boldsymbol{u})| \leq |S^*|$ by (84.1), and since $\boldsymbol{u}$ was arbitrary, there is no unifier of (E, F) with more bound variables than those of $\boldsymbol{u}^*$, i.e., $\boldsymbol{u}^*$ is maximal. And the uniqueness (up to isomorphism) of $\boldsymbol{u}^*$ follows from (84.2).

Finally, the last claim in the Lemma is immediate, since by the construction, if u_i occurs in A, then $\tau^*(u_i) \in P \cup S^*$. ⊣

The dictionary of an equation. Suppose now we are given an identity

$$f(z_1, \dots, z_m) \simeq g(z_{m+1}, \dots, z_l). \tag{63}$$

We first show how to construct a single dictionary line from it, which will determine the strict truth of (63) when all the free, basic variables in it are interpreted by distinct values in the domain. The complete set of lines for (63) will contain the lines we get in this way from all the identities which result from (63) by the identification of some of its free, basic variables.

We assume that the special variables $x_1, \dots, x_m$ we will use for the lines do not occur in the given identity.

Step 1. For each z_i which is a basic variable, we put in the antecedent of the line the equality $x_i = z_i$ and the condition $z_i\downarrow$; and for any two, distinct z_i, z_j which are introduced in the line by this process, we add the inequality $z_i \neq z_j$.

Step 2. Suppose $z_i \equiv \lambda(u_1, \dots, u_\alpha) r(a_1, \dots, a_k)$ is a genuine λ-term (i.e., $\alpha > 0$) and $u_{s_1}, \dots, u_{s_n}$ is the subsequence of $u_1, \dots, u_\alpha$ comprising all the variables of this sequence which occur in $r(a_1, \dots, a_k)$; we add to the antecedent of the line the equation

$$\lambda(u_1, \dots, u_\alpha) x_i(u_1, \dots, u_\alpha) = \lambda(u_1, \dots, u_\alpha) y_i(u_{s_1}, \dots, u_{s_n}),$$

where y_i is a fresh, extra variable of arity n. (If $n = 0$, then y_i is an extra nullary pf variable.)

Step 3. Consider any pair z_i, z_j $(i \neq j)$ of expressions which are not basic variables; we will view these as λ-expressions by identifying temporarily a term $r(\vec{x})$ with the λ-expression $\lambda(\,) r(\vec{x})$. If z_i, z_j cannot be unified, we add nothing to the line. In the opposite case, suppose

$$z_i \equiv \lambda(u_1, \dots, u_\alpha) r(a_1, \dots, a_k), \quad z_j \equiv \lambda(v_1, \dots, v_\beta) r(b_1, \dots, b_k),$$

and $(\tau, \sigma, (s_1, \dots, s_l))$ is a maximal unifier for these expressions. We add to the antecedent of the line the equation

$$\lambda(s_1, \dots, s_l)x_i(\tau(u_1), \dots, \tau(u_\alpha)) \simeq \lambda(s_1, \dots, s_l)x_j(\sigma(v_1), \dots, \sigma(v_\beta)). \tag{85}$$

To recapitulate what we said above, the complete set of lines associated with an identity is obtained by applying this procedure to every identity obtained by identifying some or all of the free basic variables of the identity.

To illustrate the procedure, consider again (77),

$$f(x, y) \simeq g(x). \tag{77}$$

There are no λ-terms, so only Step 1 comes into play and we get the following two lines:

$$x_1 = x,\ x_2 = y,\ x_3 = x,\ x\!\downarrow,\ y\!\downarrow,\ x \neq y \implies f(x_1, x_2) = g(x_3),$$
$$x_1 = x,\ x_2 = x,\ x_3 = x,\ x\!\downarrow \implies f(x_1, x_2) = g(x_3).$$

The procedure generates only one line for (78):

$$x_1 = x,\ x_3 = x,\ x\!\downarrow \implies f(x_1, x_2) = g(x_3).$$

Consider also the example

$$f(\lambda(t, s)p(x, y, t, y), q(x), x) \simeq g(p(x, y, x, y), q(y), x). \tag{86}$$

There are two free, basic variables, so we will get two lines. First from the identity as it is, the following formulas are introduced in each step:

STEP 1. $x_3 = x, x_6 = x, x\!\downarrow$.

STEP 2. $\lambda(t, s)x_1(t, s) = \lambda(t, s)y_1(t)$.

STEP 3. The most general unifier for z_1 and z_3 is given by $\tau(t) \equiv x, \tau(s) = s$ with no bound variables left, which unifies the terms to $p(x, y, x, y)$, and so the equation added to the line is $x_1(x, s) = x_4$.

So the line produced is

$$x_3 = x,\ x_6 = x,\ x\!\downarrow,\ \lambda(t, s)x_1(t, s) = \lambda(t, s)y_1(t),\ x_1(x, s) = x_4$$
$$\implies f(x_1, x_2, x_3) = g(x_4, x_5, x_6).$$

If we identify $x \equiv y$, we get the identity

$$f(\lambda(ts)p(x, x, t, t), q(x), x) \simeq g(p(x, x, x, x), q(x), x)$$

which has an additional (trivial) unification and generates the line

$$x_3 = x,\ x_6 = x,\ x\!\downarrow,\ \lambda(t, s)x_1(t, s) = \lambda(t, s)y_1(t),\ x_1(x, s) = x_4,\ x_2 = x_5$$
$$\implies f(x_1, x_2, x_3) = g(x_4,\ x_5, x_6).$$

It is not hard easy to verify directly that the extended validity of these two lines is equivalent to the strict validity of the identity.

4.13. **Lemma.** The conjunction of all the lines constructed for (63) implies (63).

Proof. We will verify that if the line produced by an identity holds, then every strict assignment which assigns distinct values to distinct basic variables satisfies the identity, from which the Lemma follows by applying it to all the substitution instances of the identity obtained by identifying basic variables.

Suppose we are given a strict assignment π to the variables of the identity. We first extend π to the special variables x_i which occur on the line by setting

$$\pi(x_i) = \pi(z_i) = \mathrm{den}(z_i, \pi); \tag{87}$$

it will be enough to show that there is a further extension of π to the special variables y_i (if any) which satisfies the antecedent of the line.

Consider how the clauses were introduced to the line by the three steps of the construction.

STEP 1. We put in $x_i = z_i$ and $z_i\!\downarrow$ if z_i is basic, and $z_i \neq z_j$ for any two distinct basic variables which occur in the equation; and π satisfies all these clauses, because it is strict and one-to-one on basic variables.

STEP 2. If clause

$$\lambda(u_1, \dots, u_\alpha)x_i(u_1, \dots, u_\alpha) = \lambda(u_1, \dots, u_\alpha)y_i(u_{s_1}, \dots, u_{s_n}),$$

is added to the antecedent of the line at this step, this is because only $u_{s_1}, \dots, u_{s_n}$ actually occur in $r(a_1, \dots, a_k)$ from the variables $u_1, \dots, u_\alpha$. This means that the partial function

$$h(u'_1 \dots, u'_\alpha) = \mathrm{den}(r(a_1, \dots, a_k), \pi\{u_1 := u'_1, \dots, u_\alpha := u'_\alpha\})$$

depends only on $u'_{s_1}, \dots, u'_{s_n}$, and so we can extend π to y_i by setting

$$\pi(y_i)(u'_{s_1}, \dots, u'_{s_n}) = \mathrm{den}(r(a_1, \dots, a_k), \pi\{u_1 := u'_1, \dots, u_\alpha := u'_\alpha\});$$

now this extension of π satisfies the equations

$$\lambda(u_1 \dots, u_\alpha)y_i(u_{s_1}, \dots, u_{s_n}) = \lambda(u_1 \dots, u_\alpha)r(a_1, \dots, a_k) = \lambda(u_1 \dots, u_\alpha)x_i(\lambda(u_1 \dots, u_\alpha),$$

and so it satisfies the new clause.

STEP 3. If the clause (85) is added to the antecedent of the line in this step, we must show that it is validated by π, and it is instructive to consider first an example. Suppose

$$z_i \equiv \lambda(u_1, u_2)r(u_1, a, b, u_2, u_1)$$
$$z_j \equiv \lambda(v_1, v_2, v_3)r(v_1, a, v_2, c, v_3).$$

The most general unifier for these terms is $(\tau, \sigma, (s))$ where

$$\tau(u_1) :\equiv s,\ \tau(u_2) :\equiv c,\ \sigma(v_1) :\equiv s,\ \sigma(v_2) :\equiv b,\ \sigma(v_3) :\equiv s,$$

and it unifies them to

$$\lambda(s)r(s, a, b, c, s);$$

so according to the recipe, the equation added to the line is

$$\lambda(s)x_i(s, c) = \lambda(s)x_j(s, b, s). \tag{88}$$

If $\pi(a) = \bar{a}, \pi(b) = \bar{b}$ and $\pi(c) = \bar{c}$, then to prove that (88) is satisfied by π we must show that for all s,

$$\pi(x_i)(s, \bar{c}) = \pi(x_j)(s, \bar{b}, s).$$

What we know is that if $\pi(r) = \bar{r}$, then for all u_1, u_2, v_1, v_2, v_3,

$$\pi(z_i)(u_1, u_2) = \bar{r}(u_1, \bar{a}, \bar{b}, u_2, u_1),$$
$$\pi(z_j)(v_1, v_2, v_3) = \bar{r}(v_1, \bar{a}, v_2, \bar{c}, v_3);$$

and with the definitions $\pi(x_i) = \pi(z_i), \pi(x_j) = \pi(z_j)$, we get from these the desired

$$\pi(x_i)(s, \bar{c}) = \bar{r}(s, \bar{a}, \bar{b}, \bar{c}, s) = \pi(x_j)(s, \bar{b}, s).$$

The general case is proved in exactly the same way, but the notation is messy. We assume that $(\tau, \sigma, \vec{s})$ is a maximal unifier of

$$z_i \equiv \lambda(u_1, \dots, u_\alpha) r(a_1, \dots, a_k) \text{ and } z_j \equiv \lambda(v_1, \dots, v_\beta) r(b_1, \dots, b_k),$$

so that

$$r(\tau a_1, \dots, \tau a_k) \equiv r(\sigma b_1, \dots, \sigma b_k), \tag{88.1}$$

where $\tau(x) \equiv x$ if x is not some u_t which occurs in $r(a_1, \dots, a_k)$ (and similarly with σ); and we must show that the assignment π defined above so that

$$\pi(x_i) = \text{den}(\lambda(u_1, \dots, u_\alpha) r(a_1, \dots, a_k), \pi),\ \pi(x_j) = \text{den}(\lambda(v_1, \dots, v_\beta) r(b_1, \dots, b_k), \pi)$$

validates the identity

$$\lambda(\vec{s}) x_i(\tau(u_1), \dots, \tau(u_\alpha) = \lambda(\vec{s}) x_j(\sigma(v_1), \dots, \sigma(v_\beta),$$

or, equivalently that every assignment π^* which agrees with π on all variables except (perhaps) on $s_1, \dots, s_l$ validates the term identity

$$x_i(\tau(u_1), \dots, \tau(u_\alpha)) \simeq x_j(\sigma(v_1), \dots, \sigma(v_\beta)). \tag{89}$$

The key to this is the so-called rule of *β-conversion*: *for any λ-term*

$$C \equiv \lambda(u_1, \dots, u_\alpha) A,$$

any variables $w_1, \dots, w_\alpha$ other than $u_1, \dots, u_\alpha$, and every strict assignment π,

$$\text{den}(C, \pi)(\pi(w_1), \dots, \pi(w_\alpha)) \simeq \text{den}(A\{u_1 :\equiv w_1, \dots, u_\alpha :\equiv w_\alpha\}, \pi),$$

where $A\{u_1 :\equiv w_1, \dots, u_\alpha :\equiv w_\alpha\}$ is constructed by (simultaneously) replacing each free occurrence of each u_i in A by w_i. Using this, we can compute, for any strict assignment π^* which agrees with π on all variables except (perhaps) $s_1, \dots, s_l$:

$$\begin{aligned}
&\text{den}(x_i(\tau(u_1), \dots, \tau(u_\alpha)), \pi^*) \\
&\quad = \text{den}(\lambda(u_1, \dots, u_\alpha) r(a_1, \dots, a_k), \pi)(\pi^*(\tau(u_1), \dots, \pi^*(\tau(u_\alpha)) \\
&\quad\quad = \text{den}(\lambda(u_1, \dots, u_\alpha) r(a_1, \dots, a_k), \pi^*)(\pi^*(\tau(u_1), \dots, \pi^*(\tau(u_\alpha)) \\
&\quad\quad\quad \text{den}(r(a_1, \dots, a_k)\{u_1 :\equiv \tau(u_1), \dots, u_\alpha :\equiv \tau(u_\alpha)\}, \pi^*) \\
&\quad\quad\quad\quad = \text{den}(r(\tau(a_1), \dots, \tau(a_k)), \pi^*),
\end{aligned}$$

where the replacement of π by π^* is justified because no s_t occurs in $\lambda(u_1, \dots, u_\alpha) r(a_1, \dots, a_k)$. The same computation for z_j gives

$$\text{den}(x_j(\sigma(v_1), \dots, \sigma(v_\beta)), \pi^*) = \text{den}(r(\sigma(b_1), \dots, \sigma(b_k), \pi^*)$$

which then with (88.1) yields the required (89). ⊣

4.14. **Lemma.** An identity implies the validity of every line it generates.

Proof. We now assume that the identity is valid and we are given an extended assignment π to the variables of the line which satisfies the antecedent: we must show that it also satisfies the consequent.

Notice that the only basic variables which occur free in the line also occur free in the identity, and π assigns distinct values other than $\perp$ to them because of the clauses we introduced in Step 1. First we extend π to the remaining free basic variables in the equation by giving a distinct,

new value other than $\perp$ to each of them. We want to extend π to all the pf variables also, so that we get

$$\pi(z_i) = \mathrm{den}(z_i, \pi) = \pi(x_i), \tag{90}$$

for every $i = 1, \dots, m$. This is already true when z_i is a basic variable, because of the equations put in the line in Step 1, and so we need worry only for the cases when z_i is a λ-term.

To explain the construction, we consider first the example in the proof of Lemma 4.13, where the following two λ-terms occur in the given equation:

$$z_i \equiv \lambda(u_1, u_2) r(u_1, a, b, u_2, u_1)$$
$$z_j \equiv \lambda(v_1, v_2, v_3) r(v_1, a, v_2, c, v_3).$$

If we set again $\bar{a} = \pi(a), \bar{b} = \pi(b), \bar{c} = \pi(c)$, then we can insure (90) for i by setting $\pi(r) = \bar{r}_i$ for any partial function $\bar{r}_i$ such that for all u_1, u_2,

$$\bar{r}_i(u_1, \bar{a}, \bar{b}, u_2, u_1) = \pi(x_i)(u_1, u_2). \tag{91}$$

Notice that there is such a partial function—in fact this equation can be considered as a definition of $\bar{r}_i$ on the set[18]

$$D_i = \{(t_1, \dots, t_5) \mid t_2 = \bar{a}, t_3 = \bar{b}, t_5 = t_1\}.$$

Moreover, if $\bar{r}_i$ is any partial function which satisfies (91) for all u_1, u_2 and we set $\pi(r) = \bar{r}_i$, then

$$\mathrm{den}(z_i, \pi) = \lambda(u_1, u_2)\bar{r}_i(u_1, \bar{a}, \bar{b}, u_2, u_1) = \pi(x_i).$$

Similarly, for j, there is some $\bar{r}_j$ defined on

$$D_j = \{(t_1, \dots, t_5) \mid t_2 = \bar{a}, t_4 = \bar{c}\}$$

so that for all v_1, v_2, v_3,

$$\bar{r}_j(v_1, \bar{a}, v_2, \bar{c}, v_3) = \pi(x_j)(v_1, v_2, v_3); \tag{92}$$

and if $\bar{r}_j$ is any partial function which satisfies (92) for all v_1, v_2, v_3 and we set $\pi(r) = \bar{r}_j$, then

$$\mathrm{den}(z_j, \pi) = \lambda(v_1, v_2, v_3)\bar{r}_j(v_1, \bar{a}, v_2, \bar{c}, v_3) = \pi(x_j).$$

Thus, it is enough to prove that the partial functions $\bar{r}_i$ and $\bar{r}_j$ determined by (91) and (92) are *compatible*, i.e.,

$$\vec{t} \in D_i \cap D_j \Longrightarrow \bar{r}(\vec{t}) = \bar{r}_j(\vec{t}),$$

since we can then set

$$\pi(r) = \bar{r}_i \cup \bar{r}_j,$$

so that $\bar{r}_i = \pi(r) \restriction D_i, \bar{r}_j = \pi(r) \restriction D_j$, and the required $\mathrm{den}(z_i, \pi) = \pi(x_i), \mathrm{den}(z_j, \pi) = \pi(x_j)$ follow immediately.

Suppose, towards a contradiction, that $\bar{r}_i$ and $\bar{r}_j$ are not compatible, so that there is a tuple $\vec{t}$ such that

$$(t_1, t_2, t_3, t_4, t_5) \in D_i \cap D_j$$

for which $\bar{r}_i(\vec{t}) \neq \bar{r}_j(\vec{t})$. By the definition of the sets D_i, D_j, we must have

$$t_2 = \bar{a}, t_3 = \bar{b}, t_5 = t_1, t_2 = \bar{a}, t_4 = \bar{c},$$

so that, in fact,

$$\vec{t} = (\bar{s}, \bar{a}, \bar{b}, \bar{c}, \bar{s})$$

[18]Notice that this is not the domain of convergence of $\bar{r}_i$, which may diverge on some of the tuples in it.

for some $\bar{s}$, and

$$\pi(x_i)(\bar{s},\bar{a}) \neq \pi(x_j)(\bar{s},\bar{b},\bar{s}). \tag{93}$$

Let s be a fresh variable and set

$$\tau(u_1) \equiv s, \tau(u_2) \equiv c, \quad \sigma(v_1) \equiv s, \sigma(v_2) \equiv b, \sigma(v_3) \equiv s$$

with $\tau(x) \equiv x$ if x is not u_1 or u_2 (and similarly for σ), and compute:

$$r(\tau(u_1),\tau(a),\tau(b),\tau(u_2),\tau(u_1)) \equiv r(s,a,b,c,s) \equiv r(\sigma(v_1),\sigma(a),\sigma(v_2),\sigma(c),\sigma(v_3)).$$

Thus the triple $(\tau,\sigma,(s))$ is a unifier for z_i and z_j; in fact (easily, in this example) it is a maximal unifier, and so in Step 3 of constructing the line for this equation we added the condition

$$\lambda(s)x_i(\tau(u_1),\tau(u_2)) = \lambda(s)x_j(\sigma(v_1),\sigma(v_2),\sigma(v_3)),$$

which after the substitutions becomes

$$\lambda(s)x_i(s,c) = \lambda(s)x_j(s,b,s);$$

so π satisfies this equation, and in particular

$$\pi(x_i)(\bar{s},\bar{c}) = \pi(x_j)(\bar{s},\bar{b},\bar{s}),$$

which contradicts (93).

Here too, the proof for the general case is based on the same idea but is substantially more complex.

To effect (90) for $z_i \equiv \lambda(u_1,\dots,u_\alpha)r(a_1,\dots,a_k)$, set for any sequence $\vec{u}' = (u'_1,\dots,u'_\alpha)$,

$$\bar{a}_t(\vec{u}') = \begin{cases} \pi(a_t), & \text{if } a_t \notin \{u_1,\dots,u_\alpha\}, \\ u'_m, & \text{if } a_t \equiv u_m, \end{cases} \tag{93.1}$$

$$D_i = \{(w_1,\dots,w_k) \mid (\text{for } t = 1,\dots,k)[a_t \in P \Longrightarrow w_t = \pi(a_t)] \\ \& \ (\text{for all } t,s)[a_t \equiv a_s \Longrightarrow w_t = w_s]\}, \tag{93.2}$$

We want to define $\bar{r}_i$ on D_i so that for all $\vec{u}'$,

$$\bar{r}_i(\bar{a}_1(\vec{u}'),\dots,\bar{a}_k(\vec{u}')) = \pi(x_i)(u'_1,\dots,u'_\alpha). \tag{93.3}$$

This is not obviously possible:[19] for example, if $z_i \equiv \lambda(u_1,u_2)r(u_1)$, then (93.3) becomes

$$\bar{r}_i(u'_1) = \pi(x_i)(u'_1,u'_2) \quad (\text{all } u'_1,u'_2)$$

which evidently cannot be satisfied by any $\bar{r}_i$ if $\pi(x_i)$ depends on both of its arguments. However, it does not: because u_2 does not occur in $r(u_1)$, and so in Step 2 of the construction of the line we introduced the clause

$$\lambda(u_1,u_2)x_i(u_1,u_2) = \lambda(u_1,u_2)y_i(u_1)$$

which must be satisfied by π, and insures that $\pi(x_i)$ does not depend on its second argument. This is the key to showing that, in general, we can define $\bar{r}_i$ so that (93.3) holds, as follows.

$$\bar{r}_i(w_1,\dots,w_k) \simeq w \iff (w_1,\dots,w_k) \in D_i \\ \& \ \exists(\vec{u}' = (u'_1,\dots,u'_\alpha))[\bar{a}_1(\vec{u}') = w_1,\dots,\bar{a}_k(\vec{u}') = w_k) \ \& \ \pi(x_i)(\vec{u}') \simeq w]. \tag{93.4}$$

This is a good definition of a partial function: because if

$$\bar{a}_t(\vec{u}') = \bar{a}_t(\vec{u}'') \quad (t = 1,\dots,k) \text{ and } u_m \text{ occurs in } r(a_1,\dots,a_k),$$

[19] This is the gap in the proof of Theorem 4.1 in the paper, where this possibility was overlooked.

then for some t, $u_m \equiv a_t$ and $u'_m = \bar{a}_t(\vec{u}') = \bar{a}_t(\vec{u}'') = u''_m$; thus $u'_m = u''_m$ for all m such that u_m occurs in $r(a_1, \dots, r_k)$, and so $\pi(x_i)(\vec{u}') \simeq \pi(x_i)(\vec{u}'')$ because of the clause inserted in the line at Step 2 of its construction.

Fact 1. *If we set $\pi(r)$ so that for all $\vec{w} \in D_i$, $\pi(r)(\vec{w}) = \bar{r}_i(\vec{w})$, then $\pi(z_i) = \pi(x_i)$.*

PROOF. We need to show that for every assignment π^* which agrees with π except (perhaps) on $u_1, \dots, u_\alpha$,

$$\pi(r)(\pi^*(a_1), \dots, \pi^*(a_k)) = \pi(x_i)(\pi^*(u_1), \dots, \pi^*(u_\alpha)).$$

Let $u'_m = \pi^*(u_m)$ (for $m = 1, \dots, \alpha$) for such a π^*, and notice first that

$$\bar{a}_t(\vec{u}') = \pi^*(a_t) \quad (t = 1, \dots, k);$$

this is because if $a_t \in P$, then $\bar{a}_t(\vec{u}') = \pi(a_t) = \pi^*(a_t)$, and if $a_t \equiv u_m$, then $\bar{a}_t(\vec{u}') = u'_m = \pi^*(u_m) = \pi^*(a_t)$. As a consequence, easily, $(\bar{a}_1(\vec{u}'), \dots, \bar{a}_k(\vec{u}')) \in D_i$, and so it is enough to show that for all $\vec{u}'$, $\bar{r}_i(\bar{a}_1(\vec{u}'), \dots, \bar{a}_k(\vec{u}')) = \pi(x_i)(\vec{u}')$—which, however, is now immediate from the definition of $\bar{r}_i$. ⊣ (Fact 1)

Thus we can extend the given π to all the pf variables which occur in the equation so that $\pi(z_i) = \pi(x_i)$ for all i, provided that $\bar{r}_i$ and $\bar{r}_j$ are not incompatible whenever z_i and z_j have the same head pf variable, which we proceed to check next.

For the next two Facts, we assume that

$$z_i \equiv (\lambda \vec{u}) r(a_1, \dots, a_k), \quad z_j \equiv \lambda(\vec{v}) r(b_1, \dots, b_k)$$

are among the immediate λ-expressions in the identity with the same head pf variable r.

Fact 2. *If there are tuples $\vec{u}' = (u'_1, \dots, u'_\alpha)$, $\vec{v}'_1 = (v'_1, \dots, v'_\beta)$ such that*

$$\bar{a}_1(\vec{u}') = \bar{b}_1(\vec{v}'), \dots, \bar{a}_k(\vec{u}') = \bar{b}_k(\vec{v}'), \tag{93.5}$$

then z_i and z_j are unifiable.

PROOF. Let $S = \{\bar{w}_1, \dots, \bar{w}_l\}$ be an enumeration of the set

$$\{u'_1, \dots, u'_\alpha, v_1, \dots, v'_\beta\} \setminus \{\pi(x) \mid x \in P\},$$

where $P = \{a_1, \dots, a_k, b_1, \dots, b_k\} \setminus \{u_1, \dots, u_\alpha, v_1, \dots, v_\beta\}$ as in (84.5). It may be, of course, that $S = \emptyset$. Choose distinct variables $s_1, \dots, s_l$ which do no occur in z_i or z_j and define the following two variable transformations:

$$\tau(u_m) = \begin{cases} u'_m, & \text{if } u_m \text{ does not occur in } r(a_1, \dots, a_k), \\ x, & \text{if } u'_m = \pi(x) \text{ for some } x \in P, \\ s_t, & \text{otherwise, if } u'_m = \bar{w}_t. \end{cases}$$

$$\sigma(v_m) = \begin{cases} v'_m, & \text{if } v_m \text{ does not occur in } r(b_1, \dots, b_k), \\ x, & \text{if } v'_m = \pi(x) \text{ for some } x \in P, \\ s_t, & \text{otherwise, if } v'_m = \bar{w}_t. \end{cases}$$

The definition is good because π assigns distinct values to distinct variables: so if, for example, $u'_m = \pi(x) = \pi(y)$ with $x, y \in P$, then $x \equiv y$. We also set $\tau(x) \equiv \sigma(x) \equiv x$ if $x \notin \{u_1, \dots, u_\alpha, v_1, \dots, v_\beta\}$, as usual, and we claim that $\boldsymbol{u} = (\tau, \sigma, (s_1, \dots, s_l))$ unifies (E, F). The only not obvious condition we need to verify is that

$$\tau(a_\gamma) \equiv \sigma(b_\gamma) \quad (\gamma = 1, \dots, k).$$

We consider cases.

CASE 1, $a_\gamma, b_\gamma \in P$, so $\tau(a_\gamma) \equiv a_\gamma, \sigma(b_\gamma) \equiv b_\gamma$. In this case

$$\bar{a}_\gamma(\vec{u}') = \pi(a_\gamma), \quad \bar{b}_\gamma(\vec{v}') = \pi(b_\gamma),$$

and so the hypothesis gives us $\pi(a_\gamma) = \pi(b_\gamma)$, which implies that $a_\gamma \equiv b_\gamma$ since π assigns distinct values o distinct variables.

CASE 2, $a_\gamma \in P, b_\gamma \equiv v_m$ for some m, so that $\tau(a_\gamma) \equiv a_\gamma$ and

$$\bar{a}_\gamma(\vec{u}') = \pi(a_\gamma), \quad \bar{b}_\gamma(\vec{v}') = v'_m;$$

now the hypothesis gives us that $\pi(a_\gamma) = v'_m$, and so $\sigma(v_m) \equiv a_\gamma \equiv \tau(a_\gamma)$.

The symmetric CASE 3 is handled similarly, and this leaves only

CASE 4, $a_\gamma \equiv u_m, b_\gamma \equiv v_n$, for suitable m, n. Now

$$\bar{a}_\gamma(\vec{u}') = u'_m, \quad \bar{b}_\gamma(\vec{v}') = v'_n,$$

and so the hypothesis gives us $u'_m = v'_n$. If $u'_m = \pi(x)$ for some variable $x \in P$, then $\tau(u_m) \equiv \sigma(v_n) \equiv x$. The alternative is that $u'_m = v'_n = \bar{w}_\gamma$ for some $\bar{w}_\gamma$ which is not the value assigned by π to any parameter $x \in P$, and then $\tau(u_m) \equiv \sigma(v_n) \equiv s_\gamma$. $\dashv$ (Fact 2)

Fact 3. *If* $\boldsymbol{u} = (\tau, \sigma, (s_1, \dots, s_l))$ *is any unifier of* (z_i, z_j), $\boldsymbol{u}^* = (\tau^*, \sigma^*, t_1, \dots, t_m))$ *is a maximal unifier of* (z_i, z_j), *and* π *satisfies the equation*

$$\lambda(t_1, \dots, t_m) x_i(\tau^*(u_1), \dots, \tau^*(u_\alpha)) = \lambda(t_1, \dots, _m) x_j(\sigma^*(v_1), \dots, \sigma^*(v_\beta)),$$

then π *also satisfies the equation*

$$\lambda(s_1, \dots, s_l) x_i(\tau(u_1), \dots, \tau(u_\alpha)) = \lambda(s_1, \dots, s_l) x_j(\sigma(v_1), \dots, \sigma(v_\beta)).$$

PROOF. Let $T^* = \{t_1, \dots, t_m\}, S = \{s_1, \dots, s_l\}$, as usual. The hypothesis means that every assignment π_h which differs from π (at most) on T^* satisfies the equation

$$x_i(\tau^*(u_1), \dots, \tau^*(u_\alpha)) \simeq x_j(\sigma^*(v_1), \dots, \sigma^*(v_\beta)); \tag{93.6}$$

and we must show that every assignment π_c which differs from π (at most) on S satisfies the equation

$$x_i(\tau(u_1), \dots, \tau(u_\alpha)) \simeq x_j(\sigma(v_1), \dots, \sigma(v_\beta)). \tag{93.7}$$

So fix such an assignment π_c, and (by appealing to Lemma 4.12), fix also a reduction ρ of $\boldsymbol{u}$ to $\boldsymbol{u}^*$. Let

$$\pi_h(x) \equiv \begin{cases} \pi(x), & \text{if } x \notin T^*, \\ \pi_c(\rho(x)), & \text{otherwise;} \end{cases}$$

this is an assignment which agrees with π on all $x \notin T^*$, and so it satisfies (93.6). Note that, for each u_m and each v_n,

$$\tau(u_m) \equiv \rho(\tau^*(u_m)), \quad \sigma(v_n) \equiv \rho(\sigma^*(v_n))$$

since ρ reduces $\boldsymbol{u}$ to $\boldsymbol{u}^*$, and also that for $m = 1, \dots, \alpha$,

$$\pi_h(\tau^*(u_m)) = \pi_c(\rho(\tau^*(u_m))),$$

this is because if $\tau^*(u_m) \notin T^*$, then $\rho(\tau^*(u_m)) \equiv \tau^*(u_m)$ and $\pi_h(\tau^*(u_m)) = \pi(\tau^*(u_m)) = \pi_c(\tau^*(u_m))$, and if $\tau^*(u_m) \in T^*$, then this equation holds by the definition of π_h. Finally, using these equations we compute:

$$\begin{aligned}
\mathrm{den}(x_i(\tau(u_1),\dots,\tau(u_\alpha),\pi_c) &\simeq \pi(x_i)(\pi_c(\tau(u_1)),\dots,\pi_c(\tau(u_\alpha))\\
&\simeq \pi(x_i)(\pi_c(\rho(\tau^*(u_1))),\dots,\pi_c(\rho(\tau^*(u_\alpha)))\\
&\simeq \pi(x_i)(\pi_h(\tau^*(u_1))\dots,\pi_h(\tau^*(u_\alpha)))\\
&\simeq \pi(x_j)(\pi_h(\sigma^*(v_1)),\dots,\pi_h(\sigma^*(v_\beta))) \quad \text{(by (93.6))}\\
&\vdots \text{ (reverse these steps for } x_j)\\
&\simeq \mathrm{den}(x_j(\sigma(v_1),\dots,\sigma(v_\beta)),\pi_c). \qquad \dashv \text{(Fact 3)}
\end{aligned}$$

Fact 4. *The partial functions $\bar{r}_i$ and $\bar{r}_j$ are compatible.*

PROOF. Assume toward a contradiction that they are not. By the definition of these partial functions, this means that there exist tuples $\vec{u}', \vec{v}'$ such that

$$(1)\ \bar{a}_1(\vec{u}') = \bar{b}_1(\vec{v}'),\dots,\bar{a}_k(\vec{u}') = \bar{b}_k(\vec{v}'), \qquad (2)\ \pi(x_i)(\vec{u}') \not\simeq \pi(x_j)(\vec{v}'). \tag{93.8}$$

Let $\boldsymbol{u} = (\tau, \sigma, (s_1,\dots,s_l))$ be the unifier of (z_i, z_j) constructed from $\vec{u}', \vec{v}'$ in the proof of Fact 2, so that (directly from its definition), $\tau(u_m) \equiv u_m$ if u_m does not occur in $r(a_1,\dots,a_k)$; $\sigma(v_m) \equiv v_m$ if v_m does not occur in $r(b_1,\dots,b_k)$; and for the significant u_m, v_m,

$$u'_m = \bar{w}_t \Longrightarrow \tau(u_m) \equiv s_t, \quad v'_m = \bar{w}_t \Longrightarrow \sigma(v_m) \equiv s_t.$$

This means that

$$\tau(u_m) \equiv \tau(u_n) \iff u'_m = u'_n, \quad \sigma(v_m) \equiv \sigma(v_n) \iff v'_m = v'_n,$$

and so we san define an assignment on these variables by

$$\pi^*(\tau(u_m)) = u'_m, \quad \pi^*(\tau(v_m)) = v'_m. \tag{93.9}$$

From Fact 3 and Step 3 in the construction of the line, we know that π satisfies the equation

$$\lambda(s_1,\dots,s_l)x_i(\tau(u_1),\dots,\tau(u_\alpha)) = \lambda(s_1,\dots,s_l)x_j(\sigma(v_1,\dots,\sigma(v_\beta)),$$

so that for every π^* which agrees with π except perhaps on $s_1,\dots,s_l$,

$$\pi(x_i)(\pi^*(\tau(u_1)),\dots,\pi^*(\tau(u_\alpha))) \simeq \pi(x_j)(\pi^*(\sigma(v_1)),\dots,\pi^*(\sigma(v_\beta)));$$

and when we apply this to the π^* defined by (93.9) we get

$$\pi(x_i)(\vec{u}') \simeq \pi(x_j)(\vec{v}')$$

which contradicts (93.8) and completes the proof of Fact 4 and the Lemma.$\dashv$ (Fact 4, Lemma 4.14)

4.15. **Structures with (some) finite basic sets**. There is an obvious, trivial way by which we can reduce the problem of intensional identity for any structure $\mathbf{A}$ of finite signature to that of another such structure $\mathbf{A}'$ in which all basic sets are infinite. If, for example, $U = \{u_1,\dots,u_n\}$ is finite and $f : U \times V \rightharpoonup W$ is a partial function among the givens, we replace f in $\mathbf{A}$ by n new partial functions

$$f_i : V \rightharpoonup W, \quad f_i(v) \simeq f(u_i, v), \quad i = 1,\dots,n.$$

The translation is a bit messier for functionals but still trivial in principle. This is, in fact what we will do if U is a *small* finite set, e.g., if $U = TV = \{\mathbf{t}, \mathbf{f}\}$ is the set of truth values. If, however, n is immense, then this reduction leads to a structure with impossibly many primitives whose dictionary is totally unmanageable. Suppose, for example, that the language is a small (in theory

formalized) fragment of the basic English currently in use as the common language of business in continental Europe. There are few basic sets in the intended interpretation, the citizens of France, the German cars, the Greek raisins, etc. but they are all immense. We may also assume few primitives, we are only interested in making simple assertions like

there are enough Greek raisins to satisfy the needs of all Frenchmen.

The problem of sense identity for sentences of such a language appears to be quite manageable, and in fact, the actual dictionaries we would use to translate this language into the national European languages are quite small. In contrast, the formal dictionary of the expanded language suggested by the trivial procedure of eliminating all the finite basic sets is absurdly large and involves specific entries detailing separately the relation between every Frenchman with every Greek raisin. The decision procedure we described allows a better solution.

4.16. **Corollary** (to the proof). Suppose $\mathbf{A} = (U_1, \dots, U_k, f_1, \dots, f_n)$ is a recursor structure of finite signature, such that every basic set U_i has at least d members. Then the decision procedure for intensional identity defined in this section will decide correctly every identity on $\mathbf{A}$ with n (free and bound) basic variables, provided that $2n + 4 \leq d$.

The Corollary suggests a method of constructing a reasonably sized "dictionary of meanings" for a structure in which some basic sets are very small—and we eliminate these—and the others are very large. The formal decision procedure for intensional identity on the basis of this dictionary is not that far from the way we would decide such questions in practice: we understand quantification over small sets by considering individual cases, while for large sets we appeal to fundamental identities relating the meanings of the primitives, including the quantifiers. The procedure will fail to resolve questions of identity of meaning which involve more quantifiers over large sets than (roughly) half the size of the structure. The proof of the Corollary follows from a careful examination of the arguments of this section, which basically require the existence of enough possible values for variables to make certain distinctions. For example, it is not hard to check that Lemma **4.3** holds, provided all the basic sets of the structure have at least 4 elements. We will omit the details.

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