HYPERARITHMETICAL SETS

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CONTENTS

By the early 1940s, ten years after Gödel's monumental [1931], the foundations of a mathematical theory of computability had been well established, primarily by the work of Alonzo Church, Alan Turing, Emil Post and Stephen Kleene. Most significant was the formulation of the *Church-Turing Thesis*, which identifies the intuitive notion of *computable function* (on the natural numbers) with the precisely defined concept of (general) *recursive function*; this was well understood and accepted (as a *law* in Emil Post's view) by all the researchers in the area, even if not yet by all logicians.[1] The Church-Turing Thesis makes it possible to give rigorous proofs of (absolute) *unsolvability* of mathematical problems whose solution asks

[1]cf. Moschovakis [1968].

for an "algorithm" or a "decision procedure". Several fundamental metamathematical relations had been shown to be *undecidable*, chief among them the relation of *first-order provability* (Hilbert's *Entscheidungsproblem*, Church [1936] and Turing [1936]). Moreover, a general *theory of computability* had also started to develop, especially with Kleene [1936].

The most obvious next steps were to

- look for unsolvability results in "ordinary mathematics", and
- study (in general) *the unsolvable*.

The first of these was (apparently) first emphasized by Post, who said in his [1944] that "[Hilbert's 10th Problem] begs for an unsolvability proof". Post [1947] and Markov [1947] proved (independently) the *unsolvability of the word problem for* (finitely generated and presented) *semigroups*, the first substantial result of this type. Martin Davis' work is an important part of this line of research which is covered extensively in other parts of this volume.

My topic is the theory of *hyperarithmetical sets*, one of the most significant developments to come out of the general theory of unsolvability in which Davis also played a very important role. I will give a survey of the development of the subject in its formative period from 1950 to 1960, starting with a discussion of its origins and with a couple of brief pointers to later developments at the end. There are few proofs, chosen partly because of the importance of the results but mostly because they illustrate simple, classical methods specific to this area which are not easy to find in the literature, especially in the treatment of *uniformity*; and I have tried to give these proofs in the spirit (if not the letter) of the methods which were available at the time—with just one, notable exception, cf. Remark 3B.3.

The Appendix collects the few basic facts from recursion and set theory that we need and fixes notation. We refer to them by App 1, App 2, etc.

1. Preamble: Kleene [1943], Post [1944] and Mostowski [1947]. The two seminal articles of Kleene and Post were published within a year of each other[2] and have had a deciding influence on the development of the theory of unsolvability up until today. Mostowski wrote his [1947] in ignorance of Kleene [1943], he discovered independently many of Kleene's results and he asked some questions which influenced profoundly the development of the subject. We will discuss it in Section 1D.

Kleene and Post approached "the undecidable" in markedly different ways: they chose different ways to measure the complexity of undecidable sets, they introduced different methods of proof and they employed distinct "styles of exposition". The results in them and in the research they inspired are closely related, of course, as they are ultimately about the same objects—the undecidable relations on the natural numbers; but there is no doubting the fact that they led to two different traditions in the theory of unsolvability with many of the best researchers in one of them (sometimes) knowing very little of what has happened in the other.

The first, key question was how to *measure the unsolvability* of a set of natural numbers.

[2]Kleene had presented much of his [1943] in a meeting of the American Mathematical Society in September 1940. I do not know when Post obtained the results in his [1944].

1A. Post's degrees of unsolvability. Post [1944] does it by comparing the complexity of two sets $A, B \subseteq \mathbb{N}$ using several methods of *reducing effectively* the relation of membership in A to that of membership in B. The strongest of these is *one-one reducibility*,

$$A \leq_e^1 B \iff \varphi_e : \mathbb{N} \rightarrowtail \mathbb{N} \text{ is a total injection and } [x \in A \iff \varphi_e(x) \in B],$$
$$A \leq_1 B \iff (\exists e)[A \leq_e^1 B],$$

close to the mildly weaker *many-one reducibility* $A \leq_m B$ where it is not required that φ_e be an injection. The weakest and most important is *Turing reducibility*,

$$A \leq_e^T B \iff \chi_A = \{e\}^B, \quad A \leq_T B \iff (\exists e)[A \leq_e^T B].$$

We will also use the strict and symmetric versions of these reducibilities,

$$A <_1 B \iff A \leq_1 B \,\&\, B \not\leq_1 A, \quad A \equiv_1 B \iff A \leq_1 B \,\&\, B \leq_1 A,$$

and similarly for $<_m, \equiv_m, <_T, \equiv_T$.

The symmetric relations induce natural notions of *degrees*, e.g.,

the 1-1 degree of $A = \mathbf{d}_1(A) = \{B : B \equiv_1 A\}$,

the Turing degree of $A = \mathbf{d}(A) = \{B : B \equiv_T A\}$;

and the central objects of study are these sets of degrees with their natural partial orders, most significantly the *poset of Turing degrees* $(\mathcal{D}, \leq_T)$ where

$$\mathbf{a} \leq_T \mathbf{b} \iff (\exists A, B \subseteq \mathbb{N})[\mathbf{a} = \mathbf{d}(A) \,\&\, \mathbf{b} = \mathbf{d}(B) \,\&\, A \leq_T B].$$

Post focusses on the study of the degrees of *recursively enumerable* sets (App 7). He introduces the "self-referential" version of Turing's *Halting Problem*

$$K = \{e : \{e\}(e)\downarrow\} = \{e : (\exists t)T_1(e, e, t)\} \tag{1}$$

and proves that it is *r.e. complete*, i.e., it is r.e. and every r.e. set is 1-1 reducible to it. In particular K is not recursive, and then the natural question is whether there are r.e. sets intermediate in complexity between the recursive sets and K. Post proves this for all of his reducibilities except for Turing's and asks what became known as *Post's Problem*: *is there an r.e. set A such that $\emptyset <_T A <_T K$?* Friedberg and Muchnik proved that there is, some ten years later, and this initiated a research program in the theory of degrees and r.e. degrees which is still vibrant today.

1B. Kleene's arithmetical hierarchy. Kleene [1943] focusses on the *arithmetical sets*, those which are first-order definable in the standard model of arithmetic

$$\mathbf{N} = (\mathbb{N}, 0, 1, +, \cdot) \tag{2}$$

and measures the complexity of a set by its *simplest definition* in $\mathbf{N}$. His crucial contribution is the choice of a useful *measure of complexity of first-order definitions* in $\mathbf{N}$: a relation $P \subseteq \mathbb{N}^n$ is Σ_k^0 (or *in* Σ_k^0) if it satisfies an equivalence of the form

$$P(\vec{x}) \iff (\exists t_1)(\forall t_2)(\exists t_3)\cdots(\mathsf{Q}_k t_k)R(\vec{x}, t_1, \ldots, t_k) \quad (k \geq 1) \tag{3}$$

where $R(\vec{x}, \vec{t})$ is recursive and Q_k is $\exists$ or $\forall$ accordingly as k is odd or even. A relation $P(\vec{x})$ is in $\Pi_k^0 = \neg\Sigma_k^0$ if its negation is in Σ_k^0, so that

$$P(\vec{x}) \iff (\forall t_1)(\exists t_2)(\forall t_3)\cdots(\mathsf{Q}_k t_k)R(\vec{x}, t_1, \ldots, t_k) \quad (k \geq 1) \tag{4}$$

$$\begin{array}{ccccccccccccccc}
 & & \Sigma^i_1 & & & & \Sigma^i_2 & & & & \Sigma^i_3 & & & & \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & & & \\
\Delta^i_1 & & & & \Delta^i_2 & & & & \Delta^i_3 & & & & & & \cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & & \\
 & & \Pi^i_1 & & & & \Pi^i_2 & & & & \Pi^i_3 & & & &
\end{array}$$

FIGURE 1. The arithmetical ($i = 0$) and analytical ($i = 1$) hierarchies.

with a recursive $R(\vec{x}, \vec{t})$, and $\Delta^0_k = \Sigma^0_k \cap \Pi^0_k$. The relations which belong to one of these classes are exactly the arithmetical ones, and that was well known after Kleene [1936]. The novelty here is that by allowing a recursive matrix in (3) and (4) rather than, say, a quantifier free one, Kleene can prove robust closure properties and to construct $\mathbb{N}$-*parametrizations* for these classes of relations:

Lemma 1B.1. (1) *Closure properties*: Σ^0_k *and* Π^0_k *are closed under recursive substitutions,* &, $\vee$ *and bounded number quantification of both kinds*; Σ^0_k *is also closed under number quantification* $\exists s$; Π^0_k *is closed under* $\forall s$; *and* Δ^0_k *is closed under negation.*

(2) *The* $\mathbb{N}$-*Parametrization Property*: *there are relations* $G^n_k \subseteq \mathbb{N}^{1+n}$ *in* Σ^0_k *and recursive injections* $S^l_n : \mathbb{N}^{1+l} \to \mathbb{N}$ *such that for every n-ary* $P(\vec{x})$ *in* Σ^0_k,

$$P(\vec{x}) \iff G^n_k(e, \vec{x}) \text{ for some } e \in \mathbb{N}, \tag{5}$$

and for all $\vec{y} = (y_1, \dots, y_l)$

$$G^{l+n}_k(e, \vec{y}, \vec{x}) \iff G^n_k(S^l_n(e, \vec{y}), \vec{x}). \tag{6}$$

These facts are very easy by induction, starting with $k = 1$ where they are immediate by the Normal Form and Enumeration Theorem for recursive partial functions App 5. They imply the *Hierarchy Theorem* for the arithmetical sets pictured in Figure 1 (with $i = 0$), and they can be used very effectively to measure the complexity of a set by placing it in the arithmetical hierarchy, sometimes exactly. Such were, in fact, their first applications.[3] Its main significance, however, was that it set the stage for its non-trivial extensions into the *analytical hierarchy*, also pictured in Figure 1 with $i = 1$, as well as the *hyperarithmetical hierarchy* which lies between them and is our main concern.

The closure of the arithmetical classes under recursive substitutions imply that for every n-ary relation $P(\vec{x})$,

$$P \in \Sigma^0_k \iff \{\langle \vec{x} \rangle : P(\vec{x})\} \in \Sigma^0_k,$$

i.e., these classes are determined by the sets in them; so we will sometimes abuse notation and use Σ^0_k to denote the class of Σ^0_k sets—and similarly for Π^0_k, Δ^0_k.

[3]For example, Davis [1950a] proves that the set $\{e : (\forall x)[\{e\}(x)\downarrow]\}$ of codes of total recursive functions is in $\Pi^0_2 \setminus \Sigma^0_2$. The Hierarchy Theorem also yields a trivial proof of *Tarski's Theorem* for $\mathbf{N}$, that *arithmetical truth is not arithmetical*.

1C. Kleene [1943] vs. Post [1944]. There is little overlap between these two papers, except that they both characterize the recursive sets as exactly those which are r.e. and have r.e. complements (Post's Theorem). Beyond that, Post limits himself to the complexity structure of r.e. sets which comprise precisely Kleene's Σ^0_1—about which Kleene says nothing non-trivial.

Both papers are brilliant examples of *concept formation*, the identification of fundamental notions which is characteristic of some of the best work in logic. Post also proves several non-trivial technical results, some by very clever constructions; there is little of this in the Kleene paper, whose technical results are proved mostly by seemingly routine computations.

Then there is the style of exposition: Post is eloquent, even colorful. He introduces suggestive, descriptive terms (*complete, creative, simple*) which give life to the formulation of his results and right in his first paragraph, he declares that his purpose is

> "to demonstrate by example that this concept [of recursive function] admits ... of an intuitive development which can be followed, if not indeed pursued, by a mathematician, layman though he be in this formal field".[4]

His exhortation to *explain* rather than *detail* proofs resonated strongly in the work of those who followed him, sometimes with beautiful results, e.g., in the classic Rogers [1967]. At the other end, Kleene is dry, formal, and more worried about whether he has a *constructive* (intuitionistic) proof than if his proof is easily comprehensible—and to some extent, these traits persisted in the writings of those who followed him.

1D. Mostowski [1947] and the analogies. Mostowski's starts with the classical notions of Descriptive Set Theory. Briefly, in modern notation and (for simplicity) only for $\mathcal{N}$:

(1) A σ-algebra is any collection $\mathcal{F} \subseteq \mathcal{P}(\mathcal{N})$ which is closed under complements and countable unions;
(2) the class **B** of Borel sets is the smallest σ-algebra which contains all the open sets;
(3) a relation $P \subseteq \mathcal{N}^m$ is $\underset{\sim}{\mathbf{\Sigma}}^1_1$ if $P = \{\vec{\alpha} : (\exists\beta)F(\vec{\alpha}, \beta)\}$ with F closed;
(4) P is $\underset{\sim}{\mathbf{\Sigma}}^1_{k+1}$ if $P = \{\vec{\alpha} : (\exists\beta)\neg Q(\vec{\alpha}, \beta)\}$ with Q in $\underset{\sim}{\mathbf{\Sigma}}^1_k$;
(5) $\underset{\sim}{\mathbf{\Pi}}^1_k = \{\mathcal{N}^m \setminus P : P \in \underset{\sim}{\mathbf{\Sigma}}^1_k\}$ and $\underset{\sim}{\mathbf{\Delta}}^1_k = \underset{\sim}{\mathbf{\Sigma}}^1_k \cap \underset{\sim}{\mathbf{\Pi}}^1_k$.

The *projective classes* $\underset{\sim}{\mathbf{\Sigma}}^1_k, \underset{\sim}{\mathbf{\Pi}}^1_k, \underset{\sim}{\mathbf{\Delta}}^1_k$ were introduced by Luzin and Sierpinski in 1925 and they fall into a hierarchy that looks exactly like the arithmetical hierarchy in Figure 1 with boldface letters and superscript 1. But the most fundamental result about them is older and concerns only the first level of this hierarchy:

Theorem 1D.1 (Suslin [1917]). *A set $A \subseteq \mathcal{N}$ is $\underset{\sim}{\mathbf{\Delta}}^1_1$ if and only if it is Borel.*

This was rightfully viewed as a "construction principle" which reduces a complementary pair of quantifications over the complex set $\mathcal{N}$ to a countable iteration of

[4]He also said that " ... with a few exceptions explicitly so noted, we have obtained formal proofs of all the consequently mathematical theorems here developed informally", and it is clear that the purely intuitive approach can only go so far: we cannot hope to prove that (say) *the word problem for semigroups is unsolvable* on the basis of our intuitions about computability, without a rigorous definition of recursive functions and an appeal to the Church-Turing Thesis.

taking countable unions and complements, starting with the simple neighborhoods of $\mathcal{N}$. Mostowski had not read Kleene [1943] but he knew Post [1944] and saw a similarity between Suslin's Theorem and Post's in the form

$$\Delta^0_1 = \text{recursive},$$

which similarly reduces Δ^0_1 definitions to "computations". He postulated the natural "analogies"

$$\begin{array}{rcl} \text{recursive function on } \mathbb{N} & \sim & \text{continuous function on } \mathcal{N}, \\ \text{recursive subsets of } \mathbb{N} & \sim & \mathbf{B}, \\ \Sigma^0_1 \text{ subsets of } \mathbb{N} & \sim & \underset{\sim}{\mathbf{\Sigma}}^1_1 \text{ subsets of } \mathcal{N}, \end{array} \tag{7}$$

and using these as motivation he defined the arithmetical hierarchy and established for it basically all the results in Kleene [1943], so that the analogies extend to all the levels of the two hierarchies. He knew that these are not perfect: not every injective, recursive image of $\mathbb{N}$ is recursive, while by a basic, classical result, *every injective, continuous image of $\mathcal{N}$ is Borel*. This, however, might be just a technical wrinkle, as *every increasing, recursive image of* $\mathbb{N}$ *is recursive*. Later, after writing this paper, he thought of another fundamental property of $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sets which could test the analogy, the following generalization of Suslin's Theorem due to Lusin:

Theorem 1D.2 ($\underset{\sim}{\mathbf{\Sigma}}^1_1$ Separation). *For any two disjoint $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sets $A, B \subseteq \mathcal{N}$, there is a Borel set C which separates them, i.e.,*

$$A \subseteq C, \quad C \cap B = \emptyset. \tag{8}$$

So is it true that any two disjoint r.e. sets can be separated by a recursive set? At some time between 1947 and 1950 he mentioned the problem to Kleene who (it turned out) had already answered it but not published his result:

Theorem 1D.3 (Kleene [1950]). *There exist two disjoint, r.e. sets $A, B \subseteq \mathbb{N}$ such that no recursive set C satisfies* (8).

So the simple minded analogies (7) fail, but they did not go away: they motivated a great deal of research in the twenty years that followed and ultimately, as we will see, a corrected version of them turned out to be an important part of the story of HYP.

2. On into the transfinite![5] For any $A \subseteq \mathbb{N}$, let

$$A' = \{e : \{e\}^A(e)\downarrow\} = \text{ the } \textit{jump} \text{ of } A. \tag{9}$$

It follows that for every B,

$$B \text{ is r.e. in } A \iff B \leq_1 A', \tag{10}$$

so that in particular $A <^T A'$, and we can get a sequence of sets of increasing Turing complexity by setting recursively

$$K_0 = \emptyset, \ K_1 = K_0', \ K_2 = K_1', \dots . \tag{11}$$

[5]For completeness, we will repeat in this section some parts of §7 – §9 of Moschovakis [2010b], which goes over some of the same ground in more detail and includes several proofs.

Now K_1 is (recursively isomorphic with) Post's complete r.e. set K and for every $k \geq 1$, easily, K_k is Σ^0_k-*complete*, i.e., a set is Σ^0_k exactly when it is 1-1 reducible to K_k. It is also easy to check that the diagonal set

$$K_\omega = \{\langle m, n\rangle : m \in K_n\} \tag{12}$$

is recursively isomorphic with the *truth set* for arithmetic

$$\text{Truth} = \{\ulcorner\theta\urcorner : \mathbf{N} \models \theta\},$$

where $\ulcorner\theta\urcorner$ is the Gödel number of the sentence θ in the language of arithmetic, relative to some standard coding. This is not arithmetical; and then one can continue and define ever more complex non-arithmetical sets,

$$K_{\omega+1} = K'_\omega, K_{\omega+2} = K'_{\omega+1}, \dots, K_{\omega\cdot 2} = \{\langle m, n\rangle : m \in K_{\omega+n}\}\dots \tag{13}$$

indexed by the ordinals $\xi < \omega^2$. The sequence $\{K_\xi : \xi < \omega^2\}$ was defined by Davis [1950a] who also showed that

$$\eta \leq \xi < \omega^2 \Longrightarrow K_\eta \leq_m K_\xi \text{ and } \xi < \eta \Longrightarrow K_\xi <_T K_\eta. \tag{14}$$

These facts are all fairly simple to verify today. They were not so easy[6] before 1955, when the theory of relative recursion had not been worked out in detail: Kleene [1943], [1953], [1955a], Davis [1950a], [1950b] and Mostowski [1947], [1951] all prove various versions of them, not always the cleanest or strongest, sometimes awkwardly and (in the case of Davis and Mostowski) mostly without knowing all of each other's or Kleene's work. Nevertheless, the later papers Davis [1950a], Mostowski [1951] and Kleene [1955a] all take the crucial step of defining natural extensions of the arithmetical hierarchy beyond its first ω classes $\Sigma^0_1, \Sigma^0_2, \dots$, "*on into the transfinite*" in Davis' exhortation with which we headed this section.

The definitions (11) – (13) of $\{K_\xi : \xi < \omega^2\}$ depend on choosing for each limit ordinal $\xi = \omega \cdot s < \omega^2$ the specific, increasing sequence $n \mapsto \omega \cdot (s \dot{-} 1) + n$ converging to ξ. This is natural enough, but not the only choice, and it is not obvious how to make a "natural" or "best" choice[7] for ordinals above ω^2. This leads us to the next, crucial bit:

2A. Notations for ordinals, S_1 and O. Following Kleene [1938], let first

$$0_O = 1, \quad (t+1)_O = 2^{t_O}, \quad e_t = \{e\}(t_O),$$

[6] For example, to prove that K_k is Σ^0_k-complete, you need the first of the following strengthenings of (10): *there are recursive injections $u(e,t), v(e)$ such that for all A, B and all e, t,*

$$(1)\ \{e\}^A(t)\!\downarrow \iff u(e,t) \in A' \text{ and } (2)\ A \leq^T_e B \Longrightarrow A' \leq^1_{v(e)} B'. \tag{15}$$

Proof: For (1), choose $\overline{m}$ so that for any A, $\{\overline{m}\}^A(e,t,y) = \{e\}^A(t)$ and set $u(e,t) = S^2_1(\overline{m}, e, t)$. For (2) you start with a recursive $v_1(e)$ such that $A \leq^T_e B \Longrightarrow \{e\}^A(t) = \{v_1(e)\}^B(t)$ and do a similar construction. That $u(e,t)$ and $v(e)$ are (absolutely) *recursive injections*—which has applications—depends on the fact that the functions $S^{l,m}_n$ in App 5 are independent of any function parameters and injective, which I cannot find in any of the early texts (including Kleene [1952]) even for $m = 0$.

[7] Spector [1956] eliminates dramatically the most obvious approach at limit ordinals: *No increasing sequence $\boldsymbol{d}_0 < \boldsymbol{d}_1 < \cdots$ of Turing degrees has a least upper bound.* Of course, this was not known to Davis, Kleene and Mostowski when they wrote these early papers.

and (by App 10), let $|\ |: \mathbb{N} \rightharpoonup$ Ordinals be the least partial function on $\mathbb{N}$ to ordinals which satisfies the following:[8]

(1) $|1| = 0$.
(2) For every t, $|2^t| = |t| + 1$.
(3) For every e, if for every t, $|e_t|\downarrow$ and $|e_t| < |e_{t+1}|$, then $|3 \cdot 5^e| = \lim_{t\to\infty} |e_t|$.

With $S_1 = \{z : |z|\downarrow\}$, the pair $\mathrm{S}_1 = (S_1, |\ |)$ is *the first* Church-Kleene *notation system* for ordinals and the only one we will use. Kleene [1938] also introduced a smaller notation system $\mathrm{S}_3 = (O, |\ |_3)$ and a partial ordering $\leq_O$ of O such that

$$O \subsetneq S_1, \quad a \in O \Longrightarrow |a|_3 = |a|, \text{ and so } a <_O b \Longrightarrow |a| < |b|, \tag{16}$$

and then used that in all his work on the topic—as did Spector and most researchers in the field. We will occasionally refer to O and $\leq_O$ when we want to quote early results exactly as they were stated, but we will not use them in any essential way and so we skip their precise definition.[9]

A countable ordinal ξ is *constructive* if $\xi = |z|$ for some $z \in S_1$. Note that directly from the definition, *the constructive ordinals form an initial segment of the set of countable ordinals*. Their supremum

$$\omega_1 = \sup\{|a| : a \in S_1\} \text{ (the Church-Kleene omega-1)} \tag{17}$$

is a "constructive analog" of the first uncountable ordinal Ω_1; it is a fundamental constant of definability theory and it can be characterized in many natural ways, including the following early result:

Theorem 2A.1 (Markwald [1954], Spector [1955]). *An ordinal ξ is constructive if and only if it is finite or the order type of a recursive wellordering of* $\mathbb{N}$.[10]

2B. The H_a-sets. By recursion on the ordinal $|a|$, we associate with each $a \in S_1$ a set $H_a \subseteq \mathbb{N}$ so that:

(H1) $H_1 = \mathbb{N}$,
(H2) $H_{2^b} = H_b'$, and
(H3) if $a = 3 \cdot 5^e$, then $x \in H_a \iff (x)_0 \in H_{e_{(x)_1}}$.

This is exactly the definition in Kleene [1955a], except that he gave it for $a \in O \subsetneq S_1$. The earlier Davis [1950a] gave an almost identical definition (for $a \in S_1$) which differs only in the details of the coding, and Mostowski [1951] gave a somewhat different and abbreviated version which seems to avoid ordinal codes, cf. Section 3C.

Davis [1950a] proves that for $a, b \in S_1$, $|a|, |b| < \omega^2$,

$$|a| \leq |b| \Longrightarrow H_a \leq_m H_b \text{ and } |b| < |a| \Longrightarrow H_b <_T H_a, \tag{18}$$

[8] Kleene's obtuse coding (the 3 and 5 in the definition) is motivated by the plans he and Church had to develop a general "constructive theory of ordinals" beyond Cantor's first and second number classes. They never got into this, but some (non-trivial and highly technical) results were proved by others, cf. Kreider and Rogers [1961], Putnam [1961], Enderton and Putnam [1970]. We will not cover this topic here.

[9] O and $\leq_O$ are defined by a (simultaneous) inductive definition as in App 10 which (in Kleene's words) "is regarded from the finitary point of view as a correction, in that it eliminates the presupposition of the classical (non-constructive) second number class". There are problems with this view, partly because many results about constructive ordinals cannot be proved (or even stated) without referring to ordinals. In any case, we will use S_1 here.

[10] A proof of this basic fact is included in Moschovakis [2010b, §8].

so that, in particular,

$$|a| = |b| < \omega^2 \Longrightarrow H_a \equiv_T H_b \quad (a, b \in S_1)$$

and asks if every constructive ordinal has this *uniqueness property*. This turned out to be a difficult problem and led some five years later to one of the first spectacular results in the area:[11]

Theorem 2B.1 (Spector [1955]). *For all $a, b \in S_1$,*

$$|a| \le |b| \Longrightarrow H_a \le_T H_b \text{ and } |b| < |a| \Longrightarrow H_b <_T H_a.$$

In particular, $|a| = |b| \Longrightarrow \boldsymbol{d}(H_a) = \boldsymbol{d}(H_b)$ and if we set $\boldsymbol{d}_{|a|} = \boldsymbol{d}(H_a)$, then $\{\boldsymbol{d}_\xi : \xi < \omega_1\}$ is an increasing sequence of Turing degrees of length ω_1.

Much more was done with constructive ordinals and the H_a-sets in the fifties and sixties, especially by Kleene who used them as his main tool for studying the hyperarithmetical sets. We will not go much into this here, for good reasons that we will explain in due course; but before we dig into our main topic, we need to discuss briefly some important, early work that we will not cover in detail.

2C. Myhill [1955]. Two sets A, B are *recursively isomorphic* if one is carried onto the other by a recursive permutation of $\mathbb{N}$,

$$A \equiv B \iff A \le_e B \text{ where } \varphi_e : \mathbb{N} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{N} \text{ is a bijection.}$$

Myhill [1955] introduces this notion and shows (among other things) that

$$\text{(19)} \qquad \textit{for all } A, B \subseteq \mathbb{N}, \textit{ if } A \equiv_1 B, \textit{ then } A \equiv B,$$

and so *any two r.e. complete sets are recursively isomorphic*. His methods also combine easily and to significant advantage with some of the results above: for example, Davis' proof of (18) naturally gives the much neater[12]

$$\text{(20)} \qquad |a| = |b| < \omega^2 \Longrightarrow H_a \equiv H_b.$$

However, none of Davis, Kleene or Mostowski knew of this article of Myhill when they wrote the papers we have been discussing.

2D. Effective grounded recursion. More significantly, neither Davis nor Mostowski refer or appeal explicitly to the following basic fact:

Theorem 2D.1 (Kleene's 2nd Recursion Theorem). *For every recursive partial function $f(e, x_1, \dots, x_n, \alpha_1, \dots, \alpha_m)$, there is a number e such that*

$$\{e\}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_m) = f(e, x_1, \dots, x_n, \alpha_1, \dots, \alpha_m).$$

For recursion on $\mathbb{N}$, this was stated unbilled and proved[13] in the last two lines of Kleene [1938, §2] and it is the main technical tool that Kleene used for all his work on constructive ordinals, hyperarithmetical sets—and much more. Myhill [1955] also used it, crucially, as did Spector [1955] in his proof of the Uniqueness Theorem 2B.1. Kleene and Spector use the 2nd Recursion Theorem to justify *effective grounded recursion*, which we can illustrate here with a relevant example.

Consider Davis' definition of the sets $\{L_a : a \in S_1\}$ which are his versions of the H_a-sets:

[11] For a discussion of the Spector Uniqueness Theorem and an outline of its proof for S_1 see Moschovakis [2010b, §9].

[12] This strong uniqueness property cannot be extended to ω^2, cf. Moschovakis [1966], Nelson [1974].

[13] Choose $\overline{k}$ such that $\{\overline{k}\}(t, \vec{x}, \vec{\alpha}) = f(S_n^{1,m}(t, t), \vec{x}, \vec{\alpha})$ and take $e = S_n^{1,m}(\overline{k}, \overline{k})$.

(L1) $L_1 = \emptyset$,
(L2) $L_{2^b} = L'_b$, and
(L3) if $a = 3 \cdot 5^e$, then $x \in L_a \iff (x)_1 \in H_{e_{(x)_2}}$.

Well, L_1 is the complement of H_1 and in the limit case Davis uses $(x)_1$ and $(x)_2$ rather than Kleene's $(x)_0$ and $(x)_1$ which, together, don't amount to much of a difference. The two definitions should be equivalent up to Turing equivalence, and they are:[14]

Lemma 2D.2. *For every $a \in S_1$, $H_a \equiv_T L_a$. In fact, there are recursive partial functions $u(a), v(a)$ which converge on S_1 and satisfy*

$$(*) \qquad H_a \leq^T_{u(a)} L_a, \quad L_a \leq^T_{v(a)} H_a \quad (a \in S_1).$$

The partial functions $u(a), v(a)$ are *uniformities* which witness respectively the reducibilities $H_a \leq_T L_a, L_a \leq_T H_a$.

Proof. The Turing equivalence $H_a \equiv_T L_a$ should be more-or-less trivial by induction on the ordinal $|a|$ and it is, when $|a|$ is 0 or a successor ordinal (granting it for its predecessor). At a limit stage $a = 3 \cdot 5^e$, however, there is no obvious way to put together the equivalences $H_{e_t} \equiv_T L_{e_t}$ supplied by the induction hypothesis to prove that $H_a \equiv_T L_a$, and it is clear that we need to formulate a stronger, "uniform" proposition which will supply a usable induction hypothesis at limit stages. For the first reducibility in $(*)$, one "recursion loading device" that works is the following:

Sublemma. There is a recursive partial function $f(i, a, x)$ which converges for all i, x when $i \leq 1$ and $a \in S_1$ and satisfies the following:

$$(**) \qquad x \in H_a \iff f(0, a, x) = 0 \vee [f(0, a, x) \neq 0 \ \&\ f(1, a, x) \in L_a].$$

Proof of the Sublemma. We set $f(0, 1, x) = 0$ and $f(1, 1, x) = 1$. If $a = 2^b$ for some b, then $f(0, a, x) = 1$ and it is not hard to define $f(1, a, x)$ from $f(i, b, x)$ so that $(**)$ holds using (15) in Footnote 6. Suppose now $a = 3 \cdot 5^e$ and $(**)$ holds for all ordinals less than $|a|$. We compute the conditions that $f(i, a, x)$ must satisfy by examining the equivalences which hold if it does:

$$\begin{aligned}
x \in H_a &\iff (x)_0 \in H_{e_{(x)_1}} \\
&\Leftrightarrow f(0, e_{(x)_1}, (x)_0) = 0 \vee [f(0, e_{(x)_1}, (x)_0) \neq 0 \ \&\ f(1, e_{(x)_1}, (x)_0) \in L_{e_{(x)_1}}] \\
&\Leftrightarrow f(0, e_{(x)_1}, (x)_0) = 0 \vee [f(0, e_{(x)_1}, (x)_0) \neq 0 \ \&\ \langle 0, f(1, e_{(x)_1}, (x)_0), (x)_1 \rangle \in L_a] \\
&\iff f(0, a, x) = 0 \vee [f(0, a, x) \neq 0 \ \&\ f(1, a, x) \in L_a]
\end{aligned}$$

where we have used the induction hypothesis in the second line and the definition of L_a in the third (with an irrelevant 0 put into the first position so that $f(1, a, b)$ codes a triple). So when $a = 3 \cdot 5^e$ we need to have

$$(***) \qquad f(0, a, x) = f(0, e_{(x)_1}, (x)_0), \quad f(1, a, x) = \langle 0, f(1, e_{(x)_1}, (x)_0), (x)_1 \rangle.$$

Now, the 2nd Recursion Theorem easily supplies us with a recursive partial function $f(i, a, x)$ which satisfies the relevant conditions for $a = 1, a = 2^b$ and $(***)$, and then the proof is completed by a routine transfinite induction on $|a|$.

$\dashv$ (Proof of the Sublemma)

[14]In the terminology of Post [1944], the proof shows that H_a and L_a are *equivalent by bounded truth tables*. Had Davis chosen to set $L_1 = \mathbb{N}$ at the basis, then these modified L_as are recursively isomorphic with Kleene's H_a sets, and by a simpler argument than the proof of this Lemma.

The corresponding Sublemma for the second reducibility in $(*)$ is proved by a similar construction, and then the two Sublemmas together imply $(*)$. $\dashv$

Briefly (and vaguely), to "compute" a function $f : D \to \mathbb{N}$ which is defined on $D \subseteq \mathbb{N}^n$ by the recursion

$$f(\vec{x}) = G(f \restriction \{\vec{x}' : \vec{x}' \prec \vec{x}\}, \vec{x}) \quad (\vec{x} \in D) \tag{21}$$

along some wellfounded relation $\prec \subset (\mathbb{N}^n \times \mathbb{N}^n)$, we use the 2nd Recursion Theorem to find a recursive partial f which converges on D and satisfies (21) *on the assumption that one such f exists*; and then we prove by induction along $\prec$ that f indeed satisfies (21). It is very important for the applications that *no definability assumptions are needed for D or $\prec$*, except as they might be used to define f; for the proof of Lemma 2D.2, for example,

$$D = \{(i, a, x) : a \in S_1\}, \quad (i, a, x) \prec (j, b, y) \iff |a| < |b|,$$

and we have no estimate of the complexity of this D and this $\prec$, certainly not now.

The method is very general and we cannot do it justice here, but it has played a very important role in the study of hyperarithmetical sets and so I thought it important to give in full at least one proof which uses it. Another, similar but more difficult example is the *uniform version* of Spector's Uniqueness Theorem 2B.1:

$$|a| \leq |b| \Longrightarrow H_a \leq_T H_b \textit{ uniformly for all } a, b \in S_1.$$

Its precise meaning is that *there is a recursive partial function $u(a, b)$*, a *uniformity*, such that

$$a, b \in S_1 \ \&\ |a| \leq |b| \Longrightarrow [u(a, b)\downarrow \ \&\ H_a \leq^T_{u(a,b)} H_b]. \tag{22}$$

This formulation not only gives useful, additional information, but is necessary for the proof of the Uniqueness Theorem (by effective grounded recursion).

In the sequel I will often refer to effective grounded recursion and uniformity, but with little detail and less explanation.[15]

3. The basic facts about HYP (1950 – 1960). A set $A \subseteq \mathbb{N}$, relation $R \subseteq \mathbb{N}^n$ or (total) function $f : \mathbb{N}^n \to \mathbb{N}$ is *hyperarithmetical* if it is recursive in some H_a; HYP is the set of all hyperarithmetical sets, and

$$\text{if } A \leq^T_e H_a, \text{ then } \langle a, e\rangle \text{ is a HYP-}\textit{code of } A. \tag{23}$$

To express succinctly (and prove) the basic properties of HYP-sets, it is useful to think of them as "bundled" with their codes by the following general notion:

3A. Codings and uniformities. A (surjective) *coding* of a set X is a pair (C, π), where $\pi : C \twoheadrightarrow X$ is a surjection of the *codeset* C onto X, and we call any $c \in C$ a *code* (or name) of the object $\pi(c) \in X$. If $C \subseteq \mathbb{N}$, we say that *the coding is in* $\mathbb{N}$. These are the only codings we will need for a while.

So $(S_1, |\,|)$ is a coding of the constructive ordinals; $(S_1, a \mapsto H_a)$ is a coding of the H_a-sets; $(S_1, a \mapsto L_a)$ is a coding of Davis' L_a-sets; and for a very elementary

[15] Cf. Moschovakis [2010b], [2010a] for a discussion (and many examples), and Moschovakis [2009, 7A.4] for a specific result which codifies many of the applications of effective grounded recursion in Descriptive Set Theory.

example, $(\mathbb{N}, e \mapsto \varphi_e)$ is a coding of the set of unary recursive partial functions. The coding of HYP we introduced by (23) is formally the pair

$$(24) \quad C = \{\langle a, e\rangle : a \in S_1 \;\&\; \{e\}^{H_a} \text{ is total}\},$$
$$\pi(\langle a, e\rangle) = \{x \in \mathbb{N} : \{e\}^{H_a}(x) = 1\}.$$

In practice we will never be so formal, in fact we will sometimes use codings which are "specified by the context" without a formal definition of C and π.

Codings are useful for expressing succinctly *uniform properties* of *coded sets*. Their general theory is technically messy, not very interesting mathematically and certainly not worth putting here.[16] We will confine ourselves to these remarks and "detail" sufficiently many claims to make the ideas clear. For example:

Lemma 3A.1. HYP *is uniformly closed under complements and relative recursion. In detail, there are recursive partial functions $u(c)$ and $v(c, e)$ such that*:

(1) *If A is* HYP *with code c, then $u(c)\downarrow$ and is a* HYP-*code of* $(\mathbb{N} \setminus A)$.
(2) *If c is a* HYP-*code of a set B and $A \leq_e^T B$, then $v(c, e)\downarrow$ and is a* HYP-*code of A*.

This is a simple lemma, as are the similar claims of uniform closure of the hyperarithmetical relations (with their natural coding) under all first-order operations on $\mathbb{N}$. There is no use of effective effective grounded recursion in these proofs, we only need appeal to uniform properties of the jump operation like (15). The next result is also quite easy, but its proof requires effective grounded recursion and some auxiliary definitions on the constructive ordinals:

Lemma 3A.2. HYP *is uniformly closed under recursive unions.*
In detail, there is a recursive partial function $u(e)$ such that if φ_e is total and for each t, $\varphi_e(t)$ is a HYP-*code of a set $A_t \subseteq \mathbb{N}$, then $u(e)\downarrow$ and is a* HYP-*code of $\bigcup_t A_t$.*

Coding invariance. Two codings $(C_1, \pi_1), (C_2, \pi_2)$ in $\mathbb{N}$ of the same set $X = \pi_1[C_1] = \pi_2[C_2]$ are *equivalent* if there are recursive partial functions $u_1(a), u_2(b)$ such that

$$a \in C_1 \Longrightarrow [u_1(a)\downarrow \;\; \& \; u_1(a) \in C_2 \;\&\; \pi_2(u_1(a)) = \pi_1(a)]$$

and similarly with 1 and 2 interchanged. It is clear that propositions like Lemmas 3A.1 and 3A.2 which hold uniformly for a certain coding also hold uniformly for every equivalent coding—and for some of them the proof might be easier.[17] We exploit this idea by establishing an elegant characterization of HYP which produces a coding for it equivalent to the classical one in (23) but much simpler.

[16] The interested reader may want to look at Moschovakis [2010a] where it was necessary to develop this generalized abstract nonsense in some detail.

[17] For a classical example, consider the coding of recursive partial functions specified by the Normal Form Theorem in App 5. Its precise definition depends on the choice of computation model that we use, Turing machines, systems of recursive equations or whatever, but all these codings are equivalent and so uniform propositions about them are *coding invariant*. Rogers [1967, §4.3–§4.5] considers this situation in some detail and formulates stronger notions of equivalence than the one we use.

3B. HYP **as effective Borel.** An *effective σ-algebra on* $\mathbb{N}$ is any collection $X \subset \mathcal{P}(\mathbb{N})$ of sets of natural numbers which admits a coding (C, π) in $\mathbb{N}$ so that the following hold:

(1) Every singleton $\{\!\{t\}\!\}$ belongs to X uniformly, i.e., for some total, recursive $u_1(t)$ and every t, $u_1(t)$ is a code of $\{\!\{t\}\!\}$ in X.

(2) X is uniformly closed under complements, i.e., there is a recursive partial function $u_2(c)$ such that

$$c \in C \Longrightarrow [u_2(c)\downarrow \ \& \ \pi(u_2(c)) = \mathbb{N} \setminus \pi(c)].$$

(3) X is uniformly closed under recursive unions, i.e., for some recursive partial function $u_3(e)$,

$$(\forall t)[\varphi_e(t)\downarrow \ \& \ \varphi_e(t) \in C] \Longrightarrow [u_3(e)\downarrow \ \& \ u_3(e) \in C \ \& \ \pi(u_3(e)) = \textstyle\bigcup_t \pi(\Phi_e(t))].$$

As in the definition of $(S_1, |\,|)$, let $\boldsymbol{b} : \mathbb{N} \rightharpoonup \mathcal{P}(\mathbb{N})$ be the least partial function on $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$, such that

(1) $\boldsymbol{b}(\langle 1, t\rangle) = \{\!\{t\}\!\}$,
(2) $\boldsymbol{b}(\langle 2, y\rangle) = \mathbb{N} \setminus \boldsymbol{b}(y)$, and
(3) if φ_e is total and for every i, $\boldsymbol{b}(\varphi_e(i))\downarrow$, then $\boldsymbol{b}(\langle 3, e\rangle) = \bigcup_i \boldsymbol{b}(\varphi_e(i))$

and set

$$B = \{i : \boldsymbol{b}(i)\downarrow\}, \quad B_i = \boldsymbol{b}(i) \ (\text{if } i \in B), \quad \mathsf{B} = \{B_i : i \in B\}, \tag{25}$$

the collection of *effective Borel* subsets of $\mathbb{N}$.

Lemma 3B.1. B *is the least effective σ-algebra on* $\mathbb{N}$, *uniformly.*

Proof. The coding $(B, i \mapsto B_i)$ witnesses that B is an effective σ-algebra on $\mathbb{N}$. To see that it is uniformly the least one, suppose (C, π) is a coding witnessing that some X is an effective σ-algebra on $\mathbb{N}$ and define by a natural effective grounded recursion a recursive partial function u such that

$$i \in B \implies [u(i)\downarrow \ \& \ u(i) \in C \ \& \ B_i = \pi(u(i))]. \qquad \dashv$$

Theorem 3B.2. HYP $= \mathsf{B}$ *uniformly, i.e.,* (C, π) *in* (24) *and* $(B, i \mapsto B_i)$ *in* (25) *are equivalent codings of* HYP.

Proof. HYP is an effective σ-algebra on $\mathbb{N}$ by Lemmas 3A.1, 3A.2 and a simple construction which puts into it every singleton, uniformly. By Lemma 3B.1 then, $\mathsf{B} \subseteq$ HYP, uniformly. To prove HYP $\subseteq \mathsf{B}$, we need to verify that *every effective σ-algebra on* $\mathbb{N}$ *is uniformly closed under the jump operation, relative recursion and diagonalization*, which is not difficult as these operations can be effectively reduced to complementation and the taking of recursive unions; we then use effective grounded recursion to define a uniform embedding of HYP into B. $\dashv$

Remark 3B.3. The theorem gives us a different view of hyperarithmetical sets and a simpler way to prove important properties of them which do not explicitly refer to the H_a-sets, and these include most of the important properties of HYP. I am not certain who should be credited for it: it was “in the air” in the mid-sixties and I think that it was probably first formulated by Shoenfield, but I cannot find now a specific citation. In any case, it was certainly not known in the 50s, and our use of it here is the most substantial anachronism in this exposition of what was proved then.

3C. Lebesgue [1905] and Mostowski [1951]. The situation is actually quite similar to one that came up in classical analysis at the turn of the last century. Recall the definition of Borel subsets of $\mathcal{N}$ in (2) of Section 1D. In modern notation, the *Borel hierarchy* $\{\utilde{\mathbf{\Sigma}}^0_\xi : \xi < \Omega_1\}$ (on $\mathcal{N}$) is defined by setting

$$(26) \qquad \utilde{\mathbf{\Sigma}}^0_1 = \text{the collection of all open subsets of } \mathcal{N}$$

and then by recursion on the countable ordinals,

$$(27) \qquad A \in \utilde{\mathbf{\Sigma}}^0_\xi \iff A = \bigcup_i (\mathcal{N} \setminus A_i) \text{ with each } A_i \in \bigcup_{\eta<\xi} \utilde{\mathbf{\Sigma}}^0_\eta \quad (\xi > 1).$$

These definitions were first given (for the reals) by Lebesgue [1905] who proved (among many other fascinating and much deeper things) that

$$(28) \qquad \mathbf{B} = \bigcup_{\xi<\Omega_1} \utilde{\mathbf{\Sigma}}^0_\xi.$$

As it happens, most of the important applications of the Borel sets to analysis (including measure theory and integration) use only the definitions and (28), which is easy and handy for proving properties of Borel sets by ordinal induction. The fine structure of the Borel hierarchy is a very interesting and much-studied topic but not as fundamental as $\mathbf{B}$.

The definition of hyperarithmetical sets in Mostowski [1951] is inspired by the classical theory of Borel sets, although he does not cite Lebesgue [1905] or any other "classical" work. It is a difficult paper to read, basically an outline: he appears to define his hierarchy directly on ordinals rather than notations (which is not possible with the tools he uses) and he refers cryptically to (what must be) effective grounded recursion as "*a rather developed technique which we do not wish to presuppose here*". Kleene [1955a, Section 9] supplies the details which are needed to make Mostowski's construction rigorous and comes up with a precise characterization of the intended hierarchy: in modern notation

$$(29) \qquad \Sigma_a = \{A \subseteq \mathbb{N} : A \text{ is r.e. in } H_a\} = \{A \subseteq \mathbb{N} : A \leq_1 H_{2^a}\} \quad (a \in S_1).$$

It is immediate from the definition that

$$\text{if } 1 \leq |a| = k < \omega, \text{ then } \Sigma_a = \Sigma^0_k.$$

Moreover, Σ_a depends only on the ordinal $|a|$ by the Spector Uniqueness Theorem 2B.1 and

$$(30) \qquad \mathsf{HYP} = \bigcup_{a\in S_1} \Sigma_a.$$

The hierarchy $\{\Sigma_a : a \in S_1\}$ has been studied even less that the Borel hierarchy $\{\utilde{\mathbf{\Sigma}}_\xi : \xi < \Omega_1\}$, partly because the topic is not easy. It is obvious that it is a hierarchy, since every Σ_a has a complete set (H_{2^a}); but to prove (for example) that *every* Σ_a *is closed under conjunction* you must use effective grounded recursion, and for more difficult questions these proofs become very complex. In any case, we will not work with it here: for what we will do, the identification $\mathsf{HYP} = \mathsf{B}$ suffices and yields simpler proofs.

3D. The analytical hierarchy; HYP $\subseteq \Delta^1_1$**.** Useful and natural as the characterization HYP $=$ B may be, it does not provide explicit definitions for the hyperarithmetical sets and relations. These require quantification over sets of natural numbers or, equivalently, the Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$.

A relation $P(\vec{x}, \vec{\beta})$ with arguments in $\mathbb{N}$ and (possibly) $\mathcal{N}$ is *analytical* if it is first-order definable in the two-sorted structure of *analysis*

(31) $$\mathbf{N}^2 = (\mathbb{N}, \mathcal{N}, 0, 1, +, \cdot, \mathrm{ap})$$

where $\mathrm{ap}(\alpha, t) = \alpha(t)$ is the *application* operation. Kleene [1955a] classifies the arithmetical and analytical relations with arguments in $\mathbb{N}$ and $\mathcal{N}$ in hierarchies which look so much like the arithmetical hierarchy over $\mathbb{N}$ that we pictured them together in Figure 1. We are mostly interested here in the "first level" of the analytical hierarchy, the *pointclasses*[18] $\Pi^1_1, \Sigma^1_1, \Delta^1_1$, but it is almost as easy to define them all. Briefly, and using the notions and notation in the Appendix:

(1) $P(\vec{x}, \vec{\beta})$ with $\vec{x} = (x_1, \dots, x_n) \in \mathbb{N}^n$ and $\vec{\beta} = (\beta_1, \dots, \beta_m) \in \mathcal{N}^m$ is Σ^0_1 if it is the domain of convergence of a recursive partial function,

$$P(\vec{x}, \vec{\beta}) \iff f(\vec{x}, \vec{\beta})\downarrow;$$

P is Π^0_k if it is the negation of a Σ^0_k relation; P is Σ^0_{k+1} if

$$P(\vec{x}, \vec{\beta}) \iff (\exists t) Q(\vec{x}, t, \vec{\beta}) \quad \text{with } Q \text{ in } \Pi^0_k;$$

and $\Delta^0_k = \Sigma^0_k \cap \Pi^0_k$.

These are the arithmetical relations with arguments in $\mathbb{N}$ and $\mathcal{N}$, those which can be defined in $\mathbf{N}^2$ without using quantification over $\mathcal{N}$.

(2) $P(\vec{x}, \vec{\beta})$ is Π^1_1 if

(32) $$P(\vec{x}, \vec{\beta}) \iff (\forall \alpha) Q(\vec{x}, \vec{\beta}, \alpha),$$

with arithmetical $Q(\vec{x}, \vec{\beta}, \alpha)$; it is Π^1_k if it is the negation of a Σ^1_k relation; and it is Σ^1_{k+1} if

$$P(\vec{x}, \vec{\beta}) \iff (\exists \alpha) Q(\vec{x}, \vec{\beta}, \alpha) \quad \text{with } Q \text{ in } \Pi^1_k;$$

and $\Delta^1_k = \Sigma^1_k \cap \Pi^1_k$.

The analytical pointclasses $\Pi^1_k, \Sigma^1_k, \Delta^1_k$ have all the closure properties of their analogs $\Pi^0_k, \Sigma^0_k, \Delta^0_k$ in the arithmetical hierarchy over $\mathbb{N}$, and they are also closed under number quantification of both kinds and under substitution of total recursive functions into $\mathbb{N}$ or $\mathcal{N}$, App 8. In addition, Π^1_k is closed under $\forall \alpha$ and Σ^1_k is closed under $\exists \alpha$. These closure properties are very easy to prove, but not without consequence:[19]

Lemma 3D.1. *The codeset B of* B $=$ HYP *defined in* (25) *is* Π^1_1.

[18]A *pointclass* in this paper is any collection Γ of relations $P(\vec{x}, \vec{\alpha})$ with arguments in $\mathbb{N}$ and $\mathcal{N}$. It is an awkward term but useful, and is has been well established since the 70s for collections of relations in various spaces typically specified by the context.

[19]They also suffice to prove that the notation system S_1 *is* Π^1_1, cf. Lemma 1 in the proof of Theorem 9.2 in Moschovakis [2010b].

PROOF. By its definition, B is the least fixed point $\overline{\Phi}$ of the monotone operator $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by

$$(33) \quad x \in \Phi(A) \iff (\exists t)[x = \langle 1, t\rangle] \vee (\exists y)[x = \langle 2, y\rangle \ \&\ y \in A] \\ \vee (\exists e)[x = \langle 3, e\rangle \ \&\ (\forall i)(\exists w)[\varphi_e(i) = w \ \&\ w \in A]]$$

so that by (59),

$$i \in B \iff (\forall A)[[(\forall x)[x \in \Phi(A) \Longrightarrow x \in A]] \Longrightarrow i \in A].$$

If we code each set A by the 0-*set* $Z_\alpha = \{x : \alpha(x) = 0\}$ of some $\alpha \in \mathcal{N}$ and set

$$(34) \qquad \Phi(x, \alpha) \iff x \in \Phi(Z_\alpha),$$

then $\Phi(x, \alpha)$ is arithmetical (just replace $u \in A$ by $\alpha(u) = 0$ in (33)); and

$$i \in B \iff (\forall \alpha)[[(\forall x)[\Phi(x, \alpha) \Longrightarrow \alpha(x) = 0]] \Longrightarrow \alpha(i) = 0],$$

so that B is Π^1_1. ⊣

This is a very general method of proof: it can be used to show that *if* Φ *is monotone on* $\mathcal{P}(\mathbb{N})$ *and the relation* $\Phi(x, \alpha)$ *associated with* Φ *by* (34) *is* Π^1_1*, then* $\overline{\Phi}$ *is* Π^1_1 and, of course, it can be generalized in many ways.

Much of the theory of Π^1_1 depends on the following refinement of its definition (32):

Theorem 3D.2 (Normal Form for Π^1_1). *Every* Π^1_1 *relation* $P(\vec{x}, \vec{\beta})$ *satisfies an equivalence*

$$(35) \qquad P(\vec{x}, \vec{\beta}) \iff (\forall \alpha)(\exists t)R(\vec{x}, \vec{\beta}, \overline{\alpha}(t))$$

where $R(\vec{x}, \vec{\beta}, u)$ *is recursive and monotone upward on its last* (sequence code) *argument, i.e.*,

$$(36) \qquad [R(\vec{x}, \vec{\beta}, u) \ \&\ u \sqsubseteq v] \Longrightarrow R(\vec{x}, \vec{\beta}, v).$$

It is easy to prove, using the closure properties of Π^1_1, the somewhat unusual "dual" of the Axiom of Choice, that for every relation $R(t, s)$,

$$(\exists t)(\forall s)R(t, s) \iff (\forall \alpha)(\exists t)R(t, \alpha(t))$$

and the Normal Form Theorem for recursive partial functions, App 5. By App 5 again, it implies the analog of (2) in Lemma 1B.1:

Lemma 3D.3 ($\mathbb{N}$-Parametrization for Π^1_1). *For all* $n, m \geq 0$*, there is a* Π^1_1 *relation*

$$G(e, \vec{x}, \vec{\beta}) \iff G^{n,m}(e, x_1, \dots, x_n, \beta_1, \dots, \beta_m)$$

such that for every Π^1_1 *relation* $P(\vec{x}, \vec{\beta})$,

$$(37) \qquad P(\vec{x}, \vec{\beta}) \iff G(e, \vec{x}, \vec{\beta}) \text{ for some } e \in \mathbb{N};$$

moreover, there are recursive injections $S^l_n : \mathbb{N}^{1+l} \to \mathbb{N}$ *such that for all tuples* $\vec{y} = y_1, \dots, y_l \in \mathbb{N}$,

$$(38) \qquad G^{l+n,m}(e, \vec{y}, \vec{x}, \vec{\beta}) \iff G^{n,m}(S^l_n(e, \vec{y}), \vec{x}, \vec{\beta}).$$

When (37) holds, we call e a Π^1_1-*code* of $P(\vec{x}, \vec{\beta})$ and a Σ^1_1-*code* of its negation $\neg P(\vec{x}, \vec{\beta})$; and if e is a Π^1_1-code and m a Σ^1_1-code of $P(\vec{x}, \vec{\beta})$, then $\langle e, m\rangle$ is a Δ^1_1-*code* of it.

To see how the Parametrization Property is used, suppose $R(\vec{x}, t)$ is a Π^1_1 relation on $\mathbb{N}$ (for simplicity) with code e and

$$P(\vec{x}) \iff (\exists t)R(\vec{x}, t).$$

Let $Q(m, \vec{x}) \iff (\exists t)G(m, \vec{x}, t)$ (with the appropriate superscripts) and let $\overline{s}$ be a Π^1_1-code of Q; then

$$\begin{aligned} P(\vec{x}) \iff (\exists t)R(\vec{x}, t) &\iff (\exists t)G(e, \vec{x}, t) \\ &\iff Q(e, \vec{x}) \iff G(\overline{s}, e, \vec{x}) \iff G(S^1_1(\overline{s}, e), \vec{x}), \end{aligned}$$

so $S^1_1(\overline{s}, e)$ is a code of $P(\vec{x})$. The upshot is that Π^1_1 is *uniformly closed under* $\exists s$, and by similar, trivial computations, Π^1_1, Σ^1_1 and Δ^1_1 are uniformly closed under all (reasonable) operations under which they are closed, including those listed above. This implies that *the collection of* Δ^1_1 *subsets of* $\mathbb{N}$ *is an effective* σ*-algebra on* $\mathbb{N}$, which with Lemma 3B.1 then yields

Theorem 3D.4 (Kleene [1955a]). HYP $\subseteq \Delta^1_1$, *uniformly. In detail, there are relations* $H_\Sigma(i, x)$ *and* $H_\Pi(i, x)$ *in* Σ^1_1 *and* Π^1_1 *respectively, such that*

$$i \in B \Longrightarrow \Big(x \in B_i \iff H_\Sigma(i, x) \iff H_\Pi(i, x)\Big).$$

Davis [1950a] and Mostowski [1951] had already shown that every HYP-relation is analytical, but Kleene's result is a considerable improvement and begs for the converse.

3E. Kleene's Theorem, HYP $= \Delta^1_1$**.** This was the most important, early result about HYP and it is still the most fundamental.

Theorem 3E.1 (Kleene [1955c]). $\Delta^1_1 \subseteq$ HYP, *uniformly, so* HYP $= \Delta^1_1$.

The foundational import of Kleene's Theorem is that it reduces existential quantification $(\exists\alpha)$ over the continuum $\mathcal{N}$ to regimented iteration of first-order quantification over $\mathbb{N}$—in the very special circumstances where a set A and its complement can both be defined by just one such quantification on arithmetical relations.

There are many proofs of Kleene's Theorem, all of them ultimately based on the Normal Form Theorem 3D.2 for Π^1_1 and using effective grounded recursion. The proof in Kleene [1955c] is quite complex and depends on several technical results about constructive ordinals and the H_a-sets. To outline briefly the much simpler argument in Spector [1955], put first

$$x \leq_f y \iff \varphi_f(x, y) = 0, \quad L = \{f : \varphi_f \text{ is total and } \leq_f \text{ is a linear order}\},$$
$$W = \{f \in L : \leq_f \text{ is a wellordering}\},$$
$$||f|| = \text{the order type of } \leq_f \quad (f \in W).$$

By Markwald's Theorem 2A.1, $\{||f|| : f \in W\}$ is exactly the set constructive ordinals, and we set

$$W_\xi = \{f \in W : ||f|| \leq \xi\} \quad (\xi < \omega_1).$$

The first move is to check that the initial segments $\{f : ||f|| \leq ||s||\}$ of W are uniformly Δ^1_1 for $s \in W$:

Lemma 3E.2. *There are binary relations $\leq_\Sigma$ and $\leq_\Pi$ in Σ^1_1 and Π^1_1 respectively, such that*

$$s \in W \Longrightarrow \Big([f \in W \;\&\; ||f|| \leq ||s||] \iff f \leq_\Sigma s \iff f \leq_\Pi s\Big).$$

Proof. Set

$$f \leq_\Sigma s \iff f, s \in L \;\&\; \text{there is an order-preserving embedding of } \leq_f \text{ into } \leq_s,$$
$$f \leq_\Pi s \iff f, s \in L \;\&\; \text{there is no order preserving embedding of } \leq_s \text{ into a proper initial segment of } \leq_f.$$

To verify that these relations do it, we code embeddings using elements of Baire space and use the closure properties of Σ^1_1 and Π^1_1. ⊣

The second move introduces what is now called the *Kleene-Brouwer* or *Luzin-Sierpinski* ordering on finite sequences. It is used in Kleene [1955c] and in many proofs of Kleene's Theorem:

Lemma 3E.3 (Spector [1955]). *W is Π^1_1-complete, uniformly.*

In detail: W is Π^1_1 and there is a recursive function $u_1(a)$ such that if a is a Π^1_1-code of a set $A \subseteq \mathbb{N}$, then $\varphi_{u_1(a)}$ is injective and

$$x \in A \iff \{u_1(a)\}(x) \in W.$$

Proof. W is Π^1_1 directly from its definition. To show that it is Π^1_1-complete, suppose that A is Π^1_1 with code a, so that by Theorem 3D.2 and Lemma 3D.3,

$$x \in A \iff G(a, x) \iff (\forall \alpha)(\exists t) R(a, x, \overline{\alpha}(t))$$

with a fixed recursive $R(a, x, v)$ (not depending on A) which is monotone upward in its last argument. Define the transitive relation

$$u \preceq^{a,x} v \iff v \sqsubseteq u \;\&\; \neg R(a, x, u)$$

and prove that

(39) $$x \in A \iff (\forall \alpha)(\exists t) R(a, x, \overline{\alpha}(t)) \iff \preceq^{a,x} \text{ is wellfounded},$$

most easily by checking its contrapositive

$$x \notin A \iff (\exists \alpha)(\forall t) \neg R(a, x, \overline{\alpha}(t)) \iff \preceq^{a,x} \text{ is not wellfounded}.$$

We then *linearize* $\preceq^{a,x}$ by setting

$$u \leq^{a,x} v \iff \neg R(a, x, u) \;\&\; \neg R(a, x, v)$$
$$\& \Big(v \sqsubseteq u \vee [u \mid v \;\&\; \min\{(u)_i : i < \mathrm{lh}(u)\} < \min\{(v)_i : i < \mathrm{lh}(v)\}]\Big);$$

verify that this is a linear ordering such that

$$x \in A \iff \leq^{a,x} \text{ is a wellordering},$$

in fact $\leq^{a,x} = \leq_{g(a,x)}$ with a recursive g such that for any a, x, $g(a, x) \in L$; and infer that

(40) $$x \in A \iff \leq^{a,x} \text{ is a wellordering} \iff g(a, x) \in W.$$

To finish the proof we need to define a recursive $u_1(a)$ such that $\{g(a,x)\}(s) = \{\{u_1(a)\}(x)\}(s)$ and $\{u_1(a)\}$ is injective for every a, and this is done by manipulating the S^l_n- functions as usual. ⊣

The third move is Spector's. It is what makes his proof simpler than Kleene's who worked with O rather than W.

Lemma 3E.4 (Boundedness, Spector [1955]). *Every Σ^1_1 subset of W is a subset of W_ξ for some $\xi < \omega_1$, uniformly.*

In detail: there is a recursive partial function $u_2(b)$ such that if b is a Σ^1_1-code of a set $A \subseteq \mathbb{N}$, then

$$A \subseteq W \Longrightarrow [u_2(b)\downarrow, u_2(b) \in W, \text{ and } A \subseteq W_{||u_2(b)||}].$$

Proof. Let $G(b,x)$ be a parametrization of the unary Π^1_1 relations by Lemma 3D.3, so that a set $A \subseteq \mathbb{N}$ is Σ^1_1 with code b if

$$A = G^c_b = \{s : \neg G(b,s)\}.$$

Fix also by the Π^1_1-completeness of W a recursive injection $g : \mathbb{N} \to \mathbb{N}$ such that

$$(*) \qquad G(x,x) \iff g(x) \in W.$$

The relation

$$P(b,f) \iff (\exists s)[\neg G(b,s) \,\&\, g(f) \leq_\Sigma s]$$

is Σ^1_1, and so by Lemma 3D.3 again, there is a recursive injection $v(b) = S^2_1(\overline{k}, b)$ (with some $\overline{k}$) such that

$$(**) \qquad (\exists s)[\neg G(b,s) \,\&\, g(f) \leq_\Sigma s] \iff \neg G(v(b), f).$$

The key observation is that

$$\textit{if } A = G^c_b \subseteq W, \textit{ then } G(v(b), v(b)) :$$

because if $A \subseteq W$ and $\neg G(v(b), v(b))$, then there is some $s \in W$ such that $g(v(b)) \leq_\Sigma s$; which gives $g(v(b)) \in W$ by Lemma 3E.2; which in turn gives $G(v(b), v(b))$ by $(*)$, contradicting the hypothesis. From $G(v(b), v(b))$ we get $g(v(b)) \in W$, by $(*)$ again, and so by taking negations in $(**)$,

$$A = G^c_b \subseteq W \Longrightarrow (\forall s)[s \in A \Longrightarrow ||s|| < ||g(v(b))||],$$

which is what we needed to show with $u_2(b) = g(v(b.b))$. ⊣

Outline of proof of Theorem 3E.1. By the two lemmas, if A is Δ^1_1 with code $\langle a, b\rangle$, then

$$(41) \qquad x \in A \iff \{u_1(a)\}(x) \in W_\xi \text{ with } \xi = ||u_2(b)||.$$

To complete the proof we need to show that $W_{||f||}$ is in B *uniformly for* $f \in W$, and this is done by a fairly straightforward effective grounded recursion along $\{(f,g) : f, g \in W \,\&\, ||f|| \leq ||g||\}$. ⊣

Spector's write-up of his proof is not quite this simple because he works with the H_a-codes rather than the B-codes of HYP and (in effect) proves the uniform inclusion $\mathsf{B} \subseteq \mathsf{HYP}$ on the fly.

Moreover, neither Kleene nor Spector claimed explicitly the full, uniform version of Kleene's Theorem 3E.1, although all the "mathematical facts" needed for it are in their papers.[20] Most likely they did not even think of it: in the spirit of the time, a result was formulated uniformly only when this was necessary, typically in order to prove it by effective grounded recursion. Uniform claims did not become important in themselves until the 70s, when the applications of these ideas to Descriptive Set Theory made them necessary. We will discuss this briefly in Section 4B.

Spector's Lemmas 3E.2 – 3E.4 are important results with many applications besides their use in proving Kleene's Theorem. We state one of them here and then one more, not quite so simple in the next section.[21]

Theorem 3E.5 (Spector [1955]). *If* $\preceq$ *is a* Δ^1_1 *wellordering with field* $F \subseteq \mathbb{N}$, *then* $\mathrm{rank}(\preceq) < \omega_1$, *uniformly.*

This is usually abbreviated by the equation

$$\delta^1_1 = \omega_1,$$

δ^1_1 being the least ordinal which is not the order type of a HYP wellordering.

PROOF. Suppose, towards a contradiction that $\preceq$ is a Δ^1_1 wellordering with $\mathrm{rank}(\preceq) \geq \omega_1$ and set

$$f \in A \iff f \in L \;\&\; \text{there is an order preserving map of } \leq_f \text{ into } \preceq.$$

This is a Σ^1_1 set and the hypotheses imply that $A = W$, which contradicts Lemma 3E.4. The uniform version is proved similarly, using the uniform version of the same Lemma. ⊣

3F. Addison [1959] and the revised analogies. Kleene's Theorem 3E.1 is an immediate consequence of the following more general

Theorem 3F.1 (Strong Separation for Σ^1_1, Addison [1959]). *For any two disjoint,* Σ^1_1 *subsets of* $\mathbb{N}$, *there is a* HYP *set* C *which separates them, i.e.,*

$$A \subseteq C, \quad C \cap B = \emptyset.$$

In fact, Addison [1959] claims more and less than this result: he states it for subsets A, B of any product space $\mathbb{N}^n \times \mathcal{N}^m$ rather than just $\mathbb{N}$ and his (abbreviated) proof is formulated quite abstractly and also gives the classical Separation Theorem 1D.2 for $\utilde{\boldsymbol{\Sigma}}^1_1$; but he does not note that the result holds uniformly (in given Σ^1_1-codes of A and B), which it does, and he only says of the separating set C that "it is Δ^1_1" skipping the punchline "and hence HYP" which he certainly knows for subsets of $\mathbb{N}$. This may be partly because there was no generally accepted definition of HYP *subset of* $\mathbb{N}^n \times \mathcal{N}^m$ at the time, or because Addison's paper is about separation and not construction principles. He also does not discuss the obvious revision of the analogies (7)

$$\begin{aligned} \text{recursive function on } \mathbb{N} \quad &\sim \quad \text{continuous function on } \mathcal{N}, \\ \mathsf{HYP} \quad &\sim \quad \mathbf{B}, \\ \Pi^1_1 \text{ sets of integers} \quad &\sim \quad \utilde{\boldsymbol{\Pi}}^1_1 \text{ subsets of } \mathcal{N}, \end{aligned} \tag{42}$$

[20] What's missing in their papers is the second part in the proof of the Boundedness Lemma 3E.4 which looks tricky at first sight but is a standard, elementary tool in this area. It computes "witnesses to counterexamples" using diagonalization in very general circumstances, and we have already used it to establish the uniform properties of the jump in Footnote 6.

[21] Cf. App 9 for the notation we use about wellorderings and ranks.

which are the working hypotheses of Mostowski [1951]. They are bolstered by the following result which is not hard to prove using Spector-type ordinal assignments and the method of proof of Kleene's Theorem 1D.3:

Theorem 3F.2. *There exists disjoint* Π^1_1*-sets* A, B *which are not* HYP*-separable, i.e., no* HYP *set* C *satisfies*

$$A \subseteq C, \quad C \cap B = \emptyset.$$

On the other hand, to my knowledge, Addison [1959] was first to refer to *Effective Descriptive Set Theory*, which suggests that more than "analogies" are in play; and he introduced the modern *lightface* $\Sigma^1_k, \ldots$ and *boldface* $\utilde{\mathbf{\Sigma}}^1_k \ldots$ notation which has been universally accepted.

3G. Relativization and the Kreisel Uniformization Theorem. We mention in App 6 the method of *proof by relativization*, which works because (roughly) recursion in some fixed parameters $\vec{\beta}$ has all the properties of "absolute" recursion. It is not simple to formulate a general metatheorem which captures all its applications—especially when uniformities are involved which should be "absolutely" recursive. It is, however, a very powerful method, heavily used by the early researchers in hyp theory, especially Kleene and Spector. We illustrate it here by proving two important and useful results.

The relative forms $\Sigma^{i,\vec{\beta}}_k, \Pi^{i,\vec{\beta}}_k, \Sigma^{i,\vec{\beta}}_k, \Delta^{i,\vec{\beta}}_k$ of the arithmetical and analytical hierarchies are defined simply by replacing "recursive" by "recursive in $\vec{\beta}$" in their definitions, and they have all the properties of their absolute forms, including Lemma 1B.1 (with absolutely recursive $S^{l,m}_n$ functions).

The same is true for the relativized system $\mathrm{S}^{\vec{\beta}}_1$ of ordinal notations: we simply replace e_t in Section 2A by $e^{\vec{\beta}}_t = \{e\}^{\vec{\beta}}(t_O)$ and write $|a|^{\vec{\beta}}$ for the ordinal with code $a \in S^{\vec{\beta}}_1$. Markwald's Theorem 2A.1 remains true: an ordinal ξ is less than

$$\omega^{\vec{\beta}}_1 = \sup\{|a|^{\vec{\beta}} : a \in S^{\vec{\beta}}_1\}$$

exactly when it is the order type of a wellordering (of part of $\mathbb{N}$) which is recursive in $\vec{\beta}$. We use these ordinals to define the relativized $H^{\vec{\beta}}_a$ sets by replacing (H2) in Section 2B by

(H2$\vec{\beta}$) $H^{\vec{\beta}}_{2^b} = \mathrm{jump}(H^{\vec{\beta}}_b; \vec{\beta}) = \{e : \{e\}(e, H^{\vec{\beta}}_b, \vec{\beta})\downarrow\}$

and we set

$$A \in \mathsf{HYP}^{\vec{\beta}} \iff (\exists a \in S^{\vec{\beta}}_1)[A \leq_T H^{\vec{\beta}}_a].$$

With these definitions, all the basic facts about HYP relativize, including Spector's Uniqueness Theorem 2B.1, the characterization of $\mathsf{HYP}^{\vec{\beta}}$ as the least σ-algebra on $\mathbb{N}$ which is *effective in* $\vec{\beta}$, Theorem 3B.2 and the uniform inclusion $\mathsf{HYP}^{\vec{\beta}} \subseteq \Delta^{1,\vec{\beta}}_1$, Theorem 3D.4. For the converse inclusion (Kleene's Theorem), we need to relativize the basic notions of Spector [1955]: we set

$$x \leq_f^{\vec\beta} y \iff \varphi_f(x, y, \vec\beta) = 0,$$
$$L^{\vec\beta} = \{f : (\forall x, y)[\varphi_f(x, y, \vec\beta)\downarrow] \text{ and } \leq_f^{\vec\beta} \text{ is a linear order}\},$$
$$W^{\vec\beta} = \{f \in L^{\vec\beta} : \leq_f^{\vec\beta} \text{ is a wellordering}\},$$
$$||f||^{\vec\beta} = \text{the order type of } \leq_f^{\vec\beta} \quad (f \in W^{\vec\beta}).$$

Using these we get immediately the relativized versions of Lemma 3E.2 and (what we need of) the relativized version of Lemma 3E.3, basically (40):

(1) *There are relations* $\leq_\Sigma^{\vec\beta}$ *and* $\leq_\Pi^{\vec\beta}$ *in* Σ_1^1 *and* Π_1^1 *respectively, such that for all* $\vec\beta$,

$$s \in W^{\vec\beta} \Longrightarrow \Big([f \in W^{\vec\beta} \;\&\; ||f||^{\vec\beta} \leq ||s||^{\vec\beta}] \iff f \leq_\Sigma^{\vec\beta} s \iff f \leq_\Pi^{\vec\beta} s\Big)\Big).$$

(2) *If* $P(\vec x, \vec\beta)$ *is* Π_1^1, *then there is a total recursive function* $f(\vec x)$ *such that*

$$P(\vec x, \vec\beta) \iff f(\vec x) \in W^{\vec\beta}.$$

These suffice to relativize Spector's proof of the non-uniform version of Kleene's Theorem 3E.1

$$\text{for every } \vec\beta,\ \mathsf{HYP}^{\vec\beta} = \Delta_1^{1,\vec\beta},$$

and a little more detailed version of (2) gives also the uniform version.

With single sets rather than tuples of functions $\vec\beta$, for simplicity, we set

$$A \leq_h B \iff A \in \mathsf{HYP}^B \iff A \text{ is hyperarithmetical in } B.$$

The *hyperdegrees* that are induced by this reducibility have been studied extensively, cf. Sacks [1990]. We will not go into this topic here, except for the following, early and important result. To appreciate what it says, notice that because W is Π_1^1-complete,

$$W \leq_h A \iff \text{every } \Pi_1^1 \text{ set is hyperarithmetical in } A.$$

Theorem 3G.1 (Spector [1955]). *For every set* $A \subseteq \mathbb{N}$,

$$W \leq_h A \iff \omega_1 < \omega_1^A,$$

and in relativized form, for all $A, B \subseteq \mathbb{N}$,

$$W^A \leq_h B \iff \omega_1^A < \omega_1^B.$$

Proof. Suppose first that $W \leq_h A$ and set

$$x \in D \iff \Big(x \in W \;\&\; (\forall y)[(y \in W \;\&\; ||y|| = ||x||) \Longrightarrow x \leq y\Big),$$
$$x \preceq y \iff x, y \in D \;\&\; ||x|| \leq ||y||;$$

now $\preceq$ is a wellordering of rank ω_1 and it is $\Delta_1^{1,A}$, so its rank is below $\delta_1^{1,A} = \omega_1^A$ by the relativized version of Spector's Theorem 3E.5.

The converse is a bit easier. ⊣

Our second example illustrates a somewhat more subtle application of the relativization technique: roughly, it proves a universal property $(\forall\vec{\beta})Q(\vec{\beta})$ by treating an arbitrary tuple $\vec{\beta}$ as a parameter, relativizing to it the proof of a simple (absolute) proposition, and then exploiting the uniform nature of the proof to infer $Q(\vec{\beta})$ with variable $\vec{\beta}$.

Theorem 3G.2 (Π^1_1-Uniformization on $\mathbb{N}$, Kreisel [1962]). *For every Π^1_1 relation $P(\vec{x}, y, \vec{\beta})$, there is a Π^1_1 relation $P^*(\vec{x}, y, \vec{\beta})$ such that*

$$P^*(\vec{x}, y, \vec{\beta}) \Longrightarrow P(\vec{x}, y, \vec{\beta}) \text{ and } (\exists y)P(\vec{x}, y, \vec{\beta}) \Longrightarrow (\exists!y)P^*(x, y, \vec{\beta}). \tag{43}$$

It follows that if $P(\vec{x}, y)$ is Π^1_1, then

$$(\forall\vec{x})(\exists y)P(\vec{x}, y) \Longrightarrow (\exists f : \mathbb{N}^n \to \mathbb{N})[f \text{ is } \mathsf{HYP} \ \& \ (\forall\vec{x})P(\vec{x}, f(\vec{x}))].$$

Proof. In the simple case where the list $\vec{\beta}$ of variables over $\mathcal{N}$ is empty, we choose a recursive $g : \mathbb{N}^n \to \mathbb{N}$ such that $P(\vec{x}, y) \iff g(\vec{x}, y) \in W$ and set

$$\begin{aligned} P^*(\vec{x}, y) \iff P(\vec{x}, y) \ \& \ (\forall u)[g(\vec{x}, y) \leq_\Pi g(\vec{x}, u)] \\ \& \ (\forall u)[g(\vec{x}, u) \leq_\Sigma g(\vec{x}, y) \Longrightarrow y \leq u]. \end{aligned}$$

This also gives the second claim: check that if $(\forall\vec{x})(\exists y)P(\vec{x}, y)$, then $P^*(\vec{x}, y)$ is the graph of a function f and it is Δ^1_1, since

$$\neg P^*(\vec{x}, y) \iff (\exists z)[P^*(\vec{x}, z) \ \& \ z \neq y)].$$

To get the more useful claim with parameters, we relativize this argument using (1) and (2) above. Given a Π^1_1 relation $P(\vec{x}, y, \vec{\beta})$, choose a recursive $g(\vec{x}, y)$ such that

$$P(\vec{x}, y, \vec{\beta}) \iff g(\vec{x}, y) \in W^{\vec{\beta}}$$

and set

$$\begin{aligned} P^*(\vec{x}, y, \vec{\beta}) \iff P(\vec{x}, y, \vec{\beta}) \ \& \ (\forall u)[g(\vec{x}, y) \leq^{\vec{\beta}}_\Pi g(\vec{x}, u)] \\ \& \ (\forall u)[g(\vec{x}, u) \leq^{\vec{\beta}}_\Sigma g(\vec{x}, y) \Longrightarrow y \leq u]. \end{aligned}$$

The check that this works is exactly as before. ⊣

The *Kondo-Addison Uniformization Theorem* for Π^1_1 relations $P(\vec{x}, \alpha, \vec{\beta})$ (Kondo [1938], Addison) is much deeper, but this simple result is also interesting and very useful.

3H. HYP**-quantification and the Spector-Gandy Theorem.** The (coded) *graph* of a function $\alpha : \mathbb{N} \to \mathbb{N}$ is the set

$$\mathrm{Graph}(\alpha) = \{\langle s, t\rangle : \alpha(s) = t\} \subset \mathbb{N},$$

and we often write "$\alpha \in \mathsf{HYP}$" when we really mean "$\mathrm{Graph}(\alpha) \in \mathsf{HYP}$", i.e., that α is hyperarithmetical. We collect here some interesting, easy (now) facts about the quantifier $(\exists\alpha \in \mathsf{HYP})$ and we also formulate the basic *Spector-Gandy Theorem* about it—which has never been easy.

It is natural to code the HYP-functions using a subset of the coding of HYP-sets as effectively Borel in (25):

$$B^1 = \{i \in B : B_i = \text{Graph}(\alpha) \text{ for some } \alpha\}, \quad \text{and if } i \in B^1, \text{ then } \beta_i(s) = t \iff \langle s, t\rangle \in B_i. \tag{44}$$

The key (easy) facts about this coding is that B^1 *is* Π^1_1 by Lemma 3D.1 and Theorem 3D.4, and that *for each* $i \in B^1$*, the relation*

$$\alpha = \beta_i \iff (\forall s, t)[\alpha(s) = t \iff \langle s, t\rangle \in B_i] \tag{45}$$

is Δ^1_1 *uniformly*, by Theorem 3D.4 again.

Theorem 3H.1. (1) HYP-*Quantification Theorem*, Kleene [1955c], [1959a]. *If*

$$P(\vec{x}) \iff (\exists \alpha \in \mathsf{HYP})Q(\vec{x}, \alpha) \tag{46}$$

and $Q(\vec{x}, \alpha)$ *is* Π^1_1, *then* $P(\vec{x})$ *is also* Π^1_1.

(2) HYP *is not a basis for* Π^0_1, Kleene [1955c]. *There is a non-empty,* Π^0_1 *set* $A \subseteq \mathcal{N}$ *which has no* HYP *members.*

(3) *Upper classification of* HYP. *As a subset of* $\mathcal{N}$, HYP *is* Π^1_1.

(4) *Lower classification of* HYP. *As a subset of* $\mathcal{N}$, HYP *is not* Σ^1_1.

Proof. (1) Compute:

$$(\exists \alpha \in \mathsf{HYP})Q(\vec{x}, \alpha) \iff (\exists i)[i \in B^1 \;\&\; (\forall \alpha)[\alpha = \beta_i \Longrightarrow Q(\vec{x}, \alpha)]].$$

(2) Towards a contradiction, assume that every non-empty, Π^0_1 set $A \subseteq \mathcal{N}$ has a HYP member and let $P \subseteq \mathbb{N}$ be an arbitrary Σ^1_1 set. By the Normal Form for Π^1_1 Theorem 3D.2 (applied to $\neg P$),

$$P(x) \iff (\exists \alpha)(\forall t)R(x, t, \alpha) \iff A_x = \{\alpha : (\forall t)R(x, t, \alpha)\} \neq \emptyset$$

with a recursive R. Since every A_x is Π^0_1, our assumption implies that

$$P(x) \iff (\exists \alpha \in \mathsf{HYP})(\forall t)R(x, t, \alpha);$$

which by (1) means that every Σ^1_1 subset of $\mathbb{N}$ is Π^1_1, which it is not.

(3) $\alpha \in \mathsf{HYP} \iff (\exists i)[i \in B^1 \;\&\; \alpha = \beta_i]$.

(4) The relation $P(i, \alpha) \iff i \in B^1 \;\&\; \alpha = \beta_i$ is Π^1_1, so by the Kreisel Uniformization Theorem 3G.2, there is a Π^1_1 relation $P^*(i, \alpha)$ such that

$$P^*(i, \alpha) \Longrightarrow i \in B^1 \;\&\; \alpha = \beta_i, \quad \alpha \in \mathsf{HYP} \Longrightarrow (\exists! i)P^*(i, \alpha).$$

Let $D(i) \iff (\exists \alpha \in \mathsf{HYP})P^*(i, \alpha)$. This is Π^1_1 by (1), but if HYP is Σ^1_1, then it is also Σ^1_1, since

$$D(i) \iff (\exists \alpha)[\alpha \in \mathsf{HYP} \;\&\; (\forall j)[P^*(j, \alpha) \Longrightarrow i = j]].$$

It follows that the function

$$\beta(i) = \begin{cases} 1 \dot{-} \beta_i(i) & \text{if } D(i), \\ 0 & \text{otherwise} \end{cases}$$

is Δ^1_1 and has no code in B^1, which is absurd. ⊣

Kleene [1959b] proved Part (1) of this theorem with a Π^0_1 relation $Q(\vec{x}, \alpha)$ and asked whether this version of (46) gives a normal form for Π^1_1. Spector [1960] proved that it does, and Gandy [1960] gave an independent proof of this basic fact after hearing of Spector's result.

Theorem 3H.2 (Spector [1960], Gandy [1960]). *Every Π^1_1 relation P on $\mathbb{N}$ satisfies an equivalence*

$$(47) \qquad P(\vec{x}) \iff (\exists \alpha \in \mathsf{HYP})(\forall t) R(\vec{x}, \overline{\alpha}(t))$$

with a recursive $R(\vec{x}, u)$. In fact, $R(\vec{x}, u)$ can be chosen so that

$$P(\vec{x}) \iff (\exists \alpha \in \mathsf{HYP})(\forall t) R(\vec{x}, \overline{\alpha}(t)) \iff (\exists ! \alpha \in \mathsf{HYP})(\forall t) R(\vec{x}, \overline{\alpha}(t)).$$

Spector's proof is difficult, as is Gandy's, both of them depending on a detailed, combinatorial analysis of Π^1_1 definitions and properties of the constructive ordinals coded by O. Easier proofs and generalizations of the first claim (without the uniqueness) were found later, cf. Moschovakis [1969], [1974], [2009].

Taken together, Kleene's HYP-Quantification and the Spector-Gandy Theorem have important foundational import, perhaps best expressed by the following

Corollary 3H.3 (Kleene, Spector). *A relation $P(\vec{x})$ on $\mathbb{N}$ satisfies*

$$P(\vec{x}) \iff (\forall \alpha) Q_1(\vec{x}, \alpha)$$

with an arithmetical $Q_1(\vec{x}, \alpha)$ if and only if it satisfies

$$P(\vec{x}) \iff (\exists \alpha \in \mathsf{HYP}) Q_2(\vec{x}, \alpha)$$

with an arithmetical $Q_2(\vec{x}, \alpha)$.

Moreover, the equivalence holds *uniformly*, i.e., Q_2 can be constructed from Q_1 and vice versa.

The Corollary reduces *one* quantification over the continuum $\mathcal{N}$ on arithmetical relations to one quantification (of the opposite kind) over the countable set $\mathsf{HYP} \subsetneq \mathcal{N}$ whose members are constructed by regimented iteration of quantification over $\mathbb{N}$.

3I. The Kleene [1959a] HYP hierarchy. This is perhaps the deepest and certainly the most difficult technical work of Kleene on hyperarithmetical sets.[22]

Theorem 3I.1 (Kleene [1959a]). *If the monotone operator Δ on $\mathcal{P}(\mathcal{N})$ is defined by* (52) *below, then*

$$(48) \qquad \eta < \xi < \omega_1 \Longrightarrow \overline{\Delta}^{\eta} \subsetneq \overline{\Delta}^{\xi} \text{ and } \mathsf{HYP} = \textstyle\bigcup_{\xi < \omega_1} \overline{\Delta}^{\xi}.$$

Even without the definition of Δ, a hierarchy of the form $\{\Delta^{\xi} : \xi < \omega_1\}$ on HYP is more satisfactory than hierarchies like (29), because it is constructed without reference to any codings: there is no need for results like Spector's Uniqueness Theorem to establish *coding invariance*. The specific operator Δ that we define next also gives a novel understanding of HYP and yields many interesting applications.

[22] It is also his last paper on the subject.

Definitions with range and basis $\mathcal{F}$. A relation $P(\vec{x})$ is Σ^1_1 *with range* $\mathcal{F} \subseteq \mathcal{P}(\mathcal{N})$ if

$$P(\vec{x}) \iff (\exists \alpha \in \mathcal{F}) Q(\vec{x}, \vec{\beta}, \alpha) \tag{49}$$

with $\vec{\beta} = \beta_1, \dots, \beta_m \in \mathcal{F}$ and an arithmetical Q; it is Σ^1_1 *with basis* $\mathcal{F}$ if

$$P(\vec{x}) \iff (\exists \alpha) Q(\vec{x}, \vec{\beta}, \alpha) \iff (\exists \alpha \in \mathcal{F}) Q(\vec{x}, \vec{\beta}, \alpha) \tag{50}$$

with $\vec{\beta} \in \mathcal{F}$ and an arithmetical Q; and it is Δ^1_1 with range or basis $\mathcal{F}$ if both P and its negation $\neg P$ are Σ^1_1 with range or basis $\mathcal{F}$ respectively.

If $P(\vec{x})$ is Σ^1_1 with basis $\mathcal{F}$, then it is also Σ^1_1 with range $\mathcal{F}$, clearly. The converse is not true: because every Π^1_1 relation is Σ^1_1 with range HYP by the Spector-Gandy Theorem 3H.2, while

$$\textit{if } P(\vec{x}) \textit{ is } \Sigma^1_1 \textit{ with basis } \mathsf{HYP}, \textit{ then } P(\vec{x}) \textit{ is } \mathsf{HYP} \tag{51}$$

by Kleene's HYP-Quantification Theorem 3H.1 (1)—and Theorem 3E.1, of course, the inclusion $\Delta^1_1 \subseteq$ HYP being basic to all this work. We let[23]

$$\Delta(\mathcal{F}) = \{A \subseteq \mathbb{N} : A \text{ is arithmetical or } \Delta^1_1 \text{ with basis } \mathcal{F}\}. \tag{52}$$

It is clear from (51) that $\Delta(\mathsf{HYP}) = \mathsf{HYP}$, and so the least fixed point $\overline{\Delta}$ of Δ is included in HYP. For the rest of (48), Kleene needs to show that

(1) $\mathsf{HYP} \subseteq \overline{\Delta}$, and

(2) if $\eta < \xi < \omega_1$, then $\overline{\Delta}^\eta \subsetneq \overline{\Delta}^\xi$.

For (1), he proves (in effect) that

$$a \in O \Longrightarrow H_a \textit{ is } \Delta^1_1 \textit{ with basis } \bigcup_{|b|<|a|} \Sigma_b$$

with Σ_a defined in (29). The key idea for (2) is to use the *ramified analytical hierarchy* comprising the iterates of the monotone operator

$$\mathrm{An}(\mathcal{F}) = \{A \subseteq \mathbb{N} : \text{for some } n,\ A \text{ is } \Sigma^1_n \text{ with range } \mathcal{F}\}$$

on $\mathcal{P}(\mathcal{N})$. Kleene shows that if $\xi < \omega_1$, then $\overline{\mathrm{An}}^\xi \subseteq \mathsf{HYP}$; and so if $\kappa(\Delta) < \omega_1$, then $\mathsf{HYP} = \overline{\Delta}$ would be a fixed point of An which contradicts the Spector-Gandy Theorem. Both proofs are by effective grounded recursion and require more detailed, delicate formulations of (1) and (2) to go through.

To formulate one of the simplest and most elegant characterizations of HYP that comes out of Theorem 3I.1, recall the two-sorted structure of analysis $\mathbf{N}^2 = (\mathbb{N}, \mathcal{N}, 0, 1, +, \cdot, \mathrm{ap})$ we used in Section 3D. Its formal language A^2 has variables $x, y, \dots, s, t, \dots$ over $\mathbb{N}$ and $\alpha, \beta, \cdots$ over $\mathcal{N}$ and symbols $0, 1, +, \cdot, \mathrm{ap}$. Its *standard interpretation* is $\mathbf{N}^2$. We are interested in general, *ω-models* of A^2-theories in which the number variables range over $\mathbb{N}$ and the function variables over some $\mathcal{F} \subseteq \mathcal{N}$, and for any formula φ we will write

$$\mathcal{F} \models \varphi \iff (\mathbb{N}, \mathcal{F}, 0, 1, +, \cdot, \mathrm{ap}) \models \forall\!\!\!\forall \varphi$$

where $\forall\!\!\!\forall \varphi$ is the universal closure of φ. As usual, we identify sets with their representing functions in such models,

$$A \in \mathcal{F} \iff \chi_A \in \mathcal{F} \quad (A \subseteq \mathbb{N}).$$

[23] We need to include all arithmetical sets in $\Delta(\mathcal{F})$, ow. $\Delta(\emptyset) = \emptyset$ and Δ would close at 0 and build up the empty set.

An A^2-formula φ is *arithmetical* if no function quantifiers occur in it. As usual, by $\varphi(x, y, \beta, \gamma)$ we will denote any formula in which the variables x, y, β, γ may occur free *but do not necessarily include all the variables which occur free in* φ.

We consider three axiom schemes in A^2:

Arithmetical comprehension. With arithmetical $\varphi(s)$ (in which α does not occur free),

$$(\Delta^0_\infty\text{-Comp}) \qquad (\exists\alpha)(\forall s)[\alpha(s) = 1 \iff \varphi(s)].$$

Δ^1_1-*comprehension.* With arithmetical $\varphi(s, \gamma)$, $\psi(s, \gamma)$ (in which α does not occur free),

$$(\Delta^1_1\text{-Comp}) \quad (\forall s)[(\exists\gamma)\varphi(s, \gamma) \iff (\forall\gamma)\psi(s, \gamma)] \Longrightarrow (\exists\alpha)(\forall s)[\alpha(s) = 1 \iff (\exists\gamma)\varphi(s, \gamma)].$$

Σ^1_1-*Choice.* With arithmetical $\varphi(s, \alpha, \gamma)$,[24]

$$(\Sigma^1_1\text{-Choice}) \qquad (\forall s)(\exists\alpha)(\exists\gamma)\varphi(s, \alpha, \gamma) \Longrightarrow (\exists\alpha)(\forall s)(\exists\gamma)\varphi(s, (\alpha)_s, \gamma).$$

Clearly, $(\Delta^1_1\text{-Comp}) \Longrightarrow (\Delta^0_\infty\text{-Comp})$, and Kreisel [1961] verified that[25]

$$(53) \qquad (\Delta^0_\infty\text{-Comp}) + (\Sigma^1_1\text{-Choice}) \Longrightarrow (\Delta^1_1\text{-Comp}).$$

Theorem 3I.2. (1) (Kleene [1959a], Kreisel [1961]) HYP *is the least model of* $(\Delta^1_1\text{-Comp})$.

(2) (Kreisel [1961]) HYP *satisfies* $(\Sigma^1_1\text{-Choice})$.

Proof. (1) If A is Σ^1_1 with range HYP, then it is Π^1_1 by the HYP-Quantification Theorem 3H.1 (1); and if A is also Π^1_1 with range HYP, then it is Δ^1_1 and hence HYP. This proves that HYP satisfies $(\Delta^1_1\text{-Comp})$, if we apply it to the set $A = \{s : (\exists\gamma)\varphi(s, \gamma)\}$ and then take $\alpha = \chi_A$. To see that it is the least model of $(\Delta^1_1\text{-Comp})$, assume that $\mathcal{F}$ satisfies $(\Delta^1_1\text{-Comp})$ and prove by induction on ξ that $\overline{\Delta}^\xi \subseteq \mathcal{F}$ using Theorem 3I.1.

(2) Suppose that $(\forall s)(\exists\alpha \in \mathsf{HYP})(\exists\gamma \in \mathsf{HYP})\varphi(s, \gamma, \alpha)$ with an arithmetical φ and set

$$P(s, i) \iff i \in B^1 \;\&\; (\forall\alpha)[\alpha = \beta_i \Longrightarrow (\exists\gamma \in \mathsf{HYP})\varphi(s, \gamma, \alpha)].$$

This is in Π^1_1, so by the Kreisel Uniformization Theorem 3G.2, it is uniformized by a Π^1_1 relation $P^*(s, i)$; we check easily that some $\alpha \in \mathsf{HYP}$ satisfies

$$(\alpha)_s = \beta_i \text{ for the unique } i \text{ which satisfies } P^*(s, i),$$

and then this α also satisfies the right-hand-side of $(\Sigma^1_1\text{-Choice})$. ⊣

Another relevant and important result that we will not discuss here in detail is the following:

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[24]We assume some formal treatment of recursive substitutions into A^2 formulas. In this case, the relevant recursive function is $(\alpha, s) \mapsto (\alpha)_s$, and we use the equivalences

$$\varphi(s, (\alpha)_s, \gamma) \iff (\exists\delta)[\delta = (\alpha)_s \;\&\; \varphi(s, \delta, \gamma)] \iff (\forall\delta)[\delta = (\alpha)_s \to \varphi(s, \delta, \gamma)].$$

These are satisfied by every model $\mathcal{F}$ of $(\Delta^0_\infty\text{-Comp})$.

[25]The converse fails, cf. Steel [1978].

Theorem 3I.3 (Gandy, Kreisel, and Tait [1960]). *A set $A \subseteq \mathbb{N}$ is* HYP *if and only if its characteristic function χ_A belongs to every $\mathcal{F} \subseteq \mathcal{P}(\mathcal{N})$ which satisfies the axiom scheme of full comprehension, i.e., for every formula $\varphi(s)$ in which α does not occur free,*

$$(\Delta^1_\infty\text{-Comp}) \qquad (\exists\alpha)(\forall s)[\alpha(s) = 1 \iff \varphi(s)].$$

Beyond these (and many other) applications, however, the importance of Theorem 3I.1 is primarily foundational. To quote Kleene [1959a],

> the definition [with basis $\mathcal{F}$] means the same to persons with various universes of functions, so long as each person's universe includes at least $\mathcal{F}$ (of which he may have no exact conception).

One can argue that it presents HYP as a *potential totality* which can be comprehended by mathematicians with varying views of "the continuum", much like $\mathbb{N}$ can be understood as a potential totality within classical and constructive mathematics alike.

3J. Inductive definability on $\mathbb{N}$. It should be clear by now that inductive definitions permeate our subject, but is was not until Spector [1961] that a neat, precise result was formulated expressing the connection.

Suppose $\Phi : \mathcal{P}(\mathbb{N}^l) \to \mathcal{P}(\mathbb{N}^l)$ is a monotone operator as in App 10 and (generalizing mildly (34)), define the *representing relation* of Φ by

$$\Phi(\vec{y}, \alpha) \iff \vec{y} \in \Phi(Z_\alpha) \text{ where } Z_\alpha = \{\vec{y}' : \alpha(\langle\vec{y}'\rangle) = 0\}. \tag{54}$$

The operator Φ is *in* a pointclass Γ (such as Π^0_1 or Π^1_1) if $\Phi(\vec{x}, \alpha)$ is in Γ; and a relation $P(\vec{x})$ is Γ-*inductive on* $\mathbb{N}$ if it is many-one reducible to the least fixed point $\overline{\Phi}$ of a monotone operator in Γ.

Lemma 3J.1 (Spector [1955]).[26] *If $\Phi(A)$ is a monotone, Π^1_1 operator on $\mathcal{P}(\mathbb{N})$ and $P \subseteq \mathbb{N}$ is Π^1_1, then*

$$x \in \Phi(P) \Longrightarrow (\exists H \subseteq P)[H \in \mathsf{HYP} \;\&\; x \in \Phi(H)].$$

This is not really difficult, but its simplest proof requires identifying the monotone Π^1_1 operators with those which are Π^1_1-*positive*, suitably defined, and it is a bit too lengthy to include here.

Theorem 3J.2. (1) (Kleene [1955b])[27]. *Every Π^1_1 relation $P(\vec{x})$ is Π^0_1-inductive on $\mathbb{N}$, in fact there is a Π^0_1 monotone operator Φ on $\mathbb{N}^{1+n}$ such that*[28]

$$P(\vec{x}) \iff (1, \vec{x}) \in \overline{\Phi}. \tag{55}$$

(2) (Spector [1961]) *If $\Phi : \mathcal{P}(\mathbb{N}^l) \to \mathcal{P}(\mathbb{N}^l)$ is Π^1_1, then its least fixed point $\overline{\Phi}$ is Π^1_1 and its closure ordinal $\kappa(\Phi) \leq \omega_1$.*

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[26] This is not quite explicit in Spector [1955], but Sacks [1990, 8.5] credits it to Spector and I think he is right.

[27] This is seriously implicit in $24 of Kleene [1955b], but the idea of the proof is there and Spector correctly credits Kleene for it.

[28] The "1" is necessary here, in fact *it is not the case that every Π^1_1 set is the least fixed point $\overline{\Phi}$ of an arithmetical monotone operator Φ on* $\mathbb{N}$, cf. Feferman [1965] and Moschovakis [1974, 8.13] (which is falsely claimed in the 1974 edition for all "countable acceptable structures"). Feferman's result was the first applications of Cohen's forcing to arithmetic.

PROOF. (1) is basically immediate from (39), which expresses Kleene's key understanding of Π^1_1 definitions: for a given $P(\vec{x})$ in Π^1_1 (and adjusting the notation in (39)), we set

$$(56) \qquad (u, \vec{x}) \in \Phi(A) \iff \text{Seq}(u) \;\&\; \Big(R(\vec{x}, u) \vee (\forall s)(u * \langle s\rangle, \vec{x}) \in A\Big),$$

prove first by induction on ξ that

$$(u, \vec{x}) \in \overline{\Phi}^{\xi} \Longrightarrow \text{Seq}(u) \;\&\; (\forall \alpha \sqsupseteq u)(\exists t) R(\vec{x}, \overline{\alpha}(t))$$

and then check easily that (with $1 = \langle\, \rangle$, the code of the empty sequence),

$$(1, \vec{x}) \notin \overline{\Phi} \Longrightarrow (\exists \alpha)(\forall t)\neg R(\vec{x}, \overline{\alpha}(t)) \Longrightarrow \neg P(\vec{x}).$$

This gives (55).

(2) That $\overline{\Phi}$ is Π^1_1 if Φ is Π^1_1, we have already proved in Lemma 3D.1. For the more difficult bound on the closure ordinal, we first check that $P = \bigcup_{\xi<\omega_1} \overline{\Phi}^{\xi}$ is Π^1_1 by effective grounded recursion and then apply the Lemma. ⊣

Like Kreisel [1961], Spector [1961] was presented at the famed *Symposium on Foundations of Mathematics* held in Warsaw in 1959. It has many more (and more difficult) results, but its main significance lies in this simple characterization of Π^1_1 (and hence HYP) in terms of inductive definability.

3K. HYP **as recursive in** $^2\mathsf{E}$**.** Starting with his [1959b], Kleene developed a theory of absolute and relative recursion for functions with arguments in the objects of *finite type over* $\mathbb{N}$, i.e., members of the sets T_i where

$$T_0 = \mathbb{N}, \quad T_{i+1} = (T_i \to \mathbb{N}) = \text{the set of functions on } T_i \text{ to } \mathbb{N}.$$

This is a technically difficult but fascinating topic, with some important applications to Descriptive Set Theory but especially to the foundations of the theory of recursion: it was the first example where there is no natural notion of *machine computable function* that can be defined independently of "recursiveness", and so it forces an examination of the meaning of *recursive definitions* in and of themselves. We cannot go into it here, but it is worth stating one of Kleene's basic results which relate it to HYP:[29]

In Kleene's words, the following type-2 object "embodies" the operation of quantification over $\mathbb{N}$:

$$^2\mathsf{E}(\alpha) = \begin{cases} 0, & \text{if } (\exists t)[\alpha(t) = 0], \\ 1, & \text{otherwise} \end{cases} \qquad (\alpha \in \mathcal{N}).$$

Theorem 3K.1 (Kleene [1959b]). *A set $A \subseteq \mathbb{N}$ is hyperarithmetical if and only if it is recursive in* $^2\mathsf{E}$.

In fact, for $A \subseteq \mathbb{N}$,

$$\begin{aligned} A \in \Pi^1_1 &\iff A \text{ is recursively enumerable in } {}^2\mathsf{E} \\ &\iff A = \{x : f(x)\!\downarrow\} \text{ for some } f : \mathbb{N} \rightharpoonup \mathbb{N} \text{ recursive in } {}^2\mathsf{E}, \end{aligned}$$

[29]Cf. Kechris and Moschovakis [1977] for a relatively simple introduction to recursion in higher types and Sacks [1990] for a full development.

which bolsters the understanding of Π^1_1 as an analog of Σ^0_1 in *recursion in* ${}^2\mathsf{E}$.[30]

4. Concluding remarks. The main results from the period 1950 – 1960 that we have surveyed established HYP as a robust class of sets, those subsets of $\mathbb{N}$ which can be defined (and can be guaranteed to exist) if we accept the structure $\mathbf{N}$ of arithmetic, quantification over $\mathbb{N}$ and recursion. The main new method introduced in this work is undoubtedly effective grounded recursion, but there are also many interesting tricks, especially in computing "witnesses to counterexamples" as in the proof of Lemma 3E.4.

There were primarily three developments which followed this work and are still extensively pursued today: *recursion in higher types* which we have already discussed and the following two.

4A. IND **and** HYP **on abstract structures.** Of the many characterizations of HYP, the easiest to formulate for an arbitrary structure $\mathbf{A} = (A, R_1, \dots, R_k)$ is Spector's *inductive definability* in Section 3J, cf. Moschovakis [1974]. Briefly, a relation $P \subseteq A^n$ is *inductive in* $\mathbf{A}$ if it is one of the mutual least fixed points of a finite system of positive, elementary (first-order) relations with arguments in A, and it is *hyperelementary in* $\mathbf{A}$ if both P and its negation $A^n \setminus P$ are inductive.

Part of the theory of HYP and Π^1_1 can be developed for $\mathsf{HYP}(\mathbf{A})$ and $\mathsf{IND}(\mathbf{A})$ for arbitrary $\mathbf{A}$; some of the results require an assumption that $\mathbf{A}$ is (almost) *acceptable*, roughly meaning that $\mathbf{A}$ admits a hyperelementary coding scheme for tuples; and suitable formulations of virtually all the results in the body of this paper can be established for all *countable, acceptable structures*, including Kleene's centerpiece that $\mathsf{IND}(\mathbf{A}) = \Pi^1_1(\mathbf{A})$ and so $\mathsf{HYP}(\mathbf{A}) = \Delta^1_1(\mathbf{A})$.

Kleene's Theorem 3I.1 holds for all acceptable structures almost exactly in the form that it is stated in Section 3I, with ω_1 replaced by the *closure ordinal* $\kappa(\mathbf{A})$ of $\mathbf{A}$, an important invariant. It is proved, however, by an entirely different argument which is different from (and perhaps simpler) than Kleene's even for the classical structure $\mathbf{N}$ of arithmetic.

The proofs, in fact, are the most interesting aspect of this generalization of HYP theory: there is little coding and no use of effective grounded recursion. These are replaced by constructs which were first used in higher type recursion (*Stage Comparison Theorems*) and ideas from the theory of *infinite open games*.

The most interesting application of inductive definability is to the structure $\mathbf{N}^2$ of analysis in (31) which is intimately related to our last topic.

4B. Effective descriptive set theory. The term was coined by Addison [1959] who formulated his results about the spaces $\mathbb{N}^n \times \mathcal{N}^m$ and might have still be thinking of "analogies" between the classical and the effective results; but in the 50+ years since then, effective descriptive set theory has evolved into a unified study of *definability on recursive Polish spaces* which include $\mathbb{N}$, $\mathcal{N}$ and the real numbers and has deep applications to parts of topology and analysis in addition to classical descriptive set theory and logic. A good part of it is covered in Moschovakis [2009], which, however, is concerned with many other things and is not sufficiently comprehensive on this topic.

[30]Kleene [1959b] does not mention this and I recall him saying (much later) that he was not certain that the notion of a recursive partial function in higher type recursion was natural, but I cannot point to a reference for this.

5. Appendix: some basic facts and notation. We list here some elementary definitions and results, primarily to establish notation.

App 1. $\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers and $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ is the *Baire space* of all unary functions on $\mathbb{N}$. This carries the natural product topology with $\mathbb{N}$ discrete, generated by the basic neighborhoods

$$\mathcal{N}_{k_0,\dots,k_t} = \{\alpha \in \mathcal{N} : \alpha(0) = k_0, \dots, \alpha(t) = k_t\}$$

and the product spaces $\mathbb{N}^n \times \mathcal{N}^m$ carry the corresponding product topologies.

In general, lower case Latin letters vary over $\mathbb{N}$ and Greek letters $\alpha, \beta, \dots$ vary over $\mathcal{N}$.

App 2. A *partial function* $f : X \rightharpoonup Y$ is a function $f : D_f \to Y$, where $D_f \subseteq X$ is the *domain of convergence* of f. We write

$$f(x)\downarrow \iff x \in D_f, \quad f(x)\uparrow \iff x \notin D_f \quad (x \in X),$$
$$f(x) = g(x) \iff [f(x)\uparrow \ \& \ g(x)\uparrow] \vee [f(x)\downarrow \ \& \ g(x)\downarrow \ \& \ f(x) = g(x)].$$

Partial functions *compose strictly*, e.g.,

$$f(g(x), h(x)) = w \iff (\exists u, v)[g(x) = u \ \& \ h(x) = v \ \& \ f(u, v) = w].$$

It is sometimes convenient to identify $f : X \rightharpoonup Y$ with its *graph*

$$\mathrm{Graph}(f) = \{(x, y) \in X \times Y : f(x) = y\}.$$

App 3. $\chi_A : X \to \mathbb{N}$ is the *characteristic function* of $A \subseteq X$ (= 1 on A and 0 on $A^c = X \setminus A$).

App 4. *Sequence coding* in $\mathbb{N}$. The following functions and relations are recursive, with p_i the $(i + 1)$'th prime number:

$\langle u_0, \dots, u_{n-1}\rangle = p_0^{u_0+1} \cdots p_{n-1}^{u_{n-1}+1}$ = the code of $(u_0, \dots, u_{n \dot{-} 1})$;

$\mathrm{Seq}(u) \Leftrightarrow$ *u is the code of some sequence*, and if it is, then $\mathrm{lh}(u)$ is its length and for $i < \mathrm{lh}(u)$, $(u)_i = u_i$;

$u \sqsubseteq v \Leftrightarrow$ *u codes an initial segment of the sequence coded by v*;

$u \sqsubsetneq v \Leftrightarrow u \sqsubseteq v \ \& \ u \neq v$;

$u \mid v \Leftrightarrow u$ and v are codes of incompatible sequences $\Leftrightarrow \neg(u \sqsubseteq v \vee v \sqsubseteq u)$;

$u * v$ = the code of the concatenation of the sequences coded by u and v;

$\overline{\alpha}(t) = \langle \alpha(0), \dots, \alpha(t \dot{-} 1)\rangle$ (= 1 if $t = 0$).

App 5. *Kleene's Normal Form and Enumeration Theorem*: Every recursive partial function(al) $f : \mathbb{N}^n \times \mathcal{N}^m \rightharpoonup \mathbb{N}$ is $\varphi_e^{n,m}$ for some e, where

$$\begin{aligned}(57) \quad \varphi_e^{n,m}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_m) &= \{e\}^{n,m}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_m) \\ &= U(\mu t T_n^m(e, x_1, \dots, x_n, t, \overline{\alpha}_1(t), \dots, \overline{\alpha}_m(t)))\end{aligned}$$

with (primitive) recursive T_n^m and U, and we will skip some or all of the superscripts n, m when they are clear from the context or irrelevant. Moreover, there are (primitive) recursive *injections* $S_n^{l,m}(e, y_1, \dots, y_l)$ such that

$$(58) \qquad \{e\}(y_1, \dots, y_l, x_1, \dots, x_n, \vec{\alpha}) = \{S_n^{l,m}(e, y_1, \dots, y_l)\}(x_1, \dots, x_n, \vec{\alpha}).$$

We call e a *code* of $\varphi_e^{n,m}$ and we use both φ_e and $\{e\}$ interchangeably, as the desire for neat typography dictates.

To avoid (implausible) confusion, we use $\{\!\{e\}\!\}$ for the singleton set whose only member is e.

App 6. *Relativization.* It is sometimes useful to fix some of the function arguments in the Normal Form Theorem and treat them as *parameters.* We write

$$\{e\}^{\vec{\beta}}(\vec{x}, \vec{\alpha}) = \{e\}(\vec{x}, \vec{\beta}, \vec{\alpha})$$

and we say that the partial function $(\vec{x}, \vec{\alpha}) \mapsto \{e\}^{\vec{\beta}}(\vec{x}, \vec{\alpha})$ is *recursive in* $\vec{\beta}$ or *relative to* $\vec{\beta}$. For recursion *relative to a set B*, we write

$$\{e\}^{B}(\vec{x}) = \{e\}(\vec{x}, \chi_{B}) \quad (B \subseteq \mathbb{N}).$$

It is often—almost always—the case that a result about (absolutely) recursive partial functions can be easily seen to be true about partial functions recursive in some $\vec{\beta}$, by *relativization*, i.e., basically adding the superscript $^{\vec{\beta}}$ to all functions in the proof; it is a simple but very useful method of proof.

App 7. *Recursively enumerable sets.* A set $A \subseteq \mathbb{N}$ is *r.e. in* $B \subseteq \mathbb{N}$ if

$$x \in A \iff \{e\}^{B}(x)\downarrow \text{ for some } e,$$

and (absolutely) *r.e.* if it is r.e. in the empty set.

App 8. *Total recursive functions into $\mathcal{N}$.* A (total) function $f : \mathbb{N}^n \times \mathcal{N}^m \to \mathcal{N}$ is recursive if

$$f(\vec{x}, \vec{\alpha}) = \lambda t f_*(t, \vec{x}, \vec{\alpha})$$

for some recursive partial $f_* : \mathbb{N}^{1+n} \times \mathcal{N}^m \rightharpoonup \mathbb{N}$. Useful examples include the *tupling* and *projection functions*:

$$\langle \alpha_0, \dots, \alpha_{k-1} \rangle = \lambda t \begin{cases} \alpha_i(s), & \text{if } t = \langle i, s \rangle \text{ for some } i < k \text{ and some } s, \\ 0, & \text{otherwise,} \end{cases}$$
$$(\beta)_i = \lambda t \beta(\langle i, t \rangle),$$

so that for $i < k$, $(\langle \alpha_0, \dots, \alpha_{k-1} \rangle)_i = \alpha_i$.

The class of total recursive functions into $\mathbb{N}$ or $\mathcal{N}$ is closed under composition—which is not true for recursive partial functions with values in $\mathcal{N}$.

App 9. *The rank of a strict, wellfounded relation.* A binary relation $\prec$ on a set F is *wellfounded* if there is no infinite descending chain $x_0 \succ x_1 \succ \cdots$ or, equivalently, if there is a function $\rho : X \to$ Ordinals such that

$$x \prec y \Longrightarrow \rho(x) < \rho(y) \quad (x, y \in F);$$

the (pointwise) least such function $\rho_{\prec}$ is the *rank function* of $\prec$ and

$$\mathrm{rank}(\prec) = \sup\{\rho_{\prec}(x) + 1 : x \in F\}.$$

When we apply this to the strict part $\prec$ of a wellordering $\preceq$, we get a (unique) similarity

$$\rho_{\preceq} : \{x : x \preceq x\} = F \rightarrowtail\!\!\!\!\rightarrow \mathrm{rank}(\preceq)$$

of $\preceq$ with an ordinal.

App 10. *Monotone inductive definitions.* An operator $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ on the subsets of a space X is monotone if

$$A \subseteq B \Longrightarrow \Phi(A) \subseteq \Phi(B) \quad (A, B \subseteq X).$$

The set

$$\overline{\Phi} = \bigcap\{A : \Phi(A) \subseteq A\} \tag{59}$$

defined inductively (or *built up*) by Φ is the *least fixed point* of Φ, and

$$\overline{\Phi} = \bigcup \overline{\Phi}^{\xi}, \text{ where for each ordinal } \xi, \overline{\Phi}^{\xi} = \Phi(\textstyle\bigcup_{\eta<\xi} \overline{\Phi}^{\eta}) \tag{60}$$

(with the usual convention that $\bigcup \emptyset = \emptyset$). Moreover,

$$\eta < \xi \Longrightarrow \overline{\Phi}^{\eta} \subseteq \overline{\Phi}^{\xi} \subseteq X,$$

and since these *iterates* cannot increase forever, there is a least ordinal $\kappa = \kappa(\Phi)$, the *closure ordinal* of Φ such that

$$\eta < \xi < \kappa(\Phi) \Longrightarrow \overline{\Phi}^{\eta} \subsetneq \overline{\Phi}^{\xi} \text{ and } \overline{\Phi} = \textstyle\bigcup_{\xi<\kappa(\Phi)} \overline{\Phi}^{\xi}. \tag{61}$$

An operator Φ is *operative on X to W* if its domain is $\mathcal{P}(X \times W)$ and

$$f : X \rightharpoonup W \Longrightarrow \Phi(\mathrm{Graph}(f)) = \mathrm{Graph}(g) \text{ for some } g : X \rightharpoonup W.$$

When this holds, then $\overline{\Phi} : X \rightharpoonup W$ is (the graph of) the least partial function fixed by the operator Φ.

REFERENCES

J. W. Addison

[1959] *Separation principles in the hierarchies of classical and effective descriptive set theory*, ***Fundamenta Methematicae***, vol. 46, pp. 123–135.

Alonzo Church

[1935] *An unsolvable problem in elementary number theory*, ***Bulletin of the American Mathematical Society***, vol. 41, pp. 332–333, This is an abstract of Church [1936].

[1936] *An unsolvable problem in elementary number theory*, ***American Journal of Mathematics***, pp. 345–363, An abstract of this paper was published in Church [1935].

Martin Davis

[1950a] *On the theory of recursive unsolvability*, ***Ph.D. Thesis***, Princeton University.

[1950b] *Relatively recursive functions and the extended Kleene hierarchy*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass, 1950, p. 723.

[1965] ***The undecidable***, Raven Press.

H. B. Enderton and Hilary Putnam

[1970] *A note on the hyperarithmetical hierarchy*, ***The Journal of Symbolic Logic***, vol. 35, pp. 429–430.

S. Feferman

[1965] *Some applications of forcing and generic sets*, ***Fundamenta Methematicae***, vol. 56, pp. 325–345.

R. Gandy, G. Kreisel, and W. Tait

[1960] *Set existence*, ***Bulletin of the Polish Academy of Sciences, Series in Science, Mathematics and Astronomy***, vol. 8, pp. 577–582.

R. O. Gandy

[1960] *Proof of Mostowski s conjecture*, ***Bulletin de l'Academie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques***, vol. 8, pp. 571 –575.

Kurt Gödel

[1931] *Über formal unentscheidbare Sätze der Principia Mathematica and verwandter Systeme, I*, ***Monatshefte für Mathematik und Physik***, pp. 173–198, English translations in Davis [1965] and Van Heijenoort [1967].

A. S. Kechris and Y. N. Moschovakis

[1977] *Recursion in higher types*, ***Handbook of mathematical logic*** (J. Barwise, editor), Studies in Logic, No. 90, North Holland, Amsterdam, pp. 681–737.

Stephen C. Kleene

[1936] *General recursive functions of natural numbers*, ***Mathematische Annalen***, vol. 112, pp. 727–742.

[1938] *On notation for ordinal numbers*, ***The Journal of Symbolic Logic***, vol. 3, pp. 150–155.

[1943] *Recursive predicates and quantifiers*, ***Transactions of the American Mathematical Society***, vol. 53, pp. 41–73.

[1950] *A symmetric form of Gödel's theorem*, ***Konin. Neder. Akad. van Weten. te Amst. Proc.Proc. of the Section of Sciences***, vol. 53, pp. 800–802.

[1952] ***Introduction to metamathematics***, D. Van Nostrand Co, North Holland Co.

[1953] *Arithmetical predicates and function quantifiers*, ***The Journal of Symbolic Logic***, vol. 18, p. 190, abstract of a talk presented at a meeting of the Association for Symbolic Logic on December 29, 1952.

[1955a] *Arithmetical predicates and function quantifiers*, ***Transactions of the American Mathematical Society***, vol. 79, pp. 312–340.

[1955b] *On the form of predicates in the theory of constructive ordinals* (second paper), ***American Journal of Mathematics***, vol. 77, pp. 405–428.

[1955c] *Hierarchies of number theoretic predicates*, ***Bulletin of the American Mathematical Society***, vol. 61, pp. 193–213.

[1959a] *Quantification of number-theoretic functions*, ***Compositio Mathematica***, vol. 14, pp. 23–40.

[1959b] *Recursive functionals and quantifiers of finite types I*, ***Transactions of the American Mathematical Society***, vol. 91, pp. 1–52.

M. Kondo

[1938] *Sur l'uniformization des complementaires analytiques et les ensembles projectifs de la second classe*, ***Japanese Journal of Mathematics***, vol. 15, pp. 197–230.

Donald L. Kreider and Hartley Rogers

[1961] *Constructive versions of ordinal number classes*, ***Transactions of the American Mathematical Society***, vol. 100, pp. 325 –369.

Georg Kreisel

[1961] *Set theoretic problems suggested by the notion of potential totality*, ***Infinitistic methods***, Pergamon, New York, pp. 103–140.

[1962] *The axiom of choice and the class of hyperarithmetic functions*, ***Indagationes Mathematicae***, vol. 24, pp. 307–319.

Henri Lebesgue

[1905] *Sur les fonctions represéntables analytiquement*, ***Journal de Mathématiques 6^e serie***, vol. 1, pp. 139–216.

A. A. Markov

[1947] *On the impossibility of certain algorithms in the theory of associative systems*, ***Coptes rendus (Doklady) de l'Academie des Sciences de USSR***, vol. 55, pp. 583–586.

Werner Markwald

[1954] *Zur Theorie der konstruktiven Wohlordnungen*, ***Mathematischen Annalen***, vol. 127, pp. 135–149.

Yiannis N. Moschovakis

[1966] *Many-one degrees of the predicates $H_a(x)$*, ***Pacific Journal of Mathematics***, vol. 18, pp. 329–342.

[1968] *Review of four papers on Church's Thesis*, ***The Journal of Symbolic Logic***, vol. 33, pp. 471–472.

[1969] *Abstract first order computability II*, ***Transactions of the American Mathematical Society***, vol. 138, pp. 465–504.

[1974] ***Elementary induction on abstract structures***, North Holland, Amsterdam, Studies in Logic, No. 77. Republished by Dover Publications, Mineola, NY, 2008, with a correction to 8.3.

[2009] ***Descriptive set theory, second edition***, Mathematical Surveys and Monographs., vol. 155, American Mathematical Society.

[2010a] *Classical descriptive set theory as a refinement of effective descriptive set theory*, ***Annals of Pure and Applied Logic***, vol. 162, pp. 243–255.

[2010b] *Kleene's amazing second recursion theorem*, ***The Bulletin of Symbolic Logic***, vol. 16, pp. 189–239.

A. Mostowski

[1947] *On definable sets of positive integers*, ***Fundamenta Methematicae***, vol. 34, pp. 81–112.

[1951] *A classification of logical systems*, ***Studia Philosophica***, vol. 4, pp. 237–274.

John Myhill

[1955] *Creative sets*, ***Zeitschrifft für Mathematische Logik und Grundlagen der Mathematik***, vol. 1, pp. 97–108.

George C. Nelson

[1974] *Many-one reducibility within the Turing degrees of the hyperarithmetic sets $Ha(x)$*, ***Transactions of the American Mathematical Society***, vol. 191, pp. 1 –44.

Emil L. Post

[1944] *Recursively enumerable sets of positive integers and their decision problems*, ***Bulletin of the American Mathematical Society***, vol. 50, pp. 284–316.

[1947] *Recursive unsolvability of a problem of Thue*, ***The Journal of Symbolic Logic***, vol. 12, pp. 1–11.

Hillary Putnam

[1961] *Uniqueness ordinals in higher constructive number classes*, ***Essays on the foundations of mathematics***, Magnes Press, Hebrew University, Jerusalem, pp. 190–206.

Hartley Rogers

[1967] ***Theory of recursive functions and effective computability***, McGraw-Hill.

Gerald E. Sacks

[1990] ***Higher recursion theory***, Perspectives in Mathematical Logic, Springer.

Clifford Spector

[1955] *Recursive wellorderings*, ***The Journal of Symbolic Logic***, vol. 20, pp. 151–163.

[1956] *On degress of recursive unsolvability*, ***Annals of Mathematics***, vol. 64, pp. 581–582.

[1960] *Hyperarithmetical quantifiers*, ***Fundamenta Methematicae***, vol. 48, pp. 313–320.

[1961] *Inductively defined sets of natural numbers*, ***Infinitistic methods***, Pergamon, New York, pp. 97–102.

John R. Steel

[1978] *Forcing with tagged trees*, ***Annals of Mathematical Logic***, vol. 15, pp. 55– 74.

M. Suslin

[1917] *Sur une definition des ensembles measurables B sans nombres transfinis*, ***Comptes Rendus Acad. Sci. Paris***, vol. 164, pp. 88–91.

ALAN M. TURING
[1936] *On computable numbers with an application to the Entscheidungsproblem*, ***Proceedings of the London Mathematical Society***, vol. 42, pp. 230–265, *A correction*, ibid. volume 43 (1937), pp. 544–546.

JEAN VAN HEIJENOORT
[1967] ***From Frege to Gödel, a source book in mathematical logic, 1879 – 1931***, Harvard University Press, Cambridge, Massachusetts, London, England.

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