Powerdomains, Powerstructures and Fairness

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Abstract. We introduce the framework of *powerstructures* for comparing models of non-determinism and concurrency, and we show that in this context the Plotkin powerdomain $\mathbf{plot}(D)$ [6] naturally occurs as a quotient of a refined and generalized *player model* $\mathbf{ipf}(D)$, following Moschovakis [2, 3]. On the other hand, Plotkin's domains for countable non-determinism $\mathbf{plot}_\omega(D)$ [7] are not comparable with these structures, as they cannot be realized concretely subsets of D.

If, as usual, we let the programs of a deterministic programming language L denote points in some *directed-complete poset* (dcpo) D, then programs in non-deterministic extensions of L should naturally correspond to non-empty subsets of D, members of the set of **players**[3]

$$\Pi = \Pi(D) =_{\mathrm{df}} \{x \subseteq D \mid x \neq \emptyset\}. \tag{1}$$

This idea immediately encounters a problem with *non-deterministic recursive definitions*. In the deterministic case, the open terms of L (its *program transformations*) denote (Scott) continuous functions on D. Their least fixed points (which exist precisely because D is a dcpo) provide a means of interpreting recursion. On $\Pi(D)$, which does not carry a natural, complete partial ordering, how are we to interpret non-deterministic program transformations so that they still have "canonical" fixed points? No known semantics solves this basic problem in the modeling of non-determinism in an entirely satisfactory way.

For a concrete example, let *Str* be the dcpo of *integer streams*, where (following [5]) a stream is a finite or infinite sequence, or a finite sequence of the form $a_1 a_2 \ldots a_n \mathbf{t}$, where the *terminator* $\mathbf{t}$ is some fixed non-integer witnessing "termination." Now $\Pi(\mathit{Str})$ is the set of *non-deterministic integer streams*, and many of the usual, non-deterministic constructs are naturally interpreted by functions on $\Pi(\mathit{Str})$ as follows:

$$x \text{ } \mathit{or} \text{ } y =_{\mathrm{df}} x \cup y \tag{2}$$

$$\mathit{merge}(x, y) =_{\mathrm{df}} \{\mu[\alpha, \beta] \mid \alpha \in x, \beta \in y, \mu\colon \mathbb{N} \to \{0, 1\}\} \tag{3}$$

$$\mathit{fairmerge}(x, y) =_{\mathrm{df}} \{\mu[\alpha, \beta] \mid \alpha \in x, \beta \in y, \mu \text{ a fair merger}\}. \tag{4}$$

⋆ During the preparation of this paper, Moschovakis was partially supported by a NSF Grant and Whitney was supported by a Fellowship from the Fanny and John Hertz Foundation.

[3] The term derives from the original construction in Moschovakis [2, 3] which was cast in game theoretic terms, for a specific domain D of *partial strategies*.

Here *or* stands for free, binary choice, $\mu[\alpha, \beta]$ stands for "interleaving α and β by the merger μ" in the obvious way, and a (strict) fair merger (following Park [5]) is any sequence of 0's and 1's which is not ultimately constant. Note that as operations on players, these merges remain distinct. If D has further structure, then additional operations of this sort can be defined, such as *state-dependent fair merges*, see [3].

To model non-deterministic recursion within domain theory, we must embed $\Pi(D)$ in some *powerdomain* D^*, and not totally arbitrarily. For example, D is embedded in $\Pi(D)$ by the natural map $d \mapsto \{d\}$, and we would want to have "liftups" of the continuous functions in $(D \to D)$ to continuous functions in $(D^* \to D^*)$ which respect composition, yield the correct least fixed points, etc. Even such simple requirements seem to force undesirable consequences about D^*, however. Consider the first and most interesting powerdomain construction $\mathbf{plot}(D)$ of Plotkin [6] (see also Smyth [8]) as an illustrative example. $\mathbf{plot}(D)$ does not faithfully model fairness because it identifies sets in $\Pi(D)$ which are equivalent under[4] the "observational Egli-Milner equivalence relation" $\simeq_{\text{em}}$. This collapses the *merge* and *fairmerge* operations on $\Pi(\mathit{Str})$, e.g., $\mathit{fairmerge}(a^\infty, b^\infty) \simeq_{\text{em}} \mathit{merge}(a^\infty, b^\infty)$, where a^∞ is the infinite string of 'a's. In addition, the equivalence relation $\simeq_{\text{em}}$ identifies certain unguarded recursions with similar, but intuitively distinct, guarded recursions, e.g., see Smyth [8].

To circumvent these imperfections of the powerdomain constructions, Moschovakis [2, 3] introduced (over some specific domains D) a model $\mathbf{ipf}(D)$ for non-determinism and concurrency in which programs are interpreted by arbitrary players, and program transformations are modeled by **implemented player functions** (**ipfs**) on $\Pi(D)$. These ipfs encode more than their values on players: there exist distinct ipfs f and g such that $f(x) = g(x)$ for all $x \in \Pi(D)$. The extra, **intensional** information carried by an ipf makes it possible to assign "canonical solutions" to systems of recursive equations, so that *the laws of recursion* are obeyed; we will make this precise further on. The *or*, *merge*, and *fairmerge* operations introduced above are naturally modeled by certain ipfs (and incidentally "unnaturally" modeled by others, distinct from but extensionally equal to the natural ones.)

Our principal aim here is to show that (with modest hypotheses on D) the Plotkin powerdomain $\mathbf{plot}(D)$ can be recovered in a natural way from $\mathbf{ipf}(D)$, while the countable powerdomains $\mathbf{plot}_\omega(D)$ appear to represent a fundamentally different modeling of fairness. For this, we will also introduce a refined construction of $\mathbf{ipf}(D)$ (for any D) and establish precise properties of $\mathbf{ipf}(D)$

[4] The terminology for various pre-orders on $\Pi(D)$ is not entirely standardized. In this paper, we will use the *lower* preorder ($x \sqsubseteq_l y$ if for all $a \in x$, there exists $b \in y$ such that $a \leq b$) and the *upper* preorder ($x \sqsubseteq^u y$ if for all $b \in y$, there exists $a \in x$ such that $a \leq b$). The usual "Egli-Milner preorder" is the conjunction of these two. However, as outlined in Smyth [8], the easiest construction of the Plotkin powerdomain for countably algebraic D is in terms of the "observational Egli-Milner" preorder, defined as $x \lesssim y$ if for all finite sets A of finite elements, $A \sqsubseteq_l x$ implies $A \sqsubseteq_l y$ and $A \sqsubseteq^u x$ implies $A \sqsubseteq^u y$. Each preorder induces an equivalence relation, for example $x \simeq_{\text{em}} y$ if $x \lesssim y$ and $y \lesssim x$.

which make it a suitable structure for modeling non-deterministic programs and program transformations. We will rely heavily on an axiomatization of the "standard" laws of recursive equations, and on a somewhat novel approach to the development of *intensional semantics* for formal languages, which has applications beyond its present use. These ideas are described in Section 1.

1 The main notions

For each *vocabulary* (signature) τ, i.e., set of function symbols with associated non-negative arities, the *expressions* of the language $\mathrm{FLR}_0(\tau)$ are given by

$$E :\equiv x \mid \mathsf{f}(E_1, \dots, E_n) \mid \ E_0 \ \texttt{where} \ \{x_1 = E_1, \dots, x_n = E_n\}\,,$$

where x is a variable (from some fixed, infinite set of variables) and $\mathsf{f} \in \tau$. Intuitively, FLR_0 has notation just for function application and for solution of simultaneous recursion equations. The `where` operator binds the variables x_1 through x_n; all other variable occurrences are free. A *closed expression* is one containing no free variables, e.g., $\mathsf{f}\big(\mathsf{g}()\big)$ or $x \ \texttt{where} \ \{x = \mathsf{f}(x)\}$ if f is unary and g is nullary.

In the *standard semantics* for FLR_0, we have a dcpo D together with some continuous functions on D to interpret the function symbols, and with each FLR_0 expression E and each assignment π : Variables $\to D$, we associate a point $\mathit{value}(E, \pi) \in D$. If E is an open FLR_0 expression and the list of variables $\mathbf{x} = x_1, \dots, x_n$ includes all the free variables of E, then

$$\varLambda(\mathbf{x})E = \lambda(\mathbf{y})\mathit{value}(E, \{x_1 := y_1, \dots, x_n := y_n\}) \tag{5}$$

is the n-ary function defined by E and $\mathbf{x}$. The more general, *intensional semantics* for FLR_0 needed here are defined directly in terms of a given *interpretation* $\varLambda$, making (5) a theorem rather than a definition in the standard case.

The *universe* of an interpretation $\varLambda$ is a set $\varPhi$ of objects with associated integer arities; $\varPhi_n$ comprises the n-ary objects of $\varPhi$, and the nullary objects in $\varPhi_0$ are its *individuals*. In a standard interpretation, $\varPhi_0 = D$ is a dcpo and $\varPhi_n$ consists of all the continuous, n-ary functions on D. The interpretation $\varLambda$ assigns to each expression E and list of variables $\mathbf{x} = x_1, \dots, x_n$ including all free variables of E an object $\varLambda(\mathbf{x})\,E$ in $\varPhi_n$, so that the following basic conditions of *compositionality* hold:

(1) $\varLambda(\mathbf{x})\,x_i$ depends only on the length of $\mathbf{x}$ and on i; in the standard case, this must be the usual projection function from D^n to D by the ith component.

(2) If $\varLambda(\mathbf{x})\,M_i = \varLambda(\mathbf{y})\,M_i'$ for $i = 1, \dots, n$, then $\varLambda(\mathbf{x})\,\mathsf{f}(M_1, \dots, M_n) = \varLambda(\mathbf{y})\,\mathsf{f}(M_1', \dots, M_n')$. In the standard case, these must be computed by ordinary function application of the interpretation of f on the given values.

(3) If $\varLambda(\mathbf{y})\,E(y_1, \dots, y_n) = \varLambda(\mathbf{z})\,E'(z_1, \dots, z_n)$ and the substitutions $E[\mathbf{M}/\mathbf{y}]$ and $E'[\mathbf{M}/\mathbf{z}]$ are free, then $\varLambda(\mathbf{x})\,E(M_1, \dots, M_n) = \varLambda(\mathbf{x})\,E'(M_1, \dots, M_n)$.

(4) If $w = \varLambda(\mathbf{x})\ E_0 \ \texttt{where} \ \{y_1 = E_1, \dots, y_n = E_n\}$, suppose first that no y_i occurs in $\mathbf{x}$. Then w depends only on $\varLambda(\mathbf{y}, \mathbf{x})\,E_i$ for i from 0 to n. In

general, let $\mathbf{x}'$ be the same as $\mathbf{x}$ except that every variable from $\mathbf{x}$ occurring as one of the y_i has been replaced by a fresh variable. Then w depends only on $\Lambda(\mathbf{y}, \mathbf{x}')\, E_i$, in the same sense as the last two requirements: if these values are equal to $\Lambda(\mathbf{u}, \mathbf{z}')\, M_i$, respectively, then $w = \Lambda(\mathbf{z})\ M_0 \ \mathtt{where}\ \{\mathbf{u} = \mathbf{M}\}$. For a standard interpretation, w must be computed by taking the least fixed point of the system $y_i = \Lambda(\mathbf{y}, \mathbf{x}')\, E_i$ for i from 1 to n, and substituting the results (which are functions of the $\mathbf{x}'$) into E_0.

An expression identity $E = M$ is **standard** if it is valid for all standard interpretations, i.e., $\Lambda(\mathbf{x})E = \Lambda(\mathbf{x})M$ for every list $\mathbf{x}$ which includes all the free variables of both E and M. The simplest example of a standard identity is

$$f(\,x\ \mathtt{where}\ \{x = f(x)\}\,) = x\ \mathtt{where}\ \{x = f(x)\} \tag{6}$$

which asserts that "the least fixed point of f is a fixed point of f". Others include the *Bekič-Scott rules* which relate simultaneous and iterated recursion, the reduction of explicit definition to recursion, etc. It can be shown that *the class of standard identities* (on a recursive, countable vocabulary) *is decidable, simply* (and usefully) *axiomatizable, and the same as the class of identities valid for all interpretations with individuals D_0,* the set of all streams of '0's.[5] This *robustness* of the standard identities suggests that they truly codify the *laws of recursive equations*—the rules we use unthinkingly when we manipulate recursive definitions—and we look for modelings of non-determinacy and concurrency among FLR_0 interpretations which satisfy them.

An (abstract, intensional) FLR_0**-structure** is a triple

$$\mathcal{A} = (\Phi_0, \{\Phi_n\}_{n\geq 1}, \Lambda),$$

where $\Phi = \bigcup \Phi_n$ is a universe and Λ is an interpretation of $\mathrm{FLR}_0(\Phi)$ into Φ which satisfies the standard identities and also

$$\Lambda(x_1, \ldots, x_n)\, f(x_1, \ldots, x_n) = f \quad \text{for each } f \in \Phi_n. \tag{7}$$

Notice that here we view Φ as both a vocabulary and universe, each $f \in \Phi_n$ being an n-ary function symbol naming itself as enforced by (7). Each dcpo D gives rise to a *standard* FLR_0*-structure*, in which $\Phi_0 = D$, Φ_n consists of the n-ary continuous functions from D to D, and Λ is the standard interpretation as described above.

In an arbitrary FLR_0-structure, we think of the elements f of Φ as *intensional functions* on $A = \Phi_0$, and every n-ary f determines an actual function $\overline{f}\colon A^n \to A$ via

$$\overline{f}(a_1, \ldots, a_n) =_{\mathrm{df}} \Lambda()\, f(a_1, \ldots, a_n). \tag{8}$$

We say that $\overline{f}$ is the *extension* of f, or that f *covers* $\overline{f}$.

A homomorphism $\rho : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ (as above) to $\mathcal{B} = (\Psi_0, \{\Psi_n\}_{n\geq 1}, \Lambda')$ is any arity-preserving map from Φ to $\Psi = \bigcup \Psi_n$ which respects the interpretations,

[5] These results will appear in a multi-authored paper *The logic of recursive equations*, now in preparation.

as follows: extend ρ (by substitution) so that it takes arbitrary expressions of $\mathrm{FLR}_0(\Phi)$ to expressions of $\mathrm{FLR}_0(\Psi)$; then it must satisfy

$$\rho(\Lambda(\mathbf{x})E) = \Lambda'(\mathbf{x})(\rho(E)).$$

Thus, homomorphisms preserve all possible compositions and recursions.

A **powerstructure** over a dcpo D is an FLR_0-structure $\mathcal{P} = (P, \Phi_{n\geq 1}, \Lambda)$ such that there is an injective FLR_0-homomorphism ρ from the standard FLR_0-structure over D to $\mathcal{P}$ satisfying the following two *finite non-determinism conditions*:

(1) The map $\{d\} \mapsto \rho(d)$ on the singletons of D extends to a surjective map $\pi : S_P \twoheadrightarrow P$, where S_P is a subset of $\Pi(D)$ closed under continuous images and finite unions.

(2) Similarly, for each arity n, the map $\{F\} \mapsto \rho(F)$ on the singletons of continuous functions extends to a map π which takes each finite set J of n-ary continuous functions to some $\pi(J) \in \Phi_n$, so that:

$$\overline{\pi(J)}(\pi x_1, \ldots, \pi x_n) = \pi\left\{F(d_1, \ldots, d_n) \mid F \in J, d_i \in x_i\right\}. \tag{9}$$

If for a particular powerstructure $\mathcal{P}$ both occurrences of "finite" in these conditions may be replaced by "countable" or "arbitrary", then the powerstructure is called *countably non-deterministic* or *fully non-deterministic*, respectively. We also say that $\mathcal{P}$ is *fine*, if the map π on individuals is actually a bijection, so that P can be identified with a set of players.

The second condition applied to singletons $\{F\}$ implies that each continuous $F : D^n \to D$ has a *lift-up* f, such that

$$\overline{f}(\pi x_1, \ldots, \pi x_n) = \pi\{F(d_1, \ldots, d_n) \mid d_1 \in x_1, \ldots, d_n \in x_n\}.$$

In addition, if $U = \{F_1, F_2\}$ where $F_1(d, e) = d, F_2(d, e) = e$, then the corresponding intensional function $\pi(U)$ covers the ("quotient" of the) binary union operation (2). If $\mathcal{P}$ is fine and fully non-deterministic over *Str* and M is the set of all functions of the form $F_\mu(\alpha_0, \alpha_1) = \mu[\alpha_0, \alpha_1]$ with μ a fair merger, then $\pi(M)$ covers *fairmerge* as defined above (4). Thus, fine, fully non-deterministic powerstructures can provide powerful and faithful models of "fair concurrency."

Note that $\mathbf{plot}(D)$ together with the continuous functions on it is a powerstructure, but not a fine one: S_P is the collection of finitely generable subsets of D and π identifies Egli-Milner equivalent sets. Neither is $\mathbf{plot}(D)$ fully non-deterministic. The powerdomains $\mathbf{plot}_\omega(D)$ enjoy the intermediate property of countable non-determinism which can be used to define *fairmerge*, although not in the direct way described above, for there are uncountably many fair mergers μ. It is not clear that countable non-determinism provides a rich enough model to handle the many extant notions of fairness; in particular, we do not expect to be able to define natural state-dependent fair merges or the fair merge of countably many streams using only countable non-determinism.

2 Main results

Theorem A. *For each domain D, there is a fine, fully non-deterministic powerstructure* $\mathbf{ipf}(D) = (\Pi(D), ipf(D), \Lambda_{ipf})$ *over D.*

In the construction of $\mathbf{ipf}(D)$, every intensional function essentially arises as f_J for some J, so every $\overline{f}$ in $\mathbf{ipf}(D)$ ends up being *set monotone*, i.e.,

$$x \subseteq y \Rightarrow \overline{f}(x) \subseteq \overline{f}(y), \tag{10}$$

and this limits the functions on $\mathbf{plot}(D)$ we can represent inside $\mathbf{ipf}(D)$. Recall that (for countably algebraic D), $\mathbf{plot}(D)$ can be defined as the quotient of the *finitely generable* subsets of D (which we will denote by $\Pi_0(D)$), under the equivalence $\simeq_{\mathrm{em}}$. Therefore, each continuous function $\phi : \mathbf{plot}(D) \to \mathbf{plot}(D)$ is *induced* by some continuous function $\phi_* : \Pi_0(D) \to \Pi_0(D)$ on the *predomain* $\Pi_0(D)$, i.e.,

$$\phi([x/\simeq_{\mathrm{em}}]) = [\phi_*(x)/\simeq_{\mathrm{em}}], \quad (x \in \Pi_0(D)). \tag{11}$$

In particular, we say that ϕ is *essentially monotone* if it is induced by some set monotone ϕ_*. The essentially monotone functions $\mathbf{em}(D)$ are closed under composition and recursion, and therefore together with $\mathbf{plot}(D)$ and the standard (least-fixed-point) interpretation comprise an FLR_0-structure $\mathbf{Pl}(D) = (\mathbf{plot}(D), \mathbf{em}(D), \Lambda_{std})$. This is a natural FLR_0-structure associated with the Plotkin powerdomain, and it includes all $\cup$-linear functions [1].

Theorem B. *If D is strongly algebraic then there is an FLR_0-substructure* $\mathbf{ipf}_0(D) = (\Pi_0(D), ipf_0(D), \Lambda_0)$ *of* $\mathbf{ipf}(D)$ *with the following properties.*

(a) *Each player function $\overline{f}$ in $\mathbf{ipf}_0(D)$ respects the Egli-Milner preorder on $\Pi_0(D)$ and is Scott continuous, so that it induces a continuous function*

$$\rho(f) = \phi : \mathbf{plot}(D) \to \mathbf{plot}(D) \tag{12}$$

on the Plotkin powerdomain by the equation $\phi([x/\simeq_{\mathrm{em}}]) = [\overline{f}(x)/\simeq_{\mathrm{em}}]$. (By the observation (10), ϕ is necessarily essentially monotone.)

(b) *If we extend the map ρ to $\Pi_0(D)$ by $\rho(x) = [x/\simeq_{\mathrm{em}}]$, it becomes an* FLR_0*-homomorphism from $\mathbf{ipf}_0(D)$ to $\mathbf{Pl}(D)$.*

(c) *If $\phi : \mathbf{plot}(D) \to \mathbf{plot}(D)$ is essentially monotone, then $\phi = \rho(f)$ is also induced by some player transformation $\overline{f}$ in $\mathbf{ipf}_0(D)$;* that is, the image of the homomorphism ρ is exactly $\mathbf{Pl}(D)$.

No similar comparison is possible between $\mathbf{ipf}(D)$ and $\mathbf{plot}_\omega(D)$, however. The obstacle is that except for extremely simple (e.g., flat) D, $\mathbf{plot}_\omega(D)$ cannot be thought of as a structure on the subsets of D, or precisely:

Theorem C. *For any domain D embedding $(1_\perp \times \mathbb{N})_\perp$, the free σ-semilattice over D is not the homomorphic image of $(\Pi(D), \sqsubseteq, \subseteq)$ with ordinary $\subseteq$ and* any *partial order $\sqsubseteq$.*

This means that $\mathbf{plot}_\omega(D)$ is not technically a powerstructure in our sense, in that it does not represent non-deterministic "programs" (FLR_0 expressions) by their set of possible "outcomes" (subset of D), but provides some altogether different, less concrete interpretation.

3 Details and proofs

To prove Theorem A, we need to define the class $ipf(D)$ of *implemented player functions* (ipfs) on an arbitrary dcpo D, specify suitable operations of composition and recursion on this class, and then show that the resulting structure $\mathbf{ipf}(D) = (\Pi(D), ipf(D), \Lambda_{ipf})$ is a fully non-deterministic powerstructure. The complete construction is quite long, but not very different from that given in detail and with many motivating examples in [3], for a specific D. Here we confine ourselves to a brief sketch, highlighting the differences arising in the general case; [9] contains a full treatment.

A unary *polyfunction* on D is a monotone function $F\colon D^I \to D$, where the *index set* I is an arbitrary set of integers and D^I is the dcpo of maps from I to D under the pointwise ordering. Each polyfunction induces a function on $\Pi(D)$

$$\bar{F}(x) = \{F(X) \mid X\colon I \to x\},$$

and we think of F as an "implementation" of $\bar{F}$. However, some polyfunctions differ inessentially by the integer "tags" they use to name their arguments: we say that $G\colon D^J \to D$ *reduces to* $F\colon D^I \to D$, written $G \preceq F$, if there is an injection $\iota\colon I \to J$ such that $G(p) = F(p \circ \iota)$ for all $p \in D^J$. Let $\asymp$ be the smallest equivalence relation extending $\preceq$, and call two polyfunctions F_1 and F_2 *equivalent* if $F_1 \asymp F_2$. It is simple to verify that *if $F \asymp G$, then $\bar{F} = \bar{G}$*. Finally, a (unary) *implemented player function* (ipf) is a nonempty set of polyfunctions closed under $\asymp$. Each ipf f induces a function $\bar{f}\colon \Pi(D) \to \Pi(D)$ (its extension) by

$$\bar{f}(x) = \bigcup_{F \in f} \bar{F}(x) = \big\{F(X) \mid F \in f, X\colon I \to x \text{ where } F : D^I \to D\big\}. \qquad (13)$$

The members of f are called its *implementations*, and a set $\mathcal{F}$ of polyfunctions*generates* f, written $f = \langle \mathcal{F} \rangle$, if f is the closure of $\mathcal{F}$ under $\asymp$, i.e.,

$$G \in f \iff (\exists F \in \mathcal{F}) \text{ such that } G \asymp F.$$

It is not difficult to see that a generating set of implementations suffices to determine the extension of f as per (13). For n-ary ipfs we use polyfunctions $F\colon D^{I_1} \times \cdots \times D^{I_n} \to D$ and proceed similarly.

Polyfunctions generalize the *infinitary behavior functions* of [3], where, however, only one index set was allowed, $I = \mathbb{N}$. A more essential difference is the present choice of polyfunction equivalence, which is less coarse than that of [3] and produces more natural modelings in the specific examples.[6] This choice of equivalence requires some extra care in the correct definition of *ipf composition* and *ipf recursion*, but these constructions are quite similar to those of [3] and we will skip them. We mention the one technical notion needed in the proofs below, to set notation.

[6] The desirability of this refinement was discussed briefly in Footnote 8 of [3], but the methods of that paper were not strong enough to prove the main results with the present, more natural equivalence relation.

An *implementation system* for a single ipf equation of the form $x = f(x)$ is a labeled infinite tree $\mathbf{F}$, whose vertices are the set $\mathbb{N}^*$ of finite sequences of natural numbers. Each vertex is labeled with an implementation F_τ of f, and so $\mathbf{F}$ determines an infinite system of recursive equations over D,

$$X_\tau = F_\tau(\lambda(i \in \mathbb{N})X_{\tau i}) \quad (\tau \in \mathbb{N}^*),$$

where τi is the result of appending i to the end of τ. We let $\{\hat{X}_\tau \mid \tau \in \mathbb{N}^*\}$ be the set of mutual fixed points of this system, and put

$$\hat{x} = \left\{ \hat{X}_\emptyset : \hat{X}_\tau \text{ are the simultaneous least fixed points of some } \mathbf{F} \right\}.$$

This $\hat{x} \in \Pi(D)$ is the "canonical" *ipf fixpoint* of the equation $x = f(x)$, and it is not hard to verify that, indeed, it is a fixed point. The construction of canonical fixpoints for systems of equations with parameters is similar but more complicated, and still very close to [3].

The proof of Theorem A now essentially consists of showing that the standard identities hold in $\mathbf{ipf}(D)$. Armed with the axiomatization mentioned in Section 1, it suffices to verify a specific, short list of identities. This method improves on that of [3], both in content (as we can handle arbitrary D and the refined equivalence relation) and in simplicity.

3.1 Comparing $\mathbf{ipf}(D)$ and $\mathbf{plot}(D)$

Turning to Theorem B, we first need to define $\mathbf{ipf}_0(D)$, which is most easily done topologically. So, place the usual *Scott topology* on all dcpos; note that for algebraic D, a base of this topology is given by the collection of sets $N_D(e) = \{d \in D \mid e \leq d\}$ for e finite. Let $\mathbb{C}$ be the Cantor set (all infinite binary sequences) with its usual topology, and call a subset X of D *compact-analytic* if it is the continuous image of $\mathbb{C}$, i.e., if there is a (topologically) continuous function $F\colon \mathbb{C} \to D$ such that $F[\mathbb{C}] = X$. Since $\mathbb{C}$ is homeomorphic to the direct product of countably many copies of itself, it is not hard to see that compact-analytic sets are closed under countable direct products and continuous images.

Now restrict attention for the remainder of this section to *strongly algebraic D*. These are the "SFP objects" of Plotkin [6]; we need just the following properties: If D is strongly algebraic, then there is an increasing sequence $D_0 \subset D_1 \subset D_2 \cdots$ of finite sets of finite elements of D whose union is all finite elements of D. Furthermore, there is a family of projections $p_n\colon D \to D_n$ such that $p_{n+1} \circ p_n = p_{n+1}$ and for all $d \in D$, $d = \sup_n p_n(D)$.

It is not difficult to check that for strongly algebraic D, the compact-analytic subsets of D coincide with the finitely generable ones.[7] Therefore, think of $\mathbf{ipf}_0(D)$ as a structure on the compact-analytic subsets of D. To provide the transformations $\mathit{ipf}_0(D)$, call an ipf f *compact-analytic*[8] if it is generated by a

[7] This statement in fact holds for all algebraic D, but the proof requires considerably more work; see [9].

[8] See also [4].

family of polyfunctions of the form $\{F_\alpha : D^{\mathbb{N}} \to D \mid \alpha \in \mathbb{C}\}$ where the function $F: \mathbb{C} \times D^{\mathbb{N}} \to D$ via $F(\alpha, p) = F_\alpha(p)$ is continuous. These ipfs have implementations continuously parametrized by the Cantor set, which one can think of as a space of "oracles" for the corresponding non-deterministic function. The closure properties of compact-analytic sets guarantee that such an f takes compact-analytic players to compact-analytic players, as $\bar{f}(x) = F[\mathbb{C} \times x^{\mathbb{N}}]$.

It is a fact that the compact-analytic ipfs $\mathit{ipf}_0(D)$ and players $\Pi_0(D)$ are closed under composition and recursion, which means that $\Pi_0(D)$, $\mathit{ipf}_0(D)$, and the (restriction of) the usual ipf interpretation form an FLR$_0$-structure $\mathbf{ipf}_0(D)$. The proof of this fact is not difficult from the definitions, and is similar to the portions of Theorems 8.2 and 8.4 of [3] which state that ipf recursion preserves "type."

Lemma. *For x, y finitely generable, $x \precsim y$ is equivalent to the conjunction of $x \sqsubseteq^u y$ and $x \sqsubseteq'_l y$, where $x \sqsubseteq'_l y$ if for every $c \in x$ and every finite $a \in D$ such that $a \le c$, there exists $d \in y$ such that $a \le d$.*

Intuitively, $x \sqsubseteq'_l y$ means that every finite approximation to x is also an approximation to y. This form of $\precsim$ will be most useful in the following proofs. Rephrasing Theorem B, part (a), we now wish to show

Claim. For any ipf f and players x and y, all compact-analytic, we have

(1) If $x \precsim y$, then $\bar{f}(x) \precsim \bar{f}(y)$.

This condition means that $\bar{f}$ takes $\simeq_{\mathrm{em}}$-equivalent players to $\simeq_{\mathrm{em}}$-equivalent ones, so it induces a monotone function on the Plotkin powerdomain.

(2) The induced function $\rho(f)$ is continuous on the Plotkin powerdomain.

Proof of claim. Let $\mathcal{A}(d)$ denote the set of finite elements less than or equal to a given $d \in D$. Since D is algebraic, $\mathcal{A}(d)$ is directed and $\sup \mathcal{A}(d) = d$. Also let $F: \mathbb{C} \times D^{\mathbb{N}} \to D$ be the continuous parametrization of f.e

Suppose that a is a finite approximation to an element $c \in \bar{f}(x)$. By definition of ipf application, $c = F_\alpha(X)$ for some $\alpha \in \mathbb{C}$ and $X: I \to x$. By continuity of F,

$$a \le c = \sup \{F_\alpha(A) \mid \forall i, A(i) \in \mathcal{A}(X(i))\}.$$

But a is finite, so some individual term of the right-hand sup must already be beyond a. That is, for a *particular* sequence $A \in D^I$ such that $A(i) \in \mathcal{A}(X(i))$, we have $a \le F_\alpha(A)$. Each $A(i)$ is finite below $X(i) \in x$, and hence there is some $Y(i) \in y$ such that $A(i) \le Y(i)$. Finally, by monotonicity of F_α, $a \le F_\alpha(Y) \in \bar{f}(y)$. In other words, $\bar{f}(x) \sqsubseteq'_l \bar{f}(y)$.

To finish the first part of the claim, show that $\bar{f}(x) \sqsubseteq^u \bar{f}(y)$: if $d = F_\alpha(Y) \in \bar{f}(y)$, choose a map $X: I \to x$ such that $X(i) \le Y(i)$ for each $i \in I$. This is possible since $x \sqsubseteq^u y$. Then $x \ni c = F_\alpha(X) \le F_\alpha(Y) = d$.

The second part of the claim asserting the continuity of $\rho(f)$ is more delicate. Note it suffices to show that for each point z in the Plotkin powerdomain, there is a sequence of finite elements a_n with supremum z such that $\rho(f)(z)$ is the supremum of $\rho(f)(a_n)$. Let $p_n: D \to D_n$ be the sequence of projections witnessing that D is strongly algebraic. Choose the set-maximal representative Z^* of z from $\Pi_0(D)$, constructed in section 7 of Smyth [8]. Unsurprisingly, the players $A_n = p_n[Z^*]$ have $\precsim$-supremum Z^*; we shall verify that the supremum of $\bar{f}(A_n)$ is $\bar{f}(Z^*)$. The key property of the choice of representative Z^* is that if $\{b_i\}_{i\in\mathbb{N}}$ is any increasing sequence with $b_i \in A_i$, then the supremum of the b_i is in Z^*.

Actually, Z^* is a supremum of the A_n in the preorders $\sqsubseteq_l'$ and $\sqsubseteq^u$, individually; so we can check that $\bar{f}(Z^*)$ is the supremum of the A_n in these two preorders individually, as well. For $\sqsubseteq_l'$, the argument goes similarly to the monotonicity argument above, but using the stronger approximation

$$X = \sup_{I'\subseteq I, |I'|<\infty} \sup\left\{A \mid A(i) \in \mathcal{A}(X(i)) \text{ and } A(i) = \bot \text{ for } i \notin I'\right\},$$

for a function $X\colon I \to Z^*$, which results from the pointwise ordering on D^I.

To show that $\bar{f}(Z^*)$ is the $\sqsubseteq^u$- supremum of the $\bar{f}(A_n)$, let Y be any $\sqsubseteq^u$-upper bound of the $\bar{f}(A_n)$, and choose any member d of Y. Then for each n, there is some α_n and map $B_n\colon \mathbb{N} \to A_n$ such that $\bar{f}(A_n) \ni F(\alpha_n, B_n) \le d$. Now, $\mathbb{C}$ is a compact topological space, and every point has a countable neighborhood base, so $\mathbb{C}$ is sequentially compact. Therefore we may safely assume that the α_n converge as n increases, to some $\alpha \in \mathbb{C}$. Next notice that the projections p_t "push down" so that $F(\alpha_n, p_t \circ B_n) \le d$ for any t and n as well. By similar compactness arguments (using Tychonoff's theorem to see that $D^{\mathbb{N}}$ is compact), one can choose a subsequence B_{n_t} so that for each $k \in \mathbb{N}$, the sequence of values $p_t(B_{n_t}(k)) \in A_t$ is eventually monotone as t increases. Hence for each k, this sequence converges to $Z(k) \in Z^*$, so that

$$F(\alpha_{n_t}, p_t \circ B_{n_t}) \to F(\alpha, W) = c \in \bar{f}(Z^*) \text{ as } t \to \infty.$$

Since each element in the left hand sequence is $\le d$, the limit $c \le d$. Hence every element of Y has an element of $\bar{f}(Z^*)$ below it, as desired. ☐

The previous claim together with the definition that $\rho(x) = [x/\simeq_{\text{em}}]$ yields a map from the universe of $\mathbf{ipf}_0(D)$ to the standard structure over the Plotkin powerdomain of D. The next part of Theorem B states that this map ρ is an FLR_0-homomorphism. To see this, it suffices to show that ρ preserves function compositions and systems of recursive equations, since all FLR_0 expressions are built up from these operations. The fact that $\rho(f) \circ \rho(g) = \rho(f \circ g)$ is easy, because $\rho(f)$ depends only on the extension $\bar{f}$. For recursion, suppose that $f(x)$ is a compact-analytic ipf with player fixed point $\hat{x}$, and that X is the least fixed point in the Plotkin powerdomain of $\rho(f)$. We wish to prove that $X = \rho(\hat{x}) = [\hat{x}/\simeq_{\text{em}}]$. (The following argument will directly generalize to systems of equations with parameters.)

First, $\rho(f)$ fixes $[\hat{x}/\simeq_{\text{em}}]$, since by definition $\rho(f)([\hat{x}/\simeq_{\text{em}}]) = [\bar{f}(\hat{x})/\simeq_{\text{em}}]$ and $\bar{f}$ fixes $\hat{x}$. Choose a compact-analytic representative x_0 of X in the Plotkin powerdomain; since X is the least fixed point of $\rho(f)$, this means that $x_0 \precsim \hat{x}$. On the other hand, we know that $\bar{f}(x_0) \simeq_{\text{em}} x_0$ since the $\simeq_{\text{em}}$-equivalence class X of x_0 is fixed by $\rho(f)$. In particular, $\bar{f}(x_0) \precsim x_0$. To finish the proof, we show that for any compact-analytic player y, $\bar{f}(y) \precsim y$ implies that $\hat{x} \precsim y$:

Suppose $\bar{f}(y) \lesssim y$. We shall show separately that $\hat{x} \sqsubseteq_l' y$ and $\hat{x} \sqsubseteq^u y$. Let $a \leq \hat{c}_\emptyset \in \hat{x}$ be a finite approximation to an element of $\hat{x}$, where $\hat{c}_\emptyset$ is the top-level fixed point for some implementation system $\mathbf{F}$ of f. By induction on k, each iterate $c_\tau^{(k)}$ of the corresponding fixed point equations satisfies that the singleton $\left\{c_\tau^{(k)}\right\} \sqsubseteq_l' y$. Since a is a finite approximation to $\hat{c}_\emptyset$, it must be less than or equal to some top-level iterate $c_\emptyset^{(K)}$ of $\mathbf{F}$, which is in turn less than some $d \in y$, as desired.

For $\hat{x} \sqsubseteq^u y$, let d be any element of y. Since $f(y) \sqsubseteq^u y$, choose some $c \in f(y)$ such that $c \leq d$. Now c must be of the form $F(d_1, d_2, \ldots)$ for some $F \in f$ and $d_1, d_2, \ldots \in y$. Proceeding by induction, once $d_\tau \in y$ is chosen, find some $F_\tau \in f$ and $d_{\tau\hat{\ }1}, d_{\tau\hat{\ }2}, \ldots \in y$ so that $d_\tau \geq F_\tau(d_{\tau\hat{\ }1}, d_{\tau\hat{\ }2}, \ldots)$. In this way construct d_τ and F_τ for all finite sequences from $\mathbb{N}$.

The F_τ of course form an implementation system $\mathbf{F}$ for the ipf equation $x = f(x)$. This system has least fixed points $\hat{d}_\tau$ so that the top-level $\hat{d}_\emptyset \in x$. But since $d_\tau \geq F_\tau(d_{\tau\hat{\ }1}, \ldots)$, it must be that $d_\tau \geq \hat{d}_\tau$ for all τ, by the general least-fixed-point properties of the $\hat{d}_\tau$. In particular, $y \ni d \geq \hat{d}_\emptyset \in x$ as desired.

Only part (c) of Theorem B remains, which says that the homomorphism ρ produces every continuous, essentially monotone function on the Plotkin powerdomain. To prove this part, we must take an arbitrary essentially monotone $g\colon \mathbf{plot}(D) \to \mathbf{plot}(D)$ and construct a compact-analytic ipf f so that $\rho(f) = g$. Restated, the goal is to find a continuous function $F\colon \mathbb{C} \times D^{\mathbb{N}} \to D$ such that $F[\mathbb{C} \times x^{\mathbb{N}}] \simeq_{\mathrm{em}} g([x/ \simeq_{\mathrm{em}}])$ for any $x \in \Pi_0(D)$. One reasonable approach is to arrange that if $X\colon \mathbb{N} \to x$ is surjective and so enumerates x, then $F(\alpha, p)$ will pick out all members of $g([x])^*$ as α varies over $\mathbb{C}$.

This approach calls for a "selection function" S which will take an $\alpha \in \mathbb{C}$ and some set R of the form $R = Y^*$ and return an element of R, so that

$$\{S(\alpha, R) \mid \alpha \in \mathbb{C}\} = R, \text{ for every } R. \tag{14}$$

The following labeled tree σ which "generates" D will help to construct S. As before, let $p_n\colon D \to D_n$ be the projections which witness the fact that D is strongly algebraic. The root of σ is labeled with $\perp$; if $n-1$ levels have been constructed, then the new children of a current leaf σ_u should consist of a set of nodes labeled with each element of D_n greater than or equal to σ_u.

We may assume that σ is actually a perfect, infinite binary tree by replacing each node having k children by a small binary tree with k leaves. Therefore, identify the nodes of σ with finite binary sequences, and the set of branches of σ with the Cantor set. We denote by σ_α the supremum of the labels along the αth branch of σ. By construction, σ_α varies over all of D as α ranges over $\mathbb{C}$. Moreover, an analogue of this property holds at every node. For any node u, let $N(u)$ denote the set of branches which extend u. Then as α varies over $N(u)$, σ_α varies over all elements of D greater than or equal to σ_u, i.e., $N_D(\sigma_u)$.

Let $\mathcal{R} = \{Y^* : Y \in \Pi_0\}$ be the range of the $*$-operation. For the purposes of our proof, any selection function $S\colon \mathbb{C} \times \mathcal{R} \to D$ satisfying (15) will do:

$$\begin{gathered} S(\alpha, R) = \sigma_\alpha \text{ if } \sigma_\alpha \in R \\ \forall u \text{ such that } \{\sigma_u\} \sqsubseteq_l R, \forall \alpha \text{ extending } u, \quad \sigma_u \leq S(\alpha, R). \end{gathered} \tag{15}$$

The following construction provides one such S. Fixing an R for the moment, find all minimal u so that no element of R is greater than or equal to σ_u. For such a u, let t be its parent. Then R does contain some element greater than σ_t, which is exactly to say that $\{\sigma_t\} \sqsubseteq_l R$. Therefore, choose the "leftmost" branch extending t whose limit is in R, and call this limit $m_{R,u}$.

Now define $S(\alpha, R) = m_{R,u}$ if $\alpha \in N(u)$, and $S(\alpha, R) = \sigma_\alpha$, otherwise. There is at most one initial sequence u of α so that $m_{R,u}$ is defined, and if there is none then $\sigma_\alpha \in R$ since R is closed under limits of approximations to itself; these observations ensure that S is well-defined and satisfies properties (14) and (15).

Finally define for $X \in D^{\mathbb{N}}$, $\alpha \in \mathbb{C}$,

$$F(\alpha, X) = S\big(\alpha, g([X[\mathbb{N}]])^*\big),$$

and let f be the ipf generated by the sections F_α. We must prove both that F is continuous and that $\rho(f) = g$. So, compute the inverse image of a generic neighborhood $N_D(a)$ under F, as follows. Let U be the set of all minimal u so that $\sigma_u \geq a$. By construction of S,

$$S^{-1}(N_D(a)) = \bigcup_{u \in U} N(u) \times \{R \mid \{\sigma_u\} \sqsubseteq_l R\}.$$

Since each $N(u)$ is open in $\mathbb{C}$, we need only show that the collection of all X so that $\{\sigma_u\} \sqsubseteq_l g([X[\mathbb{N}]])^*$ is open in $D^{\mathbb{N}}$. Fix a u and let $b = \sigma_u$ for convenience of notation. Note first that if $\{b\} \sqsubseteq_l R$, i.e., if $\exists d \in R$ s.t. $b \leq d$, then there is some finite set of finite elements $A \ni b$ so that $A \underset{\sim}{\sqsubset} R$. That is, letting $B = \{[R] : \{b\} \sqsubseteq_l R\}$, then

$$B = \bigcup_{\substack{A \text{ finite} \\ A \ni b}} N_{\mathbf{plot}(D)}(A),$$

and so B is open in $\mathbf{plot}(D)$. This means that $g^{-1}(B)$ is also open in $\mathbf{plot}(D)$, and hence is of the form

$$g^{-1}(B) = \bigcup_{A \in \mathcal{A}'} N_{\mathbf{plot}(D)}(A), \tag{16}$$

for some collection $\mathcal{A}'$ of finite sets of finite elements of D.

Unfortunately, saying that the range $X[\mathbb{N}]$ of a function from $\mathbb{N}$ to D is bigger than a given finite set of finite elements A in the order $\underset{\sim}{\sqsubset}$ is *not* an open condition on X, because *every* value of X must then be greater than or equal to some element of A. The following argument which converts (16) into a collection of neighborhoods in the order $\sqsubseteq_l$ is therefore the crux of the continuity proof.

Suppose y is given so that $A \sqsubseteq_l y$ for some $A \in \mathcal{A}'$. Clearly one can choose some subset $z \subseteq y$ such that $A \sqsubseteq^u z$ as well, so that in fact $A \underset{\sim}{\sqsubset} z$. Then

$g([A]) \lesssim g([z])$, so $\{b\} \sqsubseteq_l g([A])^* \sqsubseteq_l' g([z])^*$. But g is essentially monotone, so $g([z])^* \subseteq g([y])^*$, and hence $\{b\} \sqsubseteq_l g([y])^*$, which is to say that $g([y]) \in B$.

Conversely, if there is no $A \in \mathcal{A}'$ such that $A \sqsubseteq_l y$, then certainly no $A \in \mathcal{A}'$ has $A \lesssim y$, and so $g([y]) \notin B$. Thus equation (16) can be improved to

$$g^{-1}(B) = \bigcup_{A \in \mathcal{A}'} \{[y] \mid A \sqsubseteq_l y\} .$$

Finishing the proof of F's continuity therefore only requires showing that any set of the form $\mathcal{X} = \{X \in D^{\mathbb{N}} \mid A \sqsubseteq_l X[\mathbb{N}]\}$ for A a finite set of finite elements is open in $D^{\mathbb{N}}$. But $A \sqsubseteq_l X[\mathbb{N}]$ just means there is some finite set of natural numbers whose images under X meet $N_D(c)$ for each $c \in A$. Any candidate set of natural numbers of the appropriate size yields a finite condition on X, and so the entire set $\mathcal{X}$ is a union of basic neighborhoods in $D^{\mathbb{N}}$.

It remains to show that $\rho(f) = g$. If $x \in \Pi_0$ and $X\colon \mathbb{N} \to x$, then $X[\mathbb{N}] \subseteq x$. By the essential monotonicity of g, $g([X[\mathbb{N}]])^* \subseteq g([x])^*$. This inequality with

$$f(x) = \bigcup_{X:\mathbb{N}\to x} F[\mathbb{C} \times \{X\}] = \bigcup_{X:\mathbb{N}\to x} g([X[\mathbb{N}]])^*$$

shows that $f(x) \subseteq g([x])^*$. Equality will be achieved if for some X, $X[\mathbb{N}] \simeq_{\mathrm{em}} x$. Collecting for each n and each element a of $p_n[x]$ some element $d_a \in x$ with $a \leq d_a$ produces a countable set which is $\simeq_{\mathrm{em}}$-equivalent to x. Then an X that enumerates $\{d_a\}$ suffices.

This completes the proof of all three parts of Theorem B, providing a vivid picture of the Plotkin powerdomain as a powerstructure quotient of the player model.

3.2 Countable non-determinism

Since both the ipf structure $\mathbf{ipf}(D)$ and the powerdomains for countable non-determinism $\mathbf{plot}_\omega(D)$ seek to improve on the ability of earlier powerdomains to model fairness, it is natural to compare them. Is there a "countable non-determinism" analogue of Theorem B? Unfortunately, the answer is "no," for the simple reason that $\mathbf{plot}_\omega(D)$ does not constitute a powerstructure. The points in $\mathbf{plot}_\omega(D)$ cannot (all) be viewed as subsets of D. In other words, although it still may be true that "direct existence [of $\mathbf{plot}_\omega(D)$] along the lines of [6, 8] should be established," as Plotkin [7] suggests, no construction which stays within the subsets of D can accomplish this goal.

Let W be the domain $(1_\perp \times \mathbb{N})_\perp$, which is just a tree with root $\perp$ and countably many branches of length 2. Concretely, we take the elements of W to be $\perp$ and all pairs of the form $(0, n)$ or $(1, n)$ for any $n \in \mathbb{N}$. The ordering on W is that $\perp < (0, n) < (1, n)$ for any n, and these are the only relations. W is very close to being flat, having maximal chain length 3 as opposed to 2. Also, say that one dcpo D *embeds* another E if there is a projection from D onto a subdcpo $D' \subseteq D$ such that D' and E are isomorphic.

Proof of Theorem C. We begin by showing that $\mathcal{P} = (\Pi(W)/\simeq_{\mathrm{em}}, \sqsubseteq\!\!\!\sim, \subseteq)$ is a σ-semilattice, where $\subseteq$ is the ordinary subset relation (modulo $\simeq_{\mathrm{em}}$). First note that since W has only finite elements, $\underset{\sim}{\sqsubset}$ is just the conjunction of $\sqsubseteq^u$ and $\sqsubseteq_l$ on W. Since every $\underset{\sim}{\sqsubset}$-chain in $\Pi(W)$ has countable length, $(\Pi(W), \underset{\sim}{\sqsubset})$ is a dcpo exactly if it contains a least upper bound for each sequence. Let $x_0 \underset{\sim}{\sqsubset} x_1 \underset{\sim}{\sqsubset} \cdots$ be such a sequence, and check that its supremum is

$$x = \left\{ \sup_n a_n \mid a_n \in x_n,\, a_m \leq a_n \text{ for } m \leq n \right\}.$$

Next, $\subseteq$ must have least upper bounds of arbitrary countable sets from $\mathcal{P}$. But $\bigcup_n x_n$ is the ordinary $\subseteq$-least upper bound of $\{x_n \mid n \in \mathbb{N}\}$, so $[\bigcup_n x_n/\simeq_{\mathrm{em}}]$ is the least upper bound of $\{[x_n] \mid n \in \mathbb{N}\}$ on $\mathcal{P}$.

Finally, countable union (i.e., the operation of taking the $\subseteq$-least upper bound of countably many arguments) must be ω_1-continuous and binary union must be continuous, with respect to $\underset{\sim}{\sqsubset}$. The former is trivial because there are no uncountable chains in $\mathcal{P}$; the latter just requires that whenever $x = \sup x_n$, then $x \cup y = \sup x_n \cup y$, which is easy to check because W only has chains of finite length.

Thus, $\mathcal{P}$ is a σ-semilattice; however, it is *not* free over W. The singleton map would have to be the obvious $\{\cdot\}: w \mapsto [\{w\}]$. Now consider the map f from W into the σ-semilattice[9] $\mathcal{Q} = (\Pi(\mathbb{N}_\perp), \underset{\sim}{\sqsubset}, \subseteq)$ given by $f(\perp) = f(0, n) = \{\perp\}$ and $f(1, n) = \{n\}$. f is certainly continuous, and so if $\mathcal{P}$ were free, f would have to factor through the singleton map. The values of F at singletons determine all its other values, since F must preserve countable unions and W is itself countable. Therefore, the only possibility for F turns out to be

$$F(A) = \begin{cases} \{n \mid (1,n) \in A\} & \text{if } A \cap (\{\perp\} \cup \{(0,n) \mid n \in \mathbb{N}\}) = \emptyset, \\ \{n \mid (1,n) \in A\} \cup \{\perp\} & \text{otherwise.} \end{cases}$$

But this map F is not continuous (on the "dcpo" part $(\Pi(W), \underset{\sim}{\sqsubset})$ of the semilattices), which is a contradiction. To see the non-continuity, consider the sets $H_k = \{(1,n) \mid n < k\} \cup \{(0,n) \mid n \geq k\}$. Clearly the H_k are increasing in $\underset{\sim}{\sqsubset}$, and have least upper bound $H = \{(1,n) \mid n \in \mathbb{N}\}$. But $F(H_k) = \{\perp, 0, 1, \ldots, k-1\}$ so that $\sup_k F(H_k) = \{\perp\} \cup \mathbb{N}$, whereas $F(H) = \mathbb{N}$ is strictly bigger than $\{\perp\} \cup \mathbb{N}$ in $\mathcal{Q}$. (Intuitively, the true free σ-semilattice over W will have to include some "ideal" element, not corresponding to any set, to be the least upper bound of the sequence $\{H_k\}_{k \in \mathbb{N}}$.)

Assume by way of contradiction that the *free* σ-semilattice $\mathcal{S}$ over W is the homomorphic image of $(\Pi(W), \sqsubset, \subseteq)$ for some partial order $\sqsubset$. The homomorphism induces some equivalence relation $\sim$ on $\Pi(W)$, so that $S = (\Pi(W)/\!\sim, \sqsubset/\!\sim, \subseteq/\!\sim)$, with singleton map $w \mapsto \{w\}/\!\sim$. Since $\mathcal{P}$ is a σ-semilattice and the map $w \mapsto [\{w\}] \in \mathcal{P}$ is continuous, there is a σ-semilattice map $\phi: \mathcal{S} \to \mathcal{P}$ so

[9] See Plotkin [7] for a proof that this *is* the free σ-semilattice over $\mathbb{N}_\perp$; the proof that it is simply a σ-semilattice is analogous to the proof for W.

that the following diagram commutes:

$$\begin{array}{ccc} & W & \\ \{\cdot\} & \downarrow & \searrow \{\cdot\} \\ & \mathcal{S} & \xrightarrow{\phi} \mathcal{P} \end{array}$$

Now, $w \mapsto [\{w\}] \in \mathcal{P}$ is an injective map, so the singleton map into $\mathcal{S}$ must be as well, i.e., $\sim$ cannot identify any singletons. Since the subset relation in $\mathcal{S}$ is ordinary subset (modulo $\sim$), the $\subset$-lub of a countable set A of singletons in $\mathcal{S}$ is just $\{w \mid \{w\} \in A\}/\!\sim$. Since W is countable, every element of $\mathcal{S}$ is produced in this way, and since ϕ must preserve $\subseteq$, this means that the map ϕ is simply given by

$$\phi(x/\!\sim) = [x] \in \mathcal{P}$$

for any $x \in \Pi(W)$. Thus ϕ is clearly surjective. Suppose that $\phi(x/\!\sim) = \phi(y/\!\sim)$, which means that x is equivalent to y under $\sqsubseteq\!\!\!\!\sim$. Write x as $\{a_0, a_1, \ldots\}$ and $y = \{b_0, b_1, \ldots\}$ possibly with repetitions so that $a_i \leq b_i$. Since the singleton map to $\mathcal{S}$ is monotone, and countable union in a σ-semilattice is monotone, this representation of x and y shows that $x/\!\sim \sqsubset y/\!\sim$. Symmetrically, $y/\!\sim \sqsubset x/\!\sim$, so that ϕ is injective as well. Hence ϕ is an isomorphism of σ-semilattices witnessing $\mathcal{S} \cong \mathcal{P}$. But $\mathcal{P}$ is *not* free over W.

This argument proves the theorem for $D = W$. To extend to the case that D embeds W, just notice that any representation of the free σ-semilattice over D in the given form would yield such a representation of the free σ-semilattice over W by taking a quotient under the projection from D to W. $\square$

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