PREFACE

What this book is about. The theory of sets is a vibrant, exciting mathematical theory, with its own basic notions, fundamental results and deep open problems, and with significant applications to other mathematical theories. At the same time, axiomatic set theory is often viewed as a foundation of mathematics: it is alleged that all mathematical objects are sets, and their properties can be derived from the relatively few and elegant axioms about sets. Nothing so simple-minded can be quite true, but there is little doubt that in standard, current mathematical practice, “making a notion precise” is essentially synonymous with “defining it in set theory”. Set theory is the official language of mathematics, just as mathematics is the official language of science.

Like most authors of elementary, introductory books about sets, I have tried to do justice to both aspects of the subject.

From straight set theory, these Notes cover the basic facts about “abstract sets”, including the Axiom of Choice, transfinite recursion, and cardinal and ordinal numbers. Somewhat less common is the inclusion of a chapter on “pointsets” which focuses on results of interest to analysts and introduces the reader to the Continuum Problem, central to set theory from the very beginning. There is also some novelty in the approach to cardinal numbers, which are brought in very early (following Cantor, but somewhat deviously), so that the basic formulas of cardinal arithmetic can be taught as quickly as possible. Appendix A gives a more detailed “construction” of the real numbers than is common nowadays, which in addition claims some novelty of approach and detail. Appendix B is a somewhat eccentric, mathematical introduction to the study of natural models of various set theoretic principles, including Aczel’s Antifoundation. It assumes no knowledge of logic, but should drive the serious reader to study it.

About set theory as a foundation of mathematics, there are two aspects of these Notes which are somewhat uncommon. First, I have taken seriously this business about “everything being a set” (which of course it is not) and have tried to make sense of it in terms of the notion of faithful representation of mathematical objects by structured sets. An old idea, but perhaps this is the first textbook which takes it seriously, tries to explain it, and applies it consistently. Those who favor category theory will recognize some of its basic notions in places, shamelessly folded into a traditional set theoretical
approach to the foundations where categories are never mentioned. Second, computation theory is viewed as part of the mathematics “to be founded” and the relevant set theoretic results have been included, along with several examples. The ambition was to explain what every young mathematician or theoretical computer scientist needs to know about sets.

The book includes several historical remarks and quotations which in some places give it an undeserved scholarly gloss. All the quotations (and most of the comments) are from papers reprinted in the following two, marvellous and easily accessible source books, which should be perused by all students of set theory:


**How to use it.** About half of this book can be covered in a Quarter (ten weeks), somewhat more in a longer Semester. Chapters 1 – 6 cover the beginnings of the subject and they are written in a leisurely manner, so that the serious student can read through them alone, with little help. The trick to using the Notes successfully in a class is to cover these beginnings very quickly: skip the introductory Chapter 1, which mostly sets notation; spend about a week on Chapter 2, which explains Cantor’s basic ideas; and then proceed with all deliberate speed through Chapters 3 – 6, so that the theory of well ordered sets in Chapter 7 can be reached no later than the sixth week, preferably the fifth. Beginning with Chapter 7, the results are harder and the presentation is more compact. How much of the “real” set theory in Chapters 7 – 12 can be covered depends, of course, on the students, the length of the course, and what is passed over. If the class is populated by future computer scientists, for example, then Chapter 6 on Fixed Points should be covered in full, with its problems, but Chapter 10 on Baire Space might be omitted, sad as that sounds. For budding young analysts, at the other extreme, Chapter 6 can be cut off after 6.27 (and this too is sad), but at least part of Chapter 10 should be attempted. Additional material which can be left out, if time is short, includes the detailed development of addition and multiplication on the natural numbers in Chapter 5, and some of the less central applications of the Axiom of Choice in Chapter 9. The Appendices are quite unlikely to be taught in a course (I devote just one lecture to explain the idea of the construction of the reals in Appendix A), though I would like to think that they might be suitable for undergraduate Honors Seminars, or individual reading courses.

Since elementary courses in set theory are not offered regularly and they are seldom long enough to cover all the basics, I have tried to make these Notes accessible to the serious student who is studying the subject on their own. There are numerous, simple Exercises strewn throughout the text, which test understanding of new notions immediately after they are introduced. In class I present about half of them, as examples, and I assign some of the rest
for easy homework. The Problems at the end of each chapter vary widely in difficulty, some of them covering additional material. The hardest problems are marked with an asterisk (\(*\)).

Acknowledgments. I am grateful to the Mathematics Department of the University of Athens for the opportunity to teach there in Fall 1990, when I wrote the first draft of these Notes, and especially to Prof. A. Tsarpalias who usually teaches that Set Theory course and used a second draft in Fall 1991; and to Dimitra Kitsiou and Stratos Paschos for struggling with PCs and laser printers at the Athens Polytechnic in 1990 to produce the first “hard copy” version. I am grateful to my friends and colleagues at UCLA and Caltech (hotbeds of activity in set theory) from whom I have absorbed what I know of the subject, over many years of interaction. I am especially grateful to my wife Joan Moschovakis and my student Darren Kessner for reading large parts of the preliminary edition, doing the problems and discovering a host of errors; and to Larry Moss who taught out of the preliminary edition in the Spring Term of 1993, found the remaining host of errors and wrote out solutions to many of the problems.

The book was written more-or-less simultaneously in Greek and English, by the magic of bilingual \LaTeX and in true reflection of my life. I have dedicated it to Prof. Nikos Kritikos (a student of Caratheodory), in fond memory of many unforgettable hours he spent with me back in 1973, patiently teaching me how to speak and write mathematics in my native tongue, but also much about the love of science and the nature of scholarship. In this connection, I am also greatly indebted to Takis Koufopoulos, who read critically the preliminary Greek version, corrected a host of errors and made numerous suggestions which (I believe) improved substantially the language of the final Greek draft.

Palaion Phaliron, Greece

November 1993

About the 2nd edition. Perhaps the most important changes I have made are in small things, which (I hope) will make it easier to teach and learn from this book: simplifying proofs, streamlining notation and terminology, adding a few diagrams, rephrasing results (especially those justifying \textit{definition by recursion}) to ease their applications, and, most significantly, correcting errors, typographical and other. For spotting these errors and making numerous, useful suggestions over the years, I am grateful to Serge Bozon, Joel Hamkins, Peter Hinman, Aki Kanamori, Joan Moschovakis, Larry Moss, Thanassis Tsarpalias and many, many students.

The more substantial changes include:

— A proof of \textit{Suszlin’s Theorem} in Chapter 10, which has also been significantly massaged.

— A better exposition of ordinal theory in Chapter 12 and the addition of some material, including the basic facts about ordinal arithmetic.
— The last chapter, a compilation of solutions to the Exercises in the main part of the book – in response to popular demand. This eliminates the most obvious, easy homework assignments, and so I have added some easy problems.

I am grateful to Thanos Tsouanas, who copy-edited the manuscript and caught the worst of my mistakes.

Palaion Phaliron, Greece                July 2005
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CHAPTER 1

INTRODUCTION

Mathematicians have always used sets, e.g., the ancient Greek geometers defined a circle as the set of points at a fixed distance \( r \) from a fixed point \( C \), its center. But the systematic study of sets began only at the end of the 19th century with the work of the great German mathematician Georg Cantor, who created a rigorous theory of the concept of *completed infinite* by which we can compare infinite sets as to size. For example, let

\[
\begin{align*}
\mathbb{N} &= \{0, 1, \ldots \} = \text{the set of natural numbers}, \\
\mathbb{Z} &= \{\ldots, -1, 0, 1, \ldots \} = \text{the set of rational integers}, \\
\mathbb{Q} &= \text{the set of rational numbers (fractions)}, \\
\mathbb{R} &= \text{the points of a straight line},
\end{align*}
\]

where we also identify \( \mathbb{R} \) with the set of real numbers, each point associated with its (positive or negative) coordinate with respect to a fixed origin and direction. Cantor asked if these four sets “have the same (infinite) number of elements”, or if one of them is “more numerous” than the others. Before we make precise and answer this question in the next chapter, we review here some basic, well-known facts about sets and functions, primarily to explain the notation we will be using.

What are sets, anyway? The question is like “what are points”, which Euclid answered with

a point is that which has no parts.

This is not a rigorous mathematical definition, a reduction of the concept of “point” to other concepts which we already understand, but just an intuitive description which suggests that a point is some thing which has no extension in space. Like that of point, the concept of set is fundamental and cannot be reduced to other, simpler concepts. Cantor described it as follows:

By a set we are to understand any collection into a whole of definite and separate objects of our intuition or our thought.

Vague as it is, this description implies two basic properties of sets.

1. Every set \( A \) has **elements** or **members**. We write

\[
x \in A \iff \text{the object } x \text{ is a member of (or belongs to) } A.
\]
2. A set is determined by its members, i.e., if $A$, $B$ are sets, then

$$A = B \iff \text{A and B have the same members}$$

(1-1)

$$\iff (\forall x)[x \in A \iff x \in B].$$

This last is the **Extensionality Property**. For example, the set of students in this class will not change if we all switch places, lie down or move to another classroom; this set is completely determined by who we are, not our posture or the places where we happen to be.

Somewhat peculiar is the **empty set** $\emptyset$ which has no members. The extensionality property implies that there is only one empty set.

If $A$ and $B$ are sets, we write

$$A \subseteq B \iff (\forall x)[x \in A \implies x \in B],$$

and if $A \subseteq B$, we call $A$ a **subset** of $B$, so that for every $B$,

$$\emptyset \subseteq B, \quad B \subseteq B.$$  

A **proper subset** of $B$ is a subset distinct from $B$,

$$A \subset B \iff [A \subseteq B \& A \neq B].$$

From the extensionality property it follows that for all sets $A, B$,

$$A = B \iff A \subseteq B \& B \subseteq A.$$  

We have already used several different notations to define specific sets and we need still more, e.g.,

$$A = \{a_1, a_2, \ldots, a_n\}$$

is the (finite) set with members the objects $a_1, a_2, \ldots, a_n$. If $P$ is a condition which specifies some property of objects, then

$$A = \{x \mid P(x)\}$$

is the set of all objects which satisfy the condition $P$, so that for all $x$,

$$x \in A \iff P(x).$$

For example, if

$$P(x) \iff x \in \mathbb{N} \& x \text{ is even},$$

then $\{x \mid P(x)\}$ is the set of all even natural numbers. We use a variant of this notation when we are only interested in “collecting into a whole” members of a given set $A$ which satisfy a certain condition:

$$\{x \in A \mid P(x)\} =_{df} \{x \mid x \in A \& P(x)\},$$

---

1 We will use systematically, as abbreviations, the logical symbols

$\&$: and,  $\lor$: or,  $\neg$: not,  $\implies$: implies,  $\iff$: if and only if.

$\forall$: for all,  $\exists$: there exists,  $\exists!$: there exists exactly one.

The symbols $=_{df}$ and $\iff_{df}$ are read “equal by definition” and “equivalent by definition”. 
so that, for example, \( \{ x \in \mathbb{N} \mid x > 0 \} \) is the set of all non-zero natural numbers, while \( \{ x \in \mathbb{R} \mid x > 0 \} \) is the set of all positive real numbers.

For any two sets \( A, B \),
\[
A \cup B = \{ x \mid x \in A \lor x \in B \} \quad \text{(the union of } A, B),
\]
\[
A \cap B = \{ x \in A \mid x \in B \} \quad \text{(the intersection of } A, B),
\]
\[
A \setminus B = \{ x \in A \mid x \notin B \} \quad \text{(the difference of } A, B).\]

These “Boolean operations” are illustrated in the so-called Venn diagrams of Figure 1.1, in which sets are represented by regions in the plane. The union and the intersection of infinite sequences of sets are defined in the same way,
\[
\bigcup_{n=0}^{\infty} A_n = A_0 \cup A_1 \cup \cdots = \{ x \mid (\exists n \in \mathbb{N})[x \in A_n] \},
\]
\[
\bigcap_{n=0}^{\infty} A_n = A_0 \cap A_1 \cap \cdots = \{ x \mid (\forall n \in \mathbb{N})[x \in A_n] \}.\]

Two sets are disjoint if their intersection is empty,
\[
A \text{ is disjoint from } B \iff A \cap B = \emptyset.
\]

We will use the notations
\[
f : X \to Y \text{ or } A \xrightarrow{f} B
\]

The domain of \( f \) some member \( f(x) \) of the range \( Y \) of \( f \). Functions are also called mappings, operations, transformations and many other things. Sometimes it is convenient to use the abbreviated notation \( (x \mapsto f(x)) \) which makes it possible to talk about a function without officially naming it. For example,
\[
(x \mapsto x^2 + 1)
\]
is the function on the real numbers which assigns to each real its square increased by 1; if we call it \( f \), then it is defined by the formula
\[
f(x) = x^2 + 1 \quad (x \in \mathbb{R})
\]

\(^{2}\)In “mathematical English”, when we say “for any two objects \( x, y \)”, we do not mean that necessarily \( x \neq y \), e.g., the assertion that “for any two numbers \( x, y \), \( (x + y)^2 = x^2 + 2xy + y^2 \)” implies that “for every number \( x \), \( (x + x)^2 = 2x^2 + x^2 \)”. 

---

Figure 1.1. The Boolean operations.
so that \( f(0) = 1, f(2) = 5 \), etc. But we can say “all the values of \( x \mapsto x^2 + 1 \) are positive reals” without necessarily fixing a name for it, like \( f \).

Two functions are **equal** if they have the same domain and they assign the same value to every member of their common domain,

\[
f = g \iff (\forall x \in X)\left[f(x) = g(x)\right] \quad \left( f : X \to Y, g : X \to Z, x \in X \right).
\]

In connection with functions we will also use the notations

\[
f : X \rightarrow Y \iff \text{df } f \text{ is an injection (one-to-one)} \iff (\forall x, x' \in X)[f(x) = f(x') \implies x = x'],
\]

\[
f : X \rightarrow Y \iff \text{df } f \text{ is a surjection (onto)} \iff (\forall y \in Y)(\exists x \in X)[f(x) = y],
\]

\[
f : X \rightarrow Y \iff \text{df } f \text{ is a bijection or a correspondence} \iff (\forall y \in Y)(\exists! x \in X)[f(x) = y].
\]

For every \( f : X \to Y \) and \( A \subseteq X \), the set

\[
f[A] =_{\text{df}} \{ f(x) \mid x \in A \}
\]

is the **image** of \( A \) under \( f \), and if \( B \subseteq Y \), then

\[
f^{-1}[B] =_{\text{df}} \{ x \in X \mid f(x) \in B \}
\]

is the **pre-image** of \( B \) by \( f \).

If \( f \) is a bijection, then we can define the **inverse function** \( f^{-1} : Y \to X \) by the condition

\[
f^{-1}(y) = x \iff f(x) = y,
\]

and then the inverse image \( f^{-1}[B] \) (as we defined it above) is precisely the image of \( B \) under \( f^{-1} \).

The **composition**

\[
h =_{\text{df}} gf : X \to Z
\]

of two functions

\[
X \overset{f}{\to} Y \overset{g}{\to} Z
\]

is defined by

\[
h(x) = g(f(x)) \quad (x \in X).
\]

It is easy to prove many basic properties of sets and functions using only these definitions and the extensionality property. For example,

\[
A \cup B = B \cup A,
\]

because, for any \( x \),

\[
x \in A \cup B \iff x \in A \text{ or } x \in B
\]

\[
\iff x \in B \text{ or } x \in A
\]

\[
\iff x \in B \cup A.
\]

In some cases, the logic of the argument gets a bit complex and it is easier to prove an identity \( U = V \) by verifying separately the two implications

\[
\begin{align*}
U \implies V & \iff U \\
V \implies U & \iff V
\end{align*}
\]
\(x \in U \implies x \in V\) and \(x \in V \implies x \in U\). For example, if \(f : X \to Y\) and \(A, B \subseteq X\), then
\[
f[A \cup B] = f[A] \cup f[B].
\]
To prove this, we show first that
\[
x \in f[A \cup B] \implies x \in f[A] \cup f[B];
\]
this holds because if \(x \in f[A \cup B]\), then there is some \(y \in A \cup B\) such that \(x = f(y)\); and if \(y \in A\), then \(x = f(y) \in f[A] \subseteq f[A] \cup f[B]\), while if \(y \in B\), then \(x = f(y) \in f[B] \subseteq f[A] \cup f[B]\). Next we show the converse implication, that
\[
x \in f[A] \cup f[B] \implies x \in f[A \cup B];
\]
this holds because if \(x \in f[A]\), then \(x = f(y)\) for some \(y \in A \subseteq A \cup B\), and so \(x \in f[A \cup B]\), while if \(x \in f[B]\), then \(x = f(y)\) for some \(y \in B \subseteq A \cup B\), and so, again, \(x \in f[A \cup B]\).

**Problems for Chapter 1**

**x1.1.** For any three sets \(A, B, C\),
\[
A \cup (B \cap C) = (A \cup B) \cap (A \cup C),
A \cap (B \cup C) = (A \cap B) \cup (A \cap C),
A \setminus (A \cap B) = A \setminus B.
\]

**x1.2.** (De Morgan’s laws) For any three sets \(A, B, C\),
\[
C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B),
C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B).
\]

**x1.3.** (De Morgan’s laws for sequences) For any set \(C\) and any sequence of sets \(\{A_n\} = A_0, A_1, \ldots\),
\[
C \setminus (\bigcup_n A_n) = \bigcap_n (C \setminus A_n),
C \setminus (\bigcap_n A_n) = \bigcup_n (C \setminus A_n).
\]

**x1.4.** For every injection \(f : X \to Y\), and all \(A, B \subseteq X\),
\[
f[A \cap B] = f[A] \cap f[B],
f[A \setminus B] = f[A] \setminus f[B].
\]
Show also that these identities do not always hold if \(f\) is not an injection.

**x1.5.** For every \(f : X \to Y\), and all \(A, B \subseteq Y\),
\[
f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B],
f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B],
f^{-1}[A \setminus B] = f^{-1}[A] \setminus f^{-1}[B].
\]
x1.6. For every \( f : X \to Y \) and all sequences of sets \( A_n \subseteq X, B_n \subseteq Y \),
\[
    f^{-1}[\bigcup_{n=0}^{\infty} B_n] = \bigcup_{n=0}^{\infty} f^{-1}[B_n],
\]
\[
    f^{-1}[\bigcap_{n=0}^{\infty} B_n] = \bigcap_{n=0}^{\infty} f^{-1}[B_n],
\]
\[
    f[\bigcup_{n=0}^{\infty} A_n] = \bigcup_{n=0}^{\infty} f[A_n].
\]

x1.7. For every injection \( f : X \hookrightarrow Y \) and every sequence of sets \( A_n \subseteq X \),
\[
    f[\bigcap_{n=0}^{\infty} A_n] = \bigcap_{n=0}^{\infty} f[A_n].
\]

x1.8. The composition of injections is an injection, the composition of surjections is a surjection, and hence the composition of bijections is a bijection.
CHAPTER 2

EQUINUMEROSITY

After these preliminaries, we can formulate the fundamental definitions of Cantor about the size or cardinality of sets.

2.1. Definition. Two sets $A, B$ are equinumerous or equal in cardinality if there exists a (one-to-one) correspondence between their elements, in symbols

$A =_c B \iff \exists f (f : A \mapsto B)$.

This definition of equinumerosity stems from our intuitions about finite sets, e.g., we can be sure that a shoe store offers for sale the same number of left and right shoes without knowing exactly what that number is: the correspondence of each left shoe with the right shoe in the same pair establishes the equinumerosity of these two sets. The radical element in Cantor’s definition is the proposal to accept the existence of such a correspondence as the characteristic property of equinumerosity for all sets, despite the fact that its application to infinite sets leads to conclusions which had been viewed as counterintuitive. A finite set, for example, cannot be equinumerous with one of its proper subsets, while the set of natural numbers $\mathbb{N}$ is equinumerous with $\mathbb{N} \setminus \{0\}$ via the correspondence $(x \mapsto x + 1)$,

$$\{0, 1, 2, \ldots\} =_c \{1, 2, 3, \ldots\}.$$  

In the real numbers, also,

$$(0, 1) =_c (0, 2)$$

via the correspondence $(x \mapsto 2x)$, where as usual, for any two reals $\alpha < \beta$

$$(\alpha, \beta) = \{x \in \mathbb{R} | \alpha < x < \beta\}.$$  

We will use the analogous notation for the closed and half-closed intervals $[\alpha, \beta], [\alpha, \beta)$, etc.

2.2. Proposition. For all sets $A, B, C$,

$$A =_c A,$$

if $A =_c B$, then $B =_c A$,

if $(A =_c B \text{ & } B =_c C)$, then $A =_c C$.

Proof. To show the third implication as an example, suppose that the bijections $f : A \mapsto B$ and $g : B \mapsto C$ witness the equinumerosities of the hypothesis; their composition $gf : A \mapsto C$ then witnesses that $A =_c C$. $\dashv$
2.3. **Definition.** The set $A$ is **less than or equal to** $B$ in size if it is equinumerous with some subset of $B$, in symbols:

$$A \leq C B \iff \exists C \subset B \land A = c C.$$  

2.4. **Proposition.** $A \leq C B \iff \exists f : A \rightarrow B$.

**Proof.** If $A = c C \subset B$ and $f : A \mapsto C$ witnesses this equinumerosity, then $f$ is an injection from $A$ into $B$. Conversely, if there exists an injection $f : A \rightarrow B$, then the same $f$ is a bijection of $A$ with its image $f[A]$, so that $A = c f[A] \subset B$ and so $A \leq B$ by the definition. $\triangleright$

2.5. **Exercise.** For all sets $A, B, C$,

$$A \leq c A,$$

if $(A \leq B \land B \leq C)$, then $A \leq C$.

2.6. **Definition.** A set $A$ is **finite** if there exists some natural number $n$ such that

$$A = c \{ i \in \mathbb{N} \mid i < n \} = \{ 0, 1, \ldots, n - 1 \},$$  
otherwise $A$ is **infinite**. (Thus the empty set is finite, since $\emptyset = \{ i \in \mathbb{N} \mid i < 0 \}$.)

A set $A$ is **countable** if it is finite or equinumerous with the set of natural numbers $\mathbb{N}$, otherwise it is **uncountable**. Countable sets are also called **denumerable**, and correspondingly, uncountable sets are **non-denumerable**.

2.7. **Proposition.** The following are equivalent for every set $A$:

(1) $A$ is countable.
(2) $A \leq c \mathbb{N}$.
(3) Either $A = \emptyset$, or $A$ has an enumeration, a surjection $\pi : \mathbb{N} \rightarrow A$, so that

$$A = \pi[\mathbb{N}] = \{ \pi(0), \pi(1), \pi(2), \ldots \}.$$

**Proof.** We give what is known as a “round robin proof”.

(1) $\implies$ (2). If $A$ is countable, then either $A = c \{ i \in \mathbb{N} \mid i < n \}$ for some $n$ or $A = c \mathbb{N}$, so that, in either case, $A = c C$ for some $C \subset \mathbb{N}$ and hence $A \leq c \mathbb{N}$.

(2) $\implies$ (3). Suppose $A \neq \emptyset$, choose some $x_0 \in A$, and assume by (2) that $f : A \rightarrow \mathbb{N}$. For each $i \in \mathbb{N}$, let

$$\pi(i) = \begin{cases} x_0, & \text{if } i \notin f[A], \\ f^{-1}(i), & \text{otherwise, i.e., if } i \in f[A]. \end{cases}$$
Chapter 2. Equinumerosity

The definition works (because \( f \) is an injection, and so \( f^{-1}(i) \) is uniquely determined in the second case), and it defines a surjection \( \pi : \mathbb{N} \to A \), because \( x_0 \in A \) and for every \( x \in A \), \( x = \pi(f(x)) \).

(3) \( \implies \) (1). If \( A \) is finite then (1) is automatically true, so assume that \( A \) is infinite but it has an enumeration \( \pi : \mathbb{N} \to A \). We must find another enumeration \( f : \mathbb{N} \to A \) which is without repetitions, so that it is in fact a bijection of \( \mathbb{N} \) with \( A \), and hence \( A =^c \mathbb{N} \). The proof is suggested by Figure 2.1: we simply delete the repetitions from the given enumeration \( \pi \) of \( A \). To get a precise definition of \( f \) by recursion, notice that because \( A \) is not finite, for every finite sequence \( a_0, \ldots, a_n \) of members of \( A \) there exists some \( m \) such that \( \pi(m) \notin \{a_0, \ldots, a_n\} \). Set

\[
\begin{align*}
  f(0) &= \pi(0), \\
  m_n &= \text{the least } m \text{ such that } \pi(m) \notin \{f(0), \ldots, f(n)\}, \\
  f(n+1) &= \pi(m_n).
\end{align*}
\]

It is obvious that \( f \) is an injection, so it is enough to verify that every \( x \in A \) is a value of \( f \), i.e., that for every \( n \in \mathbb{N} \), \( \pi(n) \in f[\mathbb{N}] \). This is immediate for 0, since \( \pi(0) = f(0) \). If \( x = \pi(n+1) \) for some \( n \) and \( x \in \{f(0), \ldots, f(n)\} \), then \( x = f(i) \) for some \( i \leq n \); and if \( x \notin \{f(0), \ldots, f(n)\} \), then \( m_n = n+1 \) and \( f(n+1) = \pi(m_n) = x \) by the definition.

2.8. Exercise. If \( A \) is countable and there exists an injection \( f : B \to A \), then \( B \) is also countable; in particular, every subset of a countable set is countable.

2.9. Exercise. If \( A \) is countable and there exists a surjection \( f : A \to B \), then \( B \) is also countable.

The next, simple theorem is one of the most basic results of set theory.

2.10. Theorem (Cantor). For each sequence \( A_0, A_1, \ldots \) of countable sets, the union

\[
A = \bigcup_{n=0}^{\infty} A_n = A_0 \cup A_1 \cup \ldots
\]

is also a countable set.

In particular, the union \( A \cup B \) of two countable sets is countable.

Proof. The second claim follows by applying the first to the sequence

\[
A, B, B, \ldots
\]

For the first, it is enough (why?) to consider the special case where none of the \( A_n \) is empty, in which case we can find for each \( A_n \) an enumeration \( \pi^n : \mathbb{N} \to A_n \). If we let

\[
a^n_i = \pi^n(i)
\]

to simplify the notation, then for each \( n \)

\[
A_n = \{a^0_n, a^1_n, \ldots\},
\]

and we can construct from these enumerations a table of elements which lists all the members of the union \( A \). This is pictured in Figure 2.2, and the arrows
in that picture show how to enumerate the union:

\[ A = \{ a_0^0, a_0^1, a_1^0, a_0^2, a_1^1, \ldots \} \]

2.11. **Corollary.** *The set of rational (positive and negative) integers\[ \mathbb{Z} = \{ \ldots -2, -1, 0, 1, 2, \ldots \} \]is countable.*

**Proof.** \( \mathbb{Z} = \mathbb{N} \cup \{-1, -2, \ldots, \} \) and the set of negative integers is countable via the correspondence \( x \mapsto -(x + 1) \).

2.12. **Corollary.** *The set \( \mathbb{Q} \) of rational numbers is countable.*

**Proof.** The set \( \mathbb{Q}^+ \) of non-negative rationals is countable because

\[ \mathbb{Q}^+ = \bigcup_{n=1}^{\infty} \{ \frac{m}{n} \mid m \in \mathbb{N} \} \]

and each \( \{ \frac{m}{n} \mid m \in \mathbb{N} \} \) is countable via the enumeration \( (m \mapsto \frac{m}{n}) \). The set \( \mathbb{Q}^- \) of negative rationals is countable by the same method, and then the union \( \mathbb{Q}^+ \cup \mathbb{Q}^- \) is countable.

This corollary was Cantor’s first significant result in the program of classification of infinite sets by their size, and it was considered somewhat “paradoxical” because \( \mathbb{Q} \) appears to be so much larger than \( \mathbb{N} \). Immediately afterwards, Cantor showed the existence of uncountable sets.

2.13. **Theorem (Cantor).** *The set of infinite, binary sequences

\[ \Delta = \{(a_0, a_1, \ldots) \mid (\forall i)[a_i = 0 \lor a_i = 1]\} \]

is uncountable.*

**Proof.** Suppose (towards a contradiction) that \( \Delta \) is countable, so there exists an enumeration

\[ \Delta = \{ \alpha_0, \alpha_1, \ldots \}, \]

where for each \( n \),

\[ \alpha_n = (a_0^n, a_1^n, \ldots) \]
is a sequence of 0’s and 1’s.\footnote{To prove a proposition \( \theta \) by the method of \emph{reduction to a contradiction}, we assume its negation \( \neg \theta \) and derive from that assumption something which violates known facts, a contradiction, something \emph{absurd}: we conclude that \( \theta \) cannot be false, so it must be true. Typically we will begin such arguments with the code-phrase \emph{towards a contradiction}, which alerts the reader that the supposition which follows is the negation of what we intend to prove.} We construct a table with these sequences as before, and then we define the sequence \( \beta \) by interchanging 0 and 1 in the “diagonal” sequence \( a_0^0, a_1^1, \ldots \):

\[
\beta(n) = 1 - a_n^n.
\]

It is obvious that for each \( \alpha_n, \beta \neq \alpha_n \), since

\[
\beta(n) = 1 - \alpha_n(n) \neq \alpha_n(n),
\]

so that the sequence \( \alpha_0, \alpha_1, \ldots \) does not enumerate the entire \( \Delta \), contrary to our hypothesis.

\[\square\]

\begin{figure}
\centering
\begin{tikzpicture}
\node at (0,0) {$\alpha_0 : \ a_0^0 a_1^1 a_2^2 \cdots$};
\node at (1.5,0) {$\alpha_1 : \ a_0^1 a_1^1 a_2^1 \cdots$};
\node at (3,0) {$\alpha_2 : \ a_0^2 a_1^2 a_2^2 \cdots$};
\node at (4.5,0) {$\vdots \quad \cdots$};
\end{tikzpicture}
\caption{Cantor’s second diagonal method.}
\end{figure}

2.14. \textbf{Corollary (Cantor).} \textit{The set} \( \mathbb{R} \) \textit{of real numbers is uncountable.}

\textbf{Proof.} We define first a sequence of sets \( C_0, C_1, \ldots \), of real numbers which satisfy the following conditions:

1. \( C_0 = [0, 1] \).
2. Each \( C_n \) is a union of \( 2^n \) closed intervals and

\[
C_0 \supseteq C_1 \supseteq \cdots C_n \supseteq C_{n+1} \supseteq \cdots.
\]

3. \( C_{n+1} \) is constructed by removing the (open) middle third of each interval in \( C_n \), i.e., by replacing each \( [a, b] \) in \( C_n \) by the two closed intervals

\[
L[a, b] = [a, a + \frac{1}{3}(b - a)],
\]

\[
R[a, b] = [a + \frac{2}{3}(b - a), b].
\]

With each binary sequence \( \delta \in \Delta \) we associate now a sequence of closed intervals,

\[
F_0^\delta, F_1^\delta, \ldots .
\]
by the following recursion:

\[ F^\delta_0 = C_0 = [0, 1], \]

\[ F^\delta_{n+1} = \begin{cases} 
LF^\delta_n, & \text{if } \delta(n) = 0, \\
RF^\delta_n, & \text{if } \delta(n) = 1.
\end{cases} \]

By induction, for each \( n \), \( F^\delta_n \) is one of the closed intervals of \( C_n \) of length \( 3^{-n} \) and obviously

\[ F^\delta_0 \supseteq F^\delta_1 \supseteq \cdots, \]

so by the fundamental completeness property of the real numbers the intersection of this sequence is not empty; in fact, it contains exactly one real number, call it

\[ f(\delta) = \text{the unique element in the intersection } \bigcap_{n=0}^{\infty} F^\delta_n. \]

The function \( f \) maps the uncountable set \( \Delta \) into the set

\[ C = \bigcap_{n=0}^{\infty} C_n, \]

the so-called Cantor set, so to complete the proof it is enough to verify that \( f \) is one-to-one. But if \( n \) is the least number for which \( \delta(n) \neq \epsilon(n) \) and (for example) \( \delta(n) = 0 \), we have \( F^\delta_n = F^\epsilon_n \) from the choice of \( n \).

\[ f(\delta) \in F^\delta_{n+1} = LF^\delta_n, \quad f(\epsilon) \in F^\epsilon_{n+1} = RF^\delta_n, \quad \text{and } LF^\delta_n \cap RF^\delta_n = \emptyset, \]

so that indeed \( f \) is an injection.

The basic mathematical ingredient of this proof is the appeal to the completeness property of the real numbers, which we will study carefully in Appendix A. Some use of a special property of the reals is necessary: the rest of Cantor’s construction relies solely on arithmetical properties of numbers which are also true of the rationals, so if we could avoid using completeness we would also prove that \( \mathbb{Q} \) is uncountable, contradicting Corollary 2.12.

The fundamental importance of this theorem was instantly apparent, the more so because Cantor used it immediately in a significant application to the theory of algebraic numbers. Before we prove this corollary we need some definitions and lemmas.

2.15. Definition. For any two sets \( A, B \), the set of ordered pairs of members of \( A \) and members of \( B \) is denoted by

\[ A \times B = \{(x, y) \mid x \in A \& y \in B\}. \]
In the same way, for each \( n \geq 2 \),
\[
A_1 \times \cdots \times A_n = \{ (x_1, \ldots, x_n) \mid x_1 \in A_1, \ldots, x_n \in A_n \}, \\
A^n = \{ (x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in A \}.
\]
We call \( A_1 \times \cdots \times A_n \) the **Cartesian product** of \( A_1, \ldots, A_n \).

2.16. **Lemma.** (1) If \( A_1, \ldots, A_n \) are all countable, so is their Cartesian product \( A_1 \times \cdots \times A_n \).

(2) For every countable set \( A \), each \( A^n \) (\( n \geq 2 \)) and the union
\[
\bigcup_{n=2}^{\infty} A^n = \{ (x_1, \ldots, x_n) \mid n \geq 2, x_1, \ldots, x_n \in A \}
\]
are all countable.

**Proof.** (1) If some \( A_i \) is empty, then the product is empty (by the definition) and hence countable. Otherwise, in the case of two sets \( A, B \), we have some enumeration
\[
B = \{ b_0, b_1, \ldots \}
\]
of \( B \), obviously
\[
A \times B = \bigcup_{n=0}^{\infty} (A \times \{ b_n \}),
\]
and each \( A \times \{ b_n \} \) is equinumerous with \( A \) (and hence countable) via the correspondence \( (x \mapsto (x, b_n)) \). This gives the result for \( n = 2 \). To prove the proposition for all \( n \geq 2 \), notice that
\[
A_1 \times \cdots \times A_n \times A_{n+1} = c \ (A_1 \times \cdots \times A_n) \times A_{n+1}
\]
via the bijection
\[
f(a_1, \ldots, a_n, a_{n+1}) = ((a_1, \ldots, a_n), a_{n+1}).
\]
Thus, if every product of \( n \geq 2 \) countable factors is countable, so is every product of \( n + 1 \) countable factors, and so (1) follows by induction.

(2) Each \( A^n \) is countable by (1), and then \( \bigcup_{n=2}^{\infty} A^n \) is also countable by another appeal to Theorem 2.10.

2.17. **Definition.** A real number \( \alpha \) is **algebraic** if it is a root of some polynomial
\[
P(x) = a_0 + a_1 x + \cdots + a_n x^n
\]
with integer coefficients \( a_0, \ldots, a_n \in \mathbb{Z} \) \( (n \geq 1, a_n \neq 0) \), i.e., if
\[
P(\alpha) = 0.
\]

Typical examples of algebraic numbers are \( \sqrt{2}, (1 + \sqrt{2})^2 \) (why?) but also the real root of the equation \( x^5 + x + 1 = 0 \) which exists (why?) but cannot be expressed in terms of radicals, by a classical theorem of Abel. The basic fact (from algebra) about algebraic numbers is that a polynomial of degree \( n \geq 1 \) has at most \( n \) real roots; this is all we need for the next result.

2.18. **Corollary.** The set \( K \) of algebraic real numbers is countable (Cantor), and hence there exist real numbers which are not algebraic (Liouville).
Proof. The set $\Pi$ of all polynomials with integer coefficients is countable, because each such polynomial is determined by the sequence of its coefficients, so that $\Pi$ can be injected into the countable set $\bigcup_{n=2}^{\infty} \mathbb{Z}^n$. For each polynomial $P(x)$, the set of its roots

$$\Lambda(P(x)) = \{ \alpha \mid P(\alpha) = 0 \}$$

is finite and hence countable. It follows that the set of algebraic numbers $K$ is the union of a sequence of countable sets and hence it is countable.

This first application of the (then) new theory of sets was instrumental in ensuring its quick and favorable acceptance by the mathematicians of the period, particularly since the earlier proof of Liouville (that there exist non-algebraic numbers) was quite intricate. Cantor showed something stronger, that “almost all” real numbers are not algebraic, and he did it with a much simpler proof which used just the fact that a polynomial of degree $n$ cannot have more than $n$ real roots, the completeness of $\mathbb{R}$, and, of course, the new method of counting the members of infinite sets.

So far we have shown the existence of only two “orders of infinity”, that of $\mathbb{N}$—the countable, infinite sets—and that of $\mathbb{R}$. There are many others.

2.19. Definition. The powerset $\mathcal{P}(A)$ of a set $A$ is the set of all its subsets,

$$\mathcal{P}(A) = \{ X \mid X \text{ is a set and } X \subseteq A \}.$$

2.20. Exercise. For all sets $A, B$,

$$A =_c B \implies \mathcal{P}(A) =_c \mathcal{P}(B).$$

2.21. Theorem (Cantor). For every set $A$,

$$A <_c \mathcal{P}(A),$$

i.e., $A \leq_c \mathcal{P}(A)$ but $A \neq_c \mathcal{P}(A)$; in fact there is no surjection $\pi : A \to \mathcal{P}(A)$.

Proof. That $A \leq_c \mathcal{P}(A)$ follows from the fact that the function

$$(x \mapsto \{x\})$$

which associates with each member $x$ of $A$ its singleton $\{x\}$ is an injection. (Careful here: the singleton $\{x\}$ is a set with just the one member $x$ and it is not the same object as $x$, which is probably not a set to begin with!)

To complete the proof, we assume (towards a contradiction) that there exists a surjection

$$\pi : A \to \mathcal{P}(A),$$

and we define the set

$$B = \{ x \in A \mid x \notin \pi(x) \},$$

so that for every $x \in A$,

$$x \in B \iff x \notin \pi(x). \quad (2-1)$$

Now $B$ is a subset of $A$ and $\pi$ is a surjection, so there must exist some $b \in A$ such that $B = \pi(b)$; and setting $x = b$ and $\pi(b) = B$ in (2-1), we get

$$b \in B \iff b \notin B.$$
which is absurd.

So there are many orders of infinity, and specifically (at least) those of the sets

\[ N <_c \mathcal{P}(N) <_c \mathcal{P}(\mathcal{P}(N)) <_c \cdots. \]

If we name these sets by the recursion

\[
T_0 = N, \\
T_{n+1} = \mathcal{P}(T_n),
\]

then their union \( T_\infty = \bigcup_{n=0}^{\infty} T_n \) has a larger cardinality than each \( T_n \). Problem \textbf{x2.8}. The classification and study of these orders of infinity is one of the central problems of set theory.

Somewhat more general than powersets are function spaces.

2.22. Definition. For any two sets \( A, B \),

\[(A \to B) = \text{df} \{ f \mid f : A \to B \} = \text{the set of all functions from } A \text{ to } B.\]

2.23. Exercise. If \( A_1 =_c A_2 \text{ and } B_1 =_c B_2 \), then \( (A_1 \to B_1) =_c (A_2 \to B_2) \).

Function spaces are “generalizations” of powersets because each subset \( X \subseteq A \) can be represented by its \textit{characteristic function} \( c_X : A \to \{0, 1\} \),

\[
c_X(t) = \begin{cases} 
1, & \text{if } t \in A \cap X, \\
0, & \text{if } t \in A \setminus X,
\end{cases} \quad (t \in A). \tag{2-3}
\]

We can recover \( X \) from \( c_X \),

\[ X = \{ t \in A \mid c_X(t) = 1 \}, \]

and so the mapping \((X \mapsto c_X)\) is a correspondence of \( \mathcal{P}(A) \) with \((A \to \{0, 1\})\). Thus

\[(A \to \{0, 1\}) =_c \mathcal{P}(A) >_c A, \tag{2-4}\]

and the function space operation also leads to large, uncountable sets. The next obvious problem is to compare for size these uncountable sets, starting with the two simplest ones, \( \mathcal{P}(\mathbb{N}) \) and the set \( \mathbb{R} \) of real numbers.

2.24. Lemma. \( \mathcal{P}(\mathbb{N}) \leq_c \mathbb{R} \).

\textbf{Proof.} It is enough to prove that \( \mathcal{P}(\mathbb{N}) \leq_c \Delta \), since we have already shown that \( \Delta \leq_c \mathbb{R} \). This follows immediately from (2-4), as \( \Delta = (\mathbb{N} \to \{0, 1\}) \). \( \dashv \)
2.25. **Lemma.** \( \mathbb{R} \leq_c \mathcal{P}(\mathbb{N}) \).

**Proof.** It is enough to show that \( \mathbb{R} \leq_c \mathcal{P}(\mathbb{Q}) \), since the set of rationals \( \mathbb{Q} \) is equinumerous with \( \mathbb{N} \) and hence \( \mathcal{P}(\mathbb{N}) =_c \mathcal{P}(\mathbb{Q}) \). This follows from the fact that the function

\[
x \mapsto \pi(x) = \{ q \in \mathbb{Q} \mid q < x \} \subseteq \mathbb{Q}
\]

is an injection, because if \( x < y \) are distinct real numbers, then there exists some rational \( q \) between them, \( x < q < y \) and \( q \in \pi(y) \setminus \pi(x) \).

With these two simple Lemmas, the equinumerosity \( \mathbb{R} =_c \mathcal{P}(\mathbb{N}) \) will follow immediately from the following basic theorem.

2.26. **Theorem (Schröder-Bernstein).** For any two sets \( A, B \),

if \( A \leq_c B \) and \( B \leq_c A \), then \( A =_c B \).

**Proof.**\(^4\) We assume that there exist injections

\[
f : A \mapsto B, \quad g : B \mapsto A,
\]

and we define the sets \( A_n, B_n \) by the following recursive definitions:

\[
A_0 = A, \quad B_0 = B, \quad A_{n+1} = g f[A_n], \quad B_{n+1} = f g[B_n].
\]

\(^4\)A different proof of this theorem is outlined in Problems x4.26, x4.27.
where \( fg[X] = \{ f(g(x)) \mid x \in X \} \) and correspondingly for the function \( gf \).

By induction on \( n \) (easily)

\[
A_n \supseteq g[B_n] \supseteq A_{n+1}, \\
B_n \supseteq f[A_n] \supseteq B_{n+1},
\]

so that we have the “chains of inclusions”

\[
A_0 \supseteq g[B_0] \supseteq A_1 \supseteq g[B_1] \supseteq A_2 \cdots, \\
B_0 \supseteq f[A_0] \supseteq B_1 \supseteq f[A_1] \supseteq B_2 \cdots.
\]

We also define the intersections

\[
A^* = \bigcap_{n=0}^{\infty} A_n, \quad B^* = \bigcap_{n=0}^{\infty} B_n,
\]

so that

\[
B^* = \bigcap_{n=0}^{\infty} B_n \supseteq \bigcap_{n=0}^{\infty} f[A_n] \supseteq \bigcap_{n=0}^{\infty} A_{n+1} = B^*
\]

and since \( f \) is an injection, by Problem x1.7,

\[
f[A^*] = f[\bigcap_{n=0}^{\infty} A_n] = \bigcap_{n=0}^{\infty} f[A_n] = B^*.
\]

Thus \( f \) is a bijection of \( A^* \) with \( B^* \). On the other hand,

\[
A = A^* \cup (A_0 \setminus g[B_0]) \cup (g[B_0] \setminus A_1) \cup (A_1 \setminus g[B_1]) \cup (g[B_1] \setminus A_2) \cdots
\]

\[
B = B^* \cup (B_0 \setminus f[A_0]) \cup (f[A_0] \setminus B_1) \cup (B_1 \setminus f[A_1]) \cup (f[A_1] \setminus B_2) \cdots
\]

and these sequences are separated, i.e., no set in them has any common element with any other. To finish the proof it is enough to check that for every \( n \),

\[
f[A_n \setminus g[B_n]] = f[A_n] \setminus B_{n+1}, \\
g[B_n \setminus f[A_n]] = g[B_n] \setminus A_{n+1},
\]

from which the first (for example) is true because \( f \) is an injection and so

\[
f[A_n \setminus g[B_n]] = f[A_n] \setminus f[g[B_n]] = f[A_n] \setminus B_{n+1}.
\]

Finally we have the bijection \( \pi : A \leftrightarrow B \)

\[
\pi(x) = \begin{cases} 
  f(x), & \text{if } x \in A^* \text{ or } (\exists n)[x \in A_n \setminus g[B_n]], \\
  g^{-1}(x), & \text{if } x \notin A^* \text{ and } (\exists n)[x \in g[B_n] \setminus A_{n+1}], 
\end{cases}
\]

which verifies that \( A =_e B \) and finishes the proof.

Using the Schröder-Bernstein Theorem we can establish easily several equinumerosities which are quite difficult to prove directly.

**Problems for Chapter 2**

x2.1. For any \( \alpha < \beta \) where \( \alpha, \beta \) are reals, \( \infty \) or \( -\infty \), construct bijections which prove the equinumerosities

\[(\alpha, \beta) =_e (0, 1) =_e \mathbb{R}.\]

* x2.2. For any two real numbers \( \alpha < \beta \), construct a bijection which proves the equinumerosity

\[ [\alpha, \beta) =_e [\alpha, \beta) =_e \mathbb{R}. \]
x2.3. \( \mathcal{P}(\mathbb{N}) =_c \mathbb{R} =_c \mathbb{R}^n \), for every \( n \geq 2 \).

x2.4. For any two sets \( A, B \), \((A \to B) \leq_c \mathcal{P}(A \times B)\). Hint. Represent each \( f : A \to B \) by its graph, the set 
\[
G_f = \{(x, y) \in A \times B \mid y = f(x)\}.
\]

x2.5. \((\mathbb{N} \to \mathbb{N}) =_c \mathcal{P}(\mathbb{N})\).

*x2.6. \((\mathbb{N} \to \mathbb{R}) =_c \mathbb{R}\).

*x2.7. For any three sets \( A, B, C \), 
\[
((A \times B) \to C) =_c (A \to (B \to C))
\]
Hint. For any \( p : A \times B \to C \), define \( \pi(p) = q : A \to (B \to C) \) by the formula 
\[
q(x)(y) = p(x, y).
\]

x2.8. Using the definition (2-2), for every \( m \), 
\[
T_m <_c T_\infty = \bigcup_{n=0}^{\infty} T_n.
\]

You need to know something about continuous functions to do the last two problems.

*x2.9. The set \( C[0, 1] \) of all continuous, real functions on the closed interval \([0, 1]\) is equinumerous with \( \mathbb{R} \).

*x2.10. The set of all monotone real functions on the closed interval \([0, 1]\) is equinumerous with \( \mathbb{R} \).