The Normed Induction Theorem

Yiannis N. Moschovakis
UCLA and University of Athens

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What it is about

- An operator $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ on the powerset of a set X is monotone if

$$S \subseteq T \implies \Phi(S) \subseteq \Phi(T) \quad (S \subseteq T \subseteq X);$$

and every monotone Φ has a least fixed point $\overline{\Phi}$ characterized by

$$\Phi(\overline{\Phi}) = \overline{\Phi}, \quad (\forall S \subseteq X)[\Phi(S) \subseteq S \implies \overline{\Phi} \subseteq S]$$

- This set $\overline{\Phi}$ built up by Φ is defined explicitly by

$$\overline{\Phi} = \bigcap\{S \subseteq X \mid \Phi(S) \subseteq S\}, \tag{Exp}$$

and inductively by the ordinal recursion

$$\overline{\Phi} = \bigcup_\xi \overline{\Phi}_\xi, \text{ where } \overline{\Phi}_\xi = \Phi(\bigcup_{\eta<\xi} \overline{\Phi}_\eta). \tag{Ind}$$

- The Normed Induction Theorem gives simple—often best possible—classifications of $\overline{\Phi}$, especially in Descriptive Set Theory

Outline

- The arithmetical and analytical hierarchies on ω and $\mathcal{N}$ (3 slides)
- Borel sets and their codings (2 slides)
- Norms and the prewellordering property (2 slides)
- The main result and some of its consequences (5 slides)
- The story of O (2 slides)

- Results by several people will be discussed

- The basic references for proofs are the Second Edition of my *Descriptive Set Theory* book and the article *Kleene's amazing 2nd Recursion Theorem*, both posted on `www.math.ucla.edu/~ynm`

Notation and terminology

- $\omega = \{0, 1, \ldots, \}, s, t \in \omega$, $\mathcal{N} = (\omega \to \omega)$, the Baire space, $\alpha, \beta \in \mathcal{N}$
- A space is a product $X = X_1 \times \cdots \times X_k$ where each X_i is ω or $\mathcal{N}$
- A pointset is any $P \subseteq X = X_1 \times \cdots \times X_k$ and we write synonymously

$$x \in P \iff P(x) \iff P(x_1, \ldots, x_k)$$

- A pointclass is any collection Γ of pointsets
- Restriction: For each X, $\Gamma \restriction X = \{P \subseteq X \mid P \in \Gamma\}$
- Relativization: $\Gamma(\alpha) = \{P_\alpha \mid P \in \Gamma\}$ where $P_\alpha(x) \iff P(\alpha, x)$

$$\langle t_0, \ldots, t_{n-1} \rangle = 2^{t_0+1} 3^{t_1+1} \cdots p_{n-1}^{t_{n-1}+1}$$

$$\overline{\alpha}(t) = \langle \alpha(0), \ldots \alpha(t-1) \rangle, \quad \overline{n}(t) = n$$

$$\text{if } x = (x_1, \ldots, x_k) \in X, \text{ then } \overline{x}(t) = \langle \overline{x}_1(t), \ldots, \overline{x}_k(t) \rangle$$

$$\alpha' = \lambda t \alpha(t+1) \quad (\alpha)_i = \lambda s \alpha(\langle i, s \rangle)$$

- The results we will discuss hold for all recursive Polish spaces —and "*boldface versions*" of them hold for all Polish spaces

The arithmetical and analytical pointclasses

$$\begin{array}{ccccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & \Sigma^0_3 & & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & & & & \cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & & \Pi^0_3 & &
\end{array}$$

$$\begin{array}{ccccccccccccc}
 & & \Sigma^1_1 & & & & \Sigma^1_2 & & & & \Sigma^1_3 & & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
\cdots \subset \Delta^1_1 & & & & \Delta^1_2 & & & & \Delta^1_3 & & & & \cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
 & & \Pi^1_1 & & & & \Pi^1_2 & & & & \Pi^1_3 & &
\end{array}$$

- Σ^0_1, semirecursive: $P(x) \iff (\exists s)R(\overline{x}(s))$ with recursive $R \subseteq \omega^k$
- Π^1_1 : $P(x) \iff (\forall\alpha)Q(x,\alpha)$ with Q in Σ^0_1
  $\iff (\forall\alpha)(\exists t)R(\overline{x}(t), \overline{\alpha}(t))$ with recursive $R \subseteq \omega^{k+1}$
- Σ^1_2 : $P(x) \iff (\exists\alpha)Q(x,\alpha)$ with Q in Π^1_1
  $\iff (\exists\alpha)(\forall\beta)(\exists)R(\overline{x}(t), \overline{\alpha}_1(t), \overline{\alpha}_2(t))$ with recursive R
- The relativized pointclasses: $\Sigma^1_n(\alpha), \Pi^1_n(\alpha), \Delta^1_n(\alpha) = \Sigma^1_n(\alpha) \cap \Pi^1_n(\alpha)$
- The boldface pointclasses: $\underset{\sim}{\boldsymbol{\Sigma}}{}^1_n = \bigcup_\alpha \Sigma^1_n(\alpha)$, etc.

Closure properties and ω-parametrization

- A function $f : X \to \omega$ is recursive if $\{(x, w) \mid f(x) = w\}$ is Σ^0_1
- $f : X \to \mathcal{N}$ is recursive if $f(x) = \lambda t g(x, t)$ with a recursive g
- $f : X \to Y_1 \times \cdots \times Y_l$ is recursive if $f(x) = (f_1(x), \dots, f_l(x))$ with recursive components $f_1, \dots, f_l$
- The sequence-coding and projection functions are recursive

★ All $\Sigma^i_n, \Pi^i_n, \Delta^i_n$ are closed under recursive substitutions $\&, \vee$ and bounded number quantification $(\exists s \le t), (\forall s \le t)$

Σ^0_n is closed under $(\exists s \in \omega)$; Π^0_n is closed under $(\forall s \in \omega)$
Σ^1_n, Π^1_n are closed under $(\exists s \in \omega)$, $(\forall s \in \omega)$
Σ^1_n is closed under $(\exists y \in Y)$; Π^1_n is closed under $(\forall y \in Y)$

- The pointclasses Σ^i_n, Π^i_n are not closed under $\neg$

★ Γ is ω-parametrized if for each X, there is a $G \subseteq \omega \times X$ in Γ such that $\Gamma \restriction X = \{G_e \mid G(e, x)\}$, where $G_e(x) \iff G(e, x)$

- Theorem A. $\Sigma^0_n, \Pi^0_n, \Sigma^1_n, \Pi^1_n$ *are all ω-parametrized*

The Borel pointsets

- $\mathbf{B}\restriction X =$ the smallest σ-algebra of subsets of X which includes $\underset{\sim}{\mathbf{\Sigma}}^0_1\restriction X$
- Codes for Borel sets: $B = \bigcup_\xi B_\xi$, where by ordinal recursion,

$$\alpha \in B_\xi \iff \alpha(0) = 0 \vee [\alpha(0) \neq 0 \;\&\; (\forall i)[(\alpha')_i \in \cup_{\eta<\xi} B_\eta]]$$

i.e., B is the least fixed point of the monotone operator

$$\Phi(S) = \{\alpha \mid \alpha(0) = 0 \vee [\alpha(0) \neq 0 \;\&\; (\forall i)[(\alpha')_i \in S]]\} \quad (S \subseteq \mathcal{N})$$

- For each space X and each $\alpha \in B$ we define $B(\alpha) = B^X(\alpha)$ by an (easy) ordinal recursion, so that

$$\text{if } \alpha(0) = 0, B(\alpha) = \{x \in X \mid (\exists t)[\overline{\alpha'}(x,t) = 0]\},$$
$$\text{if } \alpha(0) \neq 0, B(\alpha) = \bigcup_i (X \setminus B((\alpha')_i))$$

Basic (easy) fact. *For each X, $\mathbf{B}\restriction X = \{B^X(\alpha) \mid \alpha \in B\}$*

How complex are B and the operation $\alpha \mapsto B^X(\alpha)$?

- The explicit definition of $B = \overline{\Phi}$ gives no useful complexity bound
- The inductive definition of $B = \overline{\Phi}$ gives (easily) that B is Σ^1_2

Theorem 1. *B is Π^1_1*

Theorem 2. *For each X, the relation*

$$M^X(\alpha, x, w) \iff \alpha \in B \\ \& \Big([w = 1 \ \& \ x \in B^X(\alpha)] \vee [w = 0 \ \& \ x \notin B^X(\alpha)]\Big)$$

is Π^1_1; and so *each $B^X(\alpha)$ is uniformly $\Delta^1_1(\alpha)$*

- Theorem 2 is old (not trivial) but Theorem 1 had not been noticed
- Getting (easy) proofs of these was a motivation for this work

Norms and the prewellordering property

- A norm on a pointset $P \subseteq X$ is any function $\sigma : P \to$ Ordinals

  and it is a Γ-norm if the following two relations on $X \times X$ are in Γ:

$$x \leq_\sigma^* y \iff P(x) \;\&\; \Big(\neg P(y) \vee \sigma(x) \leq \sigma(y)\Big)$$
$$x <_\sigma^* y \iff P(x) \;\&\; \Big(\neg P(y) \vee \sigma(x) < \sigma(y)\Big)$$

★ A pointclass Γ is normed (or has the prewellordering property) if every pointset in Γ admits a Γ-norm

- Σ_1^0 is normed: if $P(x) \iff (\exists t) R(\overline{x}(t))$, put $\sigma(x) = \mu t R(\overline{x}(t))$
- Π_1^1 and Σ_2^1 are normed

—and various versions of this fact were used in the early 20th century to derive much of the structure theory for these pointclasses

Normed pointclasses

- Σ^0_n is normed for every n (same proof as for Σ^0_1)
- If the Axiom of Costructibility holds, then every Σ^1_{n+1} is normed
- If the Axiom of Projective Determinacy holds, then Π^1_{2n+1} and Σ^1_{2n} are normed for every n

Fact: If Γ is closed under recursive substitutions, ω-parametrized and normed, then the dual pointclass $\neg\Gamma$ is not normed

This is proved by Kleene's construction of recursively inseparable disjoint r.e. sets and implies that the three results above identify all the arithmetical and analytical pointclasses which are normed, under either of the conflicting hypotheses of constructibility or projective determinacy

- There are many interesting pointclasses which are

$\boxed{\text{closed under recursive substitutions, } \omega\text{-parametrized and normed}}$

⋆ The main result

- Let Γ be a pointclass. An operator $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is Γ on Γ, if for every relation $P \subseteq Y \times X$ in Γ, the relation

$$Q(x, y) \iff x \in \Phi(\{x' \mid P(y, x')\})$$

is also in Γ.

Theorem (The Normed Induction Theorem, ynm 1974)

Suppose Γ is closed under recursive substitutions, ω-parametrized and normed:

If $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is monotone and Γ on Γ, then $\overline{\Phi}$ is in Γ

- In most applications, the hypotheses are either known or trivial
- The theorem is proved by a simple 2nd Recursion Theorem argument

⋆ *Once you prove it, you almost never need to use again the 2nd Recursion Theorem in Descriptive Set Theory*

Borel sets and their codes

The set B of Borel codes is the least fixed point of

$$\Phi(S) = \{\alpha \mid \alpha(0) = 0 \vee [\alpha(0) \neq 0 \ \& \ (\forall i)[(\alpha')_i \in S]]\} \quad (S \subseteq \mathcal{N})$$

Theorem 1. *B is Π^1_1*

Proof. Since Π^1_1 is closed under recursive substitutions, ω-parametrized and normed, it is enough to prove that Φ is Π^1_1 on Π^1_1, i.e., to verify that if $P(y, \alpha)$ is Π^1_1 and

$$Q(\alpha, y) \iff \alpha(0) = 0 \vee [\alpha(0) \neq 0 \ \& \ (\forall i)P(y, (\alpha')_i)],$$

then $Q(\alpha, y)$ is also Π^1_1; but this is obvious from the closure properties of Π^1_1 $\square$

- The NIT gives an equally trivial proof of Theorem 2, that *every $B(\alpha)$ is $\Delta^1_1(\alpha)$, uniformly for $\alpha \in B$*
- The Suslin-Kleene Theorem. *Every $\utilde{\boldsymbol{\Delta}}^1_1$ pointset is uniformly Borel*

  This can also be proved using NIT on Σ^0_1, but not quite trivially

The original motivation for the theorem

• Γ is a Spector pointclass if it is ω-parametrized, normed and closed under recursive substitutions, $\&, \vee, (\exists s \in \omega)$ and $(\forall s \in \omega)$

• Π^1_1 *is the smallest Spector pointclass*

• Σ^1_2 *is the smallest Spector pointclass closed under* $(\exists \alpha \in \mathcal{N})$

• *The pointclass* IND *of all* (positive, elementary) *inductive pointsets is the smallest Spector pointclass closed under both* $(\forall \alpha \in \mathcal{N})$ *and* $(\exists \alpha \in \mathcal{N})$

• *The pointclass* Envelope($^3\mathbf{E}$) *of all sets Kleene-semirecursive in the type-3 object which embodies existential quantifier* $^3\mathbf{E}$ *over* $\mathcal{N}$ *is the smallest Spector pointclass which is closed under* $^3\mathbf{E}$

• A pointset is inductive if it is Σ_1-definable over *the smallest admissible set* which contains $\mathcal{N}$

• Envelope($^3\mathbf{E}$) is closed under $(\forall \alpha \in \mathcal{N})$, but it is not closed under $(\exists \alpha \in \mathcal{N})$, and so $\boxed{\bigcup_n \Sigma^1_n \subsetneq \text{Envelope}(^3\mathbf{E}) \subsetneq \text{IND}}$

Largest fixed points

- Every monotone operator $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ has a largest fixed point $\underline{\Phi}$ characterized by

$$\Phi(\underline{\Phi}) = \underline{\Phi}, \quad (\forall S \subseteq X)[S \subseteq \Phi(S) \implies S \subseteq \underline{\Phi}],$$

in fact

$$\underline{\Phi} = X \setminus (\text{the least fixed point of } \check{\Phi}),$$

where $\check{\Phi}$ is the operator dual to Φ,

$$\check{\Phi}(S) = X \setminus \Phi(X \setminus S),$$

Corollary (to the Normed Induction Theorem)

If Γ is ω-parametrized and closed under recursive substitutions, the dual pointclass $\neg\Gamma$ is normed and $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is Γ on Γ, then the largest fixed point $\underline{\Phi}$ of Φ is in Γ

The effective Cantor-Bendixson Theorem

Theorem 3 (Kreisel, 1959). *If $F \subseteq X$ is a closed, Σ^1_1 set and*

$$F = k(F) \cup s(F)$$

is the unique decomposition of F into a perfect set $k(F)$ and a countable set $s(F)$, then the kernel $k(F)$ is Σ^1_1

Proof $k(F)$ is the largest fixed point of the Cantor derivative

$$\Phi_F(S) = \{x \in S \mid x \text{ is a limit point of } S \cap F\} \quad (S \subseteq X)$$

which is easily Σ^1_1 on Σ^1_1, and $\neg\Sigma^1_1 = \Pi^1_1$ is normed

• Kreisel proved much more, including the fact that this complexity result is best possible: *there is a Π^0_1 set $F \subseteq \mathcal{N}$ whose kernel is Σ^1_1 but not Π^1_1.*

The story of O

- O is a set of numbers, a canonical notation system for countable ordinals that Kleene introduced in 1938, following Church. Kleene used O extensively throughout his life, as the basic tool in the study of hyperarithmetical sets
- In 1944, Kleene published a proof that *O is Π^0_2* . . .
- . . . and in 1955, he published a proof that *O is Π^1_1 and not Σ^1_1*

  which this time was correct!
- Kleene's 1944 error is due to a misunderstanding(!) of inductive definitions with "non-constructive" clauses and it is easy to understand it using the Normed Induction Theorem
- Since the inductive definition of O is quite complex, we will illustrate the point using a set A which is much easier to define

The story of A

- Let $\varphi_e(t) = U(\mu y T_1(e,t,y))$ be the recursive partial function with code e, and let $A = \overline{\Phi} \subseteq \omega$ be the least fixed point of

$$\Phi(S) = \{m \mid m = 1 \vee (\exists e)[m = 2^e \ \& \ (\forall t)[\varphi_e(t)\downarrow \ \ \& \ \varphi_e(t) \in S]]\}$$

so that

$$\begin{aligned} m \in \Phi(S) \iff m = 1 \vee (\exists e \leq m)\Big(& m = 2^e \ \& \ (\forall t)(\exists y) T_1(e,t,y) \\ & \& \ (\forall t)(\forall y)[T_1(e,t,y) \implies U(y) \in S]\Big) \end{aligned}$$

- Φ is Π^1_1 on Π^1_1, so its least fixed point $A = \overline{\Phi}$ is Π^1_1, since Π^1_1 is normed
- Φ is Π^0_2 on Π^0_2, so its largest fixed point $\underline{\Phi}$ is Π^0_2, since Σ^0_2 is normed
- In the 1944 paper, Kleene does the similar (somewhat more complex) computation for the operator Ψ for which $\overline{\Psi} = O$, and naively assumes that Ψ has only one fixed point, i.e., O—which then must be Π^0_2