You may refer to and use every result in the Notes and the slides of the lectures for Parts 1 and 2 of the class and every problem in the first eight homework assignments whose solutions are posted.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the "architecture" of your argument.

There are 6 problems or parts of problems and they are all worth the same, but they vary in difficulty: do first those which are easy.

Problem 1. Consider the following proposition about an arbitrary infinite structure $\mathbf{A}$:

(∗) *If* $R(x, y)$ *is an elementary relation on* $\mathbf{A}$ *and*

$$P(x) \iff \text{the set } \{y \mid R(x, y)\} \text{ is infinite,}$$

then $P(x)$ *is also elementary on* $\mathbf{A}$.

For each of the following structures, determine whether (∗) is true or false and outline a proof of your claim.

(1a) $\mathbf{A} = \mathbf{N} = (\mathbb{N}, 0, 1, S, +, \cdot)$ is the standard structure of arithmetic.

Solution. True:

$$P(x) \iff (\forall u)(\exists y)[u \le y \wedge R(x, y)].$$

where $x \le y \iff (\exists t)[x + t = y]$.

(1b) $\mathbf{A} = (\mathbb{Z}, \le)$, the rational integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ with their usual ordering.

Solution. True:

$$P(x) \iff (\forall u)(\exists y)[u \le y \wedge R(x, y)] \vee (\forall u)(\exists y)[y \le u \wedge R(x, y)].$$

To prove the implication $(\Leftarrow)$ of this equivalence, we take cases on the two disjuncts on the right. If the first disjunct $(\forall u)(\exists y)[u \le y \wedge R(x, y)]$ is true, then the set $\{y \ge 0 \mid R(x, y)\}$ has no maximum and so it is infinite; and if the second disjunct $(\forall u)(\exists y)[y \le u \wedge R(x, y)]$ is true, then the set $\{y \le 0 \mid R(x, y)\}$ has no minimum, and so its is again infinite.

To prove the converse implication $(\Rightarrow)$ of the claimed equivalence, we assume that for some fixed x, both disjuncts on the the right hand side are false. It follows that there are numbers u_r, u_l such that

$$\text{for all } y, R(x,y) \Longrightarrow [y \le u_r \wedge y \ge u_l],$$

so that either the set $\{y \mid R(x,y)\}$ is empty, if $u_r < u_l$, or it is contained in the interval

$$[u_l, u_l + 1, \dots, u_r - 1, u_r]$$

and is finite.

(1c) $\mathbf{A}$ is an arbitrary, infinite structure.

Solution. This is not always true. One counterexample is a non-standard model of arithmetic

$$\mathbf{N}^* = (\mathbb{N}^*, 0, 1, S^*, +^*, \cdot^*).$$

If we take

$$R(x,y) \iff y \le^* x \iff (\exists z)[y +^* z = x],$$

then clearly,

$$P(x) \iff x \text{ has infinitely many elements below it} \iff x \in \mathbb{N}^* \setminus \mathbb{N},$$

so that if Proposition $(*)$ were true for $\mathbf{N}^*$, then the non-standard part $\mathbb{N}^* \setminus \mathbb{N}$ of $\mathbf{N}^*$ would be elementary; but it would then have a least member which its does not, since $(\forall y)[y \ne 0 \to (\exists u)[u < y]]$ is true in $\mathbf{N}$ and so also true in $\mathbf{N}^*$.

Problem 2. To do this problem you will need to use the fact that

$(*)$ *Every set can be ordered,*

i.e., for every set A there exists a binary relation $x \le y$ which is a linear ordering of A. This is a non-trivial (but basic) set-theoretic fact about sets which we will just assume here. (It is proved in every standard course in set theory).

Recall also the first order definition of what it means for $\le$ to be a linear ordering,

$$\begin{aligned}\mathsf{LO} \equiv \forall x[x \le x] &\wedge \forall x \forall y \forall z[(x \le y \wedge y \le z) \to x \le z] \\ &\wedge \forall x \forall y[(x \le y \wedge y \le x) \to x = y] \wedge \forall x \forall y[x \le y \vee y \le x].\end{aligned}$$

Prove that if τ is any finite vocabulary without the binary relational symbol $\le$, T is a τ-theory and χ is a τ-sentence, then

$$\text{if } T, \mathsf{LO} \vdash \chi, \text{ then } T \vdash \chi.$$

SOLUTION. Let χ be any τ-sentence and assume that $T, \mathsf{LO} \vdash \chi$.

Let $\mathbf{A}$ be any model of T with universe A; choose by $(*)$ an ordering $\le$ of A; and let $(\mathbf{A}, \le)$ be the expansion of $\mathbf{A}$ with that ordering $\le$, so that

$$(\mathbf{A}, \le) \models T \cup \{\mathsf{LO}\}.$$

By the Hypothesis and the Soundness Theorem,

$$(\mathbf{A}, \le) \models \chi;$$

by (1) of the Compositionality Theorem 3G.1 in the Notes,

$$\mathbf{A} \models \chi;$$

and since $\mathbf{A}$ was an arbitrary model of T, by the Completeness Theorem, $T \vdash \chi$, as required.

Problem 3. True or false: for any formula χ of the Propositional Calculus PL, any sequence $p_1, \dots, p_n$ of distinct propositional variables which includes all the variables which occur in χ and any sequence $\phi_1, \dots, \phi_n$ of $\mathsf{LPCI}(\tau)$-formulas,

$$\text{if } \chi \text{ is a tautology, then } \vdash \chi\{p_1 :\equiv \phi_1, \dots, p_n :\equiv \phi_n\}.$$

You must prove your answer.

SOLUTION. This is true, as follows.

By the hypothesis that χ is a tautology and the Completeness Theorem for the Propositional Calculus, there is a PL-proof

$$\chi_0, \chi_2, \dots, \chi_k \equiv \chi.$$

If some propositional variable q other than $p_1, \dots, p_n$ occurs in some χ_i, replace it with p_1, set

$$\psi\{\vec{p} :\equiv \vec{\phi}\} :\Longleftrightarrow_{\text{df}} \{p_1 :\equiv \phi_1, \dots, p_n :\equiv \phi_n\}$$

to save typing and then prove that the sequence of LPCI-formulas

$$\chi_0\{\vec{p} :\equiv \vec{\phi}\}, \chi_2\{\vec{p} :\equiv \vec{\phi}\}, \dots, \chi_k\{\vec{p} :\equiv \vec{\phi}\} \equiv \chi\{\vec{p} :\equiv \vec{\phi}\}$$

is a proof in LPCI; the argument is trivial, because

$$\text{if } \psi \text{ is a } \mathsf{PL}\text{-axiom, then } \psi\{\vec{p} :\equiv \vec{\phi}\} \text{ is an } \mathsf{LPCI}\text{-axiom,}$$

and the only Inference Rule in PL is Modus Ponens, which is also an Inference Rule of LPCI.

Problem 4. Prove that if $g(t, x)$ is arithmetical and the function $f(t, x)$ satisfies the equations

$$f(0, x) = x + 1,$$
$$f(t + 1, x) = f(t, g(t, x)),$$

then f is also arithmetical.

Solution. The definition of f *is not* by primitive recursion from g, check it out; it is a simple case of what Rósza Petér called definition by Nested Recursion. The proof that *if* g *is arithmetical then so is* f is similar to the proof for the Ackermann function in Problem x2.23 (which can be found in the solutions to Homework 5).

A sequence of triples

$$\alpha = \Big((n_0, x_0, w_0), (n_1, x_1, w_1), \dots , (n_k, x_k, w_k)\Big)$$

is a *Petér derivation* for f, if one of the following two conditions holds for each $i \le k$:

(1) $n_i = 0$ and $w_i = x_i + 1$.
(2) $n_i > 0$ and there is a $j < i$ such that

$$n_i = n_j + 1; \quad w_i = w_j, \quad x_j = g(n_j, x_i).$$

Lemma 1. If α *is a Petér derivation, then*

$$\text{for each } i \le k, \quad (i, x_i, w_i) \in \alpha \Longrightarrow f(i, x_i) = w_i.$$

Proof is by (complete) induction on $i \le k$ for a given Petér derivation.

Case 1. If (n_i, x_i, w_i) is in α and $n_i = 0$, then $w_i = x_i + 1$ (by the conditions on α) and (by the hypothesis on f),

$$f(n_i, x_i) = f(0, x_i) = x_i + 1 = w_i.$$

Case 2. If $n_i > 0$, then there is some $j < i$ satisfying (2) and

$$\begin{aligned} w_i = w_j &\quad \text{(by the conditions on } \alpha) \\ &= f(n_j, x_j) \quad \text{(by the induction hypothesis)} \\ &= f(n_i - 1, g(n_i - 1, x_i)) \quad \text{(by the conditions on } \alpha) \\ &= f(n_i, x_i) \quad \text{(by the hypothesis on } f). \end{aligned}$$

Lemma 2. For all t *and* x*, there is a Petér derivation* α *which includes a triple* $(t, x, f(t, x))$.

This is proved by a routine induction on t.

Now the two Lemmas together give

$f(t,x) = w$

$\iff$ there is a Petér derivation which contains the triple (t,x,w),

and from this we can get an arithmetical definition of the graph of f using the tuple codings supplied by the β function.