You may refer to and use every result in PL and every problem in the first three homework assignments whose solution is posted.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the "architecture" of your argument.

There are 10 problems or parts of problems and they are all worth the same, but they vary in difficulty: do first those which are easy.

Problem 1. Construct the truth table for the formula $(p \to q) \to p$, where p and q are distinct variables.

SOLUTION.

p	q	$(p \to q)$	$(p \to q) \to p$
0	0	1	0
0	1	1	0
1	0	0	1
1	1	1	1

Problem 2. Write the following formula correctly spelled, i.e., with all the parentheses it needs: $(\neg\neg\neg p) \to (p \to q)$.

SOLUTION. $((\neg(\neg(\neg p))) \to (p \to q))$

Problem 3. For each of the following claims, determine whether it is true or not for all formulas ϕ, ψ and outline a proof of your claim.

(3a) $\neg\phi \models (\phi \to \psi)$

SOLUTION. This is true. Proof: By the definition of logical consequence in 2D,

$$\neg\phi \models (\phi \to \psi) \iff \text{for every assignment } v, \quad \text{if } v \models \neg\phi, \text{ then } v \models \neg\phi \to (\phi \to \psi);$$

and by the Tarski conditions, for any v:

$$\begin{aligned} v \models \neg\phi \to (\phi \to \psi) &\iff \text{either } v \not\models \neg\phi \text{ or } v \models (\phi \to \psi) \\ &\iff \text{either } v \models \phi \text{ or } v \models \neg\phi \text{ or } v \models \psi \end{aligned}$$

and the condition on the last line is obviously true for all v.

(3b) $(\phi \to \psi) \to \psi \models \psi$

Solution. This is not always true; it is equivalent by the Deduction Theorem to

$$\models (\phi \to \psi) \to \psi) \to p,$$

so equivalent to

$$v \models (\phi \to \psi) \to \psi) \to p$$

for every v; but this fails when $v \models \phi$ but $v \models \neg\psi$.

(3c) For every assignment v, if $v \models (\phi \wedge \psi) \vee \neg\psi$, then $v \models (\psi \to \phi)$

Solution. This is true. To check it, consider cases on a given v and the truth values it assigns to the parts ϕ, ψ of the hypothesis.

If $v \models \neg\psi$, then $v \models (\psi \to \phi)$, whether $v \models \phi$ or not; while if $v \models (\phi \wedge \psi)$, then clearly $v \models (\psi \to \phi)$.

(3d) $\neg\phi \vdash (\phi \to \psi)$

Solution. This is the same as part (1a), except for the "proves" sign $\vdash$ in place of the "validates" sign $\models$. The Completeness Theorem, however, guarantees that

$$\text{if } \neg\phi \models (\phi \to \psi) \text{ then } \neg\phi \vdash (\phi \to \psi),$$

and so the claim is true.

Problem 4. Recall that two formulas ϕ and ψ are (logically) *equivalent* if $\phi \models \psi$ and $\psi \models \phi$.

(4a) Prove or give a counterexample:

Every formula is equivalent to one in which the negation symbol $\neg$ does not occur.

Solution. This is false, for the formula $\neg p$ among many others. To see this, we prove by structural induction the following proposition about formulas:

$(*)$ *If there is no negation in ϕ and all its variables are in the list $p_1, \dots, p_n$, then*

$$p_1, \dots, p_n \models \phi.$$

We skip the steps of the induction which are easy.

(4b) Prove or give a counterexample:

Every formula is equivalent to one in which the only connective that occurs is the negation symbol $\neg$.

Solution. This is false, for example for the formula $p \wedge q$; because the only formulas that can be constructed using only the unary connective $\neg$ have only one variable in them, and so cannot restrict both p and q.

(4c) Let $\bullet$ be any one of the binary connectives $\wedge, \vee, \rightarrow$. Prove (in outline, skipping details) or give a counterexample:

Every formula is equivalent to one in which the only connectives that occur are $\neg$ *and* $\bullet$.

Solution. This is true for every binary connective $\bullet$. The relevant definitions are

$$\begin{array}{lll} \vee & (\phi \wedge \psi) :\equiv \neg((\neg\phi) \vee (\neg\psi)), & (\phi \rightarrow \psi) :\equiv \neg\phi \vee \psi \\ \wedge & (\phi \vee \psi) :\equiv \neg((\neg\phi) \wedge (\neg\psi)), & (\phi \rightarrow \psi) :\equiv \neg\phi \vee \psi \\ \rightarrow & (\phi \vee \psi) :\equiv (\neg\phi) \rightarrow \psi, & (\phi \wedge \psi) :\equiv \neg((\neg\phi) \vee (\neg\psi)). \end{array}$$

Problem 5. Let

$$\phi_1 \equiv (p \rightarrow q)$$

and define by recursion

$$\phi_{n+1} :\equiv (\phi_n \rightarrow p),$$

so that, for example,

$$\phi_2 \equiv ((p \rightarrow q) \rightarrow p), \quad \phi_3 \equiv (((p \rightarrow q) \rightarrow p) \rightarrow p), \ \ldots$$

Is it True or False that $\models \phi_{27}$? You must prove your answer.

Solution. This is true, in fact $\models \phi_n$ for every odd $n \geq 3$. We prove this by induction, starting with $\models \phi_3$ which is Peirce's Law. At the induction step, with n odd,

$$\phi_{n+2} \equiv (\phi_{n+1} \rightarrow p) \equiv (\phi_n \rightarrow p) \rightarrow p.$$

Now, for any assignment v, if $v \models p$ then $v \models \phi_{n+2}$, since $\phi_{n+1} \equiv (\psi \rightarrow p)$ for some ψ; and if $v \models \neg p$, then $v \models \neg(\phi_n \rightarrow p)$ since by the induction hypothesis $v \models \phi_n$, and so $v \models \phi_{n+2}$.