Math 114L, Spring 2021, Solutions to HW #9

---

x3.1. Prove (3) Theorem 1.1, that the relation

$$\mathrm{Seq}(u) \iff u = 1 \vee (\exists n, x_0, \dots, x_{n-1})[u = f_n(x_0, \dots, x_{n-1})]$$

is arithmetical.

Solution. If $\mathrm{Seq}(u)$, then either $u = 1$ (which is the code of the empty sequence), or

$$u = p_0^{x_0+1} \cdot p_1^{x_1+1} \cdots p_n^{x_n+1}$$

for some numbers $x_0, x_1, \dots, x_n$ where $p_0 = 2, p_1 = 3, \dots$ is the sequence of prime numbers; the characteristic property of such numbers u is that *if a prime divides* u*, then every smaller prime divides* u, so that we have the equivalence

$$\mathrm{Seq}(u) \iff u = 1 \vee (\forall i, j)\Big([i < j \wedge p_j \text{ divides } u] \rightarrow p_i \text{ divides } u\Big).$$

Now we have proved in LPCI that the function $j \mapsto p_j$ and the relation n divides u are arithmetical, and then Theorem 3J.1 of LPCI implies that $\mathrm{Seq}(u)$ is arithmetical.

x3.2. Prove (5) of Theorem 1.1, that there is a binary, arithmetical function $\mathrm{proj}(u, i)$ such that

$$\text{if } u = \langle x_0, \dots, x_{n-1}\rangle \text{ and } i < n, \text{ then } \mathrm{proj}(u, i) = x_i.$$

Solution. Using notation from the preceding parts of Theorem 1.1, put

$$\begin{aligned} R(u, i, w) \iff_{\mathrm{df}} &[(\neg\mathrm{Seq}(u) \vee \mathrm{lh}(u) = 0) \wedge w = 0] \\ &\vee [\mathrm{Seq}(u) \wedge i < \mathrm{lh}(u) \wedge p_i^{w+1} \text{ divides } u \wedge p_i^{w+2} \text{ does not divide } u], \end{aligned}$$

check easily that $R(u, i, w$ is the graph of a function, it is arithmetical (by Theorem 3J.1 of LPCI again) and so we can set

$$\mathrm{proj}(u, i) = w \iff_{\mathrm{df}} R(u, i, w).$$

x3.3. Prove Lemma 1.2, that the concatenation function on $\mathbb{N}^{<\omega}$ is arithmetical in the codes and also that it is associative, (1-1).

Solution. The concatenation function is arithmetical in the codes because it (easily) satisfies the equivalence

$$\begin{aligned} u * v = w \iff &\Big((\neg\mathrm{Seq}(u) \vee \neg\mathrm{Seq}(v)) \wedge w = 0\Big) \\ &\vee \Big(\mathrm{Seq}(u) \wedge \mathrm{Seq}(v) \wedge \mathrm{Seq}(w) \wedge [\mathrm{lh}(w) = \mathrm{lh}(u) + \mathrm{lh}(v)] \\ &\wedge (\forall i < \mathrm{lh}(u))[(w)_i = (u)_i \wedge (\forall i < \mathrm{lh}(v))[(w)_{\mathrm{lh}(u)+i} = (v)_i]]\Big), \end{aligned}$$

so it is arithmetical. For the second part, we need to prove that for all u, v, w,

$$(u * v) * w = u * (v * w).$$

If any one of u, v or w, then (easily) both sides of this equation take the value 0; and if none of them are $= 0$, then all three of them code sequences and the (basic) property (2) of Lemma 1.2 does it.

x3.6. Prove the claims about Formula(f) and Sentence(c) in Lemma 1.7.

Solution. The proof is very similar to that for TermDer(y) and Term(t) given in full detail in the proof of Lemma 1.7, and we will just summarize it here.

Looking at the definition of *formula derivation* on page 4, we put first:

$$\begin{aligned}
\mathrm{TermEq}(f) &\iff_{\mathrm{df}} t \text{ is the code of a prime formula } s = t\\
&\iff \mathrm{Seq}(f) \wedge \mathrm{lh}(f) = 3\\
&\qquad \wedge \mathrm{Term}((f)_0) \wedge (f)_1 = \mathrm{sc}(=) \wedge \mathrm{Term}((f)_2)\\
\mathrm{FormNeg}(a, b) &\iff_{\mathrm{df}} \text{for some strings } \alpha \text{ and } \beta\\
&\qquad b = \#(\beta), a = \#(\alpha) \text{ and } \beta \text{ is the code of } (\neg\beta)\\
&\iff b = \langle \mathrm{sc}((), \mathrm{sc}(\neg)\rangle * a * \langle \mathrm{sc}())\rangle\\
\mathrm{FormProp}(a, b, c) &\iff_{\mathrm{df}} a, b, c \text{ code strings } \alpha, \beta, \gamma\\
&\qquad \text{and } \gamma \text{ is the code of } (\alpha \bullet \beta)\\
&\qquad \text{where } \bullet \text{ is } \wedge, \vee \text{ or } \rightarrow\\
&\iff (\exists i)\Big(c = \langle \mathrm{sc}(()\rangle * a * \langle i \rangle * b * \langle \mathrm{sc}())\rangle\\
&\qquad\quad \wedge\ i \in \{\mathrm{sc}(\wedge), \mathrm{sc}(\vee), \mathrm{sc}(\rightarrow)\}\Big)\\
\mathrm{FormQuant}(a, k, b) &\iff_{\mathrm{df}} a, b \text{ code strings } \alpha, \beta\\
&\qquad \beta \text{ is the code of } \exists \mathrm{v}_k(\alpha) \text{ or } \exists \mathrm{v}_k(\alpha)\\
&\iff (\exists q)\Big(b = \langle q, \mathrm{sc}(\mathrm{v}_k)\rangle * b\\
&\qquad\quad \wedge[q = \mathrm{sc}(\exists) \vee q = \mathrm{sc}(\forall)]\Big)
\end{aligned}$$

These relations are all arithmetical by Theorem 3J.1, and by the definition,

$$\begin{aligned}
\mathrm{FormulaDer}(y) \iff_{\mathrm{df}} \mathrm{Seq}(y) \wedge (\forall i < \mathrm{lh}(y))\Big(&\mathrm{TermEq}((y)_i)\\
&\vee(\exists j < i)\mathrm{FormNeg}((y)_j, (y)_i)\\
&\vee(\exists j, l < i)\mathrm{FormProp}((y)_j, (y)_l, (y)_i)\\
&\vee(\exists j < i)(\exists k)\mathrm{FormQuant}((y)_j, k, (y)_i)\Big)
\end{aligned}$$

```
Let me know of errors or better solutions.
```

Again, these relations are all arithmetical by Theorem 3J.1, and hence so are the relations Formula(f) and Sentence(c) by their definitions given in Lemma 1.7.

x3.7. Outline a proof of Part (1) of Lemma 2.6.

Solution. As defined in 6B.4 of LPCI, Peano arithmetic has four axioms and one axiom scheme and to prove that it is arithmetical by Definition 2.3 we must show that

$$\#(\mathsf{PA}) = \{\#(\vec{\forall}\psi) \mid \vec{\forall}(\psi) \text{ is the universal code of an axiom of } \mathsf{PA}\}$$

is arithmetical; here $\vec{\forall}(\psi)$ is defined on page 8. So

$$\begin{aligned} c \in \#(\mathsf{PA}) \iff \quad & c = \vec{\forall}(\psi) \text{ where } \psi \text{ is one of the four axioms of } \mathsf{PA} \\ & \text{or } c = \vec{\forall}(\psi) \text{ where for some formula } \phi, \\ & \psi \equiv \Big(\Big[\phi(0,\vec{y}) \wedge (\forall x)[\phi(x,\vec{y}) \rightarrow \phi(S(x),\vec{y})]\Big] \rightarrow (\forall x)\phi(x,\vec{y})\Big). \end{aligned}$$

The only part in this equivalence which is not trivially arithmetical is the last one, and it is (or should be by now) quite routine to prove it by the methods we used in Lemma 1.7 and Problem x3.6.

x3.8. Outline a proof of Part (2) of Lemma 2.6.

Solution. The proof relation of a theory T is defined in Definition 2.5 and it is very easy to prove that it is arithmetical (if T is arithmetical) by the methods we have used in Lemma 1.7 and Problem x3.6.