Math 114L, Spring 2021, Solutions to HW #8

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x2.44. Prove the Compactness Theorem 7D.1.

Solution. Suppose every finite subset of a theory T has a model; then every finite subset of T is consistent; so T is consistent Lemma 2C.9; and so T has a model by the Completeness Theorem II, 7C.10.

x2.45. Prove the Skolem-Löwenheim Theorem 7D.2.

Solution. If a theory T has a model, it is consistent, and then (by the proof of the Completeness Theorem) it has a model $\mathbf{A}$ whose universe is a subset

$$A \subset \mathrm{Const}_\tau \cup \{d_0, d_1, \dots\} = \{c_0, \dots, c_{n-1}, d_0, d_1, \dots\} = \{e_0, e_1, \dots\}$$

of the set of constants in the expanded signature τ^*; now A can be enumerated, by deleting from the sequence $e_0, e_1, \dots$ those constants which are not in A.

x2.48. Prove that if a τ-theory T has arbitrarily large, finite models, then it has an infinite model.

Solution. Suppose T has arbitrarily large finite models and let

$$T^* = T \cup \{\exists \mathrm{v}_1 \exists \mathrm{v}_2 (\mathrm{v}_1 \neq \mathrm{v}_2), \exists \mathrm{v}_1 \exists \mathrm{v}_2 \exists \mathrm{v}_3 (\mathrm{v}_1 \neq \mathrm{v}_2 \wedge \mathrm{v}_1 \neq \mathrm{v}_3 \wedge \mathrm{v}_2 \neq \mathrm{v}_3), \\ \dots, \exists \mathrm{v}_1 \cdots \exists \mathrm{v}_n \textstyle\bigwedge\!\!\!\bigwedge_{1 \le i < j \le n} (\mathrm{v}_i \neq \mathrm{v}_j), \dots\}$$

Every finite subset of T^* is a subset of $T \cup \{\chi_2, \dots, \chi_n\}$ where each χ_i (with $i \geq 2$) asserts that there are at least i distinct elements in the universe and so has a model, any model of T with at least n elements; so T^* has a model, by the Compactness Theorem, which is a model of T and is infinite.

x2.49. For the empty signature τ (for which the τ-structures are just sets) decide whether the following properties of τ-structures are basic elementary or elementary, and prove your answer.

1. A is finite.
2. A is infinite.

Solution. 1. The class of all finite structure (A) is not elementary; because if for some theory T (in the empty vocabulary)

$$(*) \qquad (A) \models T \iff A \text{ is finite},$$

then T has arbitrarily large finite models and so it would have an infinite model by Problem x2.48, which contradicts $(*)$.

2. The class of all infinite structures (in the empty vocabulary) is elementary, axiomatized by the theory

$$T_{\mathrm{inf}} = \{\chi_2, \chi_3, \dots\}$$

where (as usual) for $n \geq 2$, $\chi_n \equiv \exists \mathrm{v}_1 \cdots \exists \mathrm{v}_n \bigwedge\!\!\!\bigwedge_{1 \leq i < j \leq n} (\mathrm{v}_i \neq \mathrm{v}_j)$. It is not basic elementary because if

$$A \text{ is infinite } \iff \mathbf{A} \models \phi, \text{ then } A \text{ is finite } \iff \mathbf{A} \models \neg\phi,$$

contradicting 1.

Note: With a bit more care, this result can be proved for arbitrary vocabularies τ; you just need to show that for any τ, there are arbitrarily large τ-structures (in which all the primitives are interpreted trivially).

x2.50. For the signature $\tau = (E)$ with just one, binary relation symbol, prove that the class of structures which are symmetric, connected graphs is not elementary.

Solution. Recall the definitions in Section §1 and assume towards a contradiction that there is a theory T such that

$$(G, E) \text{ is a connected, symmetric graph } \iff (G, E) \models T.$$

Let a, b be two distinct constants and let

$$T^* = T \cup \{d(a,b) > 2, d(a,b) > 3, \ldots d(a,b) > n, \ldots\}$$

where the distance $d(a,b)$ is defined in §1 and each condition $d(a,b) > n$ is defined by a sentence,

$$d(a,b) > n$$
$$\iff (G, E, a, b) \models \forall \mathrm{v}_0 \cdots \forall \mathrm{v}_n \neg \Big(\bigwedge\!\!\!\bigwedge_{0 \leq i < n} E(\mathrm{v}_i, \mathrm{v}_{i+1}) \wedge \mathrm{v}_1 = a \wedge \mathrm{v}_n = b \Big).$$

Every finite subset of T^* includes $d(a,b) > i$ only for $i \leq n$ for some n and has models, for example the finite, symmetric graph

$$a - 0 - 1 - \cdots - n - b$$

which has $n+3$ elements, each joined by an edge with the next. By the Compactness Theorem then, T^* has a model (G^*, E^*, a^*, b^*) in which the elements interpreting the constants a, b are infinitely far apart, i.e., they are not connected by a (finite) path; the reduct (G^*, E^*) is then a disconnected, symmetric graph which satisfies T, contradicting our assumption.

x2.53. Let $\mathbf{N}^*$ be a non-standard model of true arithmetic as in Section 7E, i.e., $\mathbf{N}^*$ is elementarily equivalent but not isomorphic with $\mathbf{N}$. Prove that if we define on $\mathbf{N}^*$ the relation

$$xE^*y \iff (x +^* 1 = y) \vee (y +^* 1 = x),$$

then the following two relations (from Problem x2.16*) are not elementary in $\mathbf{N}^*$—and hence not elementary in the graph $(\mathbb{N}^*, E^*)$:

(3) $P(x,y) \iff d(x,y) < \infty$.
(4) $P(x,y,z) \iff d(x,y) \leq d(x,z)$.

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Let me know of errors or better solutions.
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HINT: The *standard part* of $\mathbf{N}^*$ is an initial segment of $\mathbf{N}^*$ which is isomorphic with $\mathbf{N}$. We may assume that it is $\mathbf{N}$ and put

$$\mathrm{Inf} = \mathbb{N}^* \setminus \mathbb{N} = \text{ the set of "infinite numbers" in } \mathbb{N}^*.$$

This set is not empty. For (3), prove and use the fact that Inf is not elementary; and for (4) prove and use the stronger fact, that Inf is not *elementary from a parameter*, i.e., for every extended formula $\chi(u, v)$ of arithmetic and every $z \in \mathbb{N}^*$,

$$\mathrm{Inf} \neq \{x \in \mathbb{N}^* \mid \chi^{\mathbf{N}^*}[x, z]\}.$$

Solution. A subset $X \subseteq A$ of the universe of a τ-structure $\mathbf{A}$ is *elementary from the parameter* $z \in A$ if there is an extended τ-formula $\chi(u, v)$ such that

$$x \in X \iff \chi^{\mathbf{A}}[x, z].$$

The usual ordering on the natural numbers is defined by

$$x \leq y \iff (\exists t)[x + t = y],$$

and it is a *wellordering*, i.e., *every non-empty subset of* $\mathbb{N}$ *has a least member.* This holds, in particular, for subsets of $\mathbb{N}$ which are elementary from a parameter; which means that *for every extended formula* $\chi(u, v)$ *as above*

$$\mathbf{N} \models (\forall v)\Big((\exists u)\chi(u, v) \to (\exists u)[\chi(u, v) \wedge (\forall u')[\chi(u', v) \to u \leq u']]\Big)$$

and so the same holds for $\mathbf{N}^*$.

Let $\leq^*$ be the natural ordering of $\mathbb{N}^*$ and for any $\chi(u, v)$ as above and any $y \in \mathbb{N}^*$, put

$$X^{\chi, y} = \{x \in \mathbb{N}^* \mid \chi^{\mathbf{N}^*}[x, y];$$

the claim above means that for every $\chi(u, v)$ and $y \in \mathbb{N}^*$,

if $X^{\chi, y}$ *is not empty, then it has a* $\leq^*$*-least member.*

In particular, if $\mathrm{Inf} = \mathbb{N}^* \setminus \mathbb{N}$ were definable from a parameter in $\mathbf{N}^*$, then it would have a $\leq^*$-least member, which it does not.

For (3) and (4),

$$d(x, y) = \text{ the length of the shortest path which joins } x \text{ to } y \text{ in } E^* \quad (x, y \in \mathbb{N}^*)$$

and it is $= \infty$ if there is no such path, e.g., if $x \in \mathbb{N}$ and $y \in \mathrm{Inf}$.

We can now prove (3) and (4) following the hint.

(3) $x \in \mathrm{Inf} \iff d(x, 0) < \infty$; so $d(x, y) < \infty$ cannot be elementary.

(4) If, towards a contradiction, the relation

$$P(x, y, z) \iff d(x, y) \leq d(x, z)$$

were elementary in $\mathbf{N}^*$, then its negation

$$\neg P(x,y,z) \iff d(x,z) < d(x,y)$$

would also be elementary; but for any $y^* \in \mathrm{Inf}$, easily,

$$d(x,0) < d(x,y^*) \iff x \in \mathbb{N},$$

so $\mathbb{N}$ is elementary from a parameter in $\mathbf{N}^*$, so Inf is also elementary from a parameter, which it is not.