Math 114L, Spring 2021, Solutions to HW #7

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x2.32. For the signature $\tau = (E)$ with just one binary relation symbol, decide whether the following properties of τ-structures (graphs) are basic elementary or elementary, and if your answer is positive, define the relevant theory:

1. (G, E) is a symmetric graph.
2. (G, E) is symmetric and connected.

Solution. The class of symmetric graphs is basic elementary by 6B.1, but the class of connected graphs is not even elementary.

x2.33. Prove that isomorphic structures are elementarily equivalent,

$$\mathbf{A} \cong \mathbf{B} \Longrightarrow \mathbf{A} \approx \mathbf{B}.$$

Solution. It is enough to prove that for every τ-formula ϕ the following is true: for all τ-structures $\mathbf{A}, \mathbf{B}$, every isomorphism $\sigma : \mathbf{A} \rightarrowtail\!\!\!\!\rightarrow \mathbf{B}$ and every assignment $\pi : \{v_0, v_1, \dots, \} \to A$,

$$(*) \qquad \text{if } \mathbf{A}, \pi \models \phi, \text{ then } \mathbf{B}, \sigma \circ \pi \models \phi,$$

where $\sigma \circ \pi(v_n) = \sigma(\pi(v_n))$. Notice that if ϕ is a sentence, then π is irrelevant, i.e., $(*)$ gives

$$\mathbf{A} \models \phi \Longrightarrow \mathbf{B} \models \phi,$$

so that if $\mathbf{A} \models T$ then $\mathbf{B} \models T$ as required.

To prove $(*)$, we first check by induction on terms that *if $\sigma : \mathbf{A} \rightarrowtail\!\!\!\!\rightarrow \mathbf{B}$ is an isomorphism and π is an assignment into $\mathbf{A}$, then*

$$\sigma(\text{value}^{\mathbf{A}}(t, \pi)) = \text{value}^{\mathbf{B}}(t, \sigma \circ \pi).$$

This is quite trivial, for example for a variable v_i,

$$\sigma(\text{value}^{\mathbf{A}}(v_i, \pi)) = \sigma(\pi(v_i)),$$

similarly for constants, and at the induction step,

$$\begin{aligned}
\sigma(\text{value}^{\mathbf{A}}(f(t_1, \dots, t_n), \pi)) &= \sigma(f^{\mathbf{A}}(\text{value}^{\mathbf{A}}(t_1, \pi), \dots, \text{value}^{\mathbf{A}}(t_n, \pi))) \\
&= f^{\mathbf{B}}(\sigma(\text{value}^{\mathbf{A}}(t_1, \pi)), \dots, \sigma(\text{value}^{\mathbf{A}}(t_n, \pi))) \\
&= f^{\mathbf{B}}(\text{value}^{\mathbf{B}}(t_1, \sigma \circ \pi), \dots, \text{value}^{\mathbf{B}}(t_n, \sigma \circ \pi)),
\end{aligned}$$

where for the last inference we used the induction hypothesis.

The implication $(*)$ is proved by a similar and equally simple structural induction on the formula, using the result for terms at the basis, e.g.,

$$\begin{aligned}
\mathbf{A}, \pi \models s = t &\iff \text{value}^{\mathbf{A}}(s, \pi) = \text{value}^{\mathbf{A}}(t, \pi) \\
&\iff \sigma(\text{value}^{\mathbf{A}}(s, \pi)) = \sigma(\text{value}^{\mathbf{A}}(t, \pi)) \text{ (because } \sigma \text{ is a bijection)}
\end{aligned}$$

$$\iff \text{value}^{\mathbf{B}}(s, \sigma \circ \pi) = \text{value}^{\mathbf{B}}(t, \sigma \circ \pi) \iff \mathbf{B}, \sigma \circ \pi \models s = t.$$

The remaining cases of this inductive proof are all equally simple, so we will just outline the computation for one of them, skipping the justifications—which are easy once the notation is understood:

$$\begin{aligned} \mathbf{A}, \pi \models \exists v \phi \iff & \text{ there is some } x \in A \text{ such that } \mathbf{A}, \pi\{v := x\} \models \phi \\ \iff & \text{ there is some } x \in A \text{ such that } \mathbf{B}, \sigma \circ (\pi\{v := x\}) \models \phi \\ \iff & \text{ there is some } y \in B \text{ such that} \\ & \mathbf{B}, (\sigma \circ \pi)\{v := y\}) \models \phi \text{ (because } \sigma \text{ is a bijection)} \\ \iff & \mathbf{B}, \sigma \circ \pi \models \exists v \phi. \end{aligned}$$

x2.34. Prove that every bijection $\sigma : A \rightarrowtail\!\!\!\!\twoheadrightarrow A$ of a set A with itself is an automorphism of the trivial structure (A) with no primitives. Use this fact to identify all unary and binary (A)-elementary relations.

Solution. The key observation is that every bijection $\sigma : A \rightarrowtail\!\!\!\!\twoheadrightarrow A$ is an isomorphism of the trivial structure (A), since nothing needs to be preserved. We can use this together with (3) of Problem x2.13 to check the following:

(a) The only unary elementary relations in (A) are the constants truth and falsity,

$$P(x) \iff x = x, \quad Q(x) \iff x \neq x.$$

To see this suppose $R(x)$ is elementary in (A) and it is true of some $x \in A$. Let y be any member of A and let $\sigma : A \rightarrowtail\!\!\!\!\twoheadrightarrow A$ be defined by interchanging x and y, i.e.,

$$\sigma(u) = v \iff [u = x \wedge v = y] \vee [u = y \wedge v = x] \vee [u \neq x \wedge u \neq y \wedge u = v].$$

It follows that $R(x) \iff R(y)$, and so $R(y)$ is true of all y. (The same argument shows that if $R(x)$ is false for some x, then it is false for every y.)

(b) The only binary elementary relations in (A) are

$$P_1(x, y) \iff \mathbf{tt}, \quad P_2(x, y) \iff \mathbf{ff},$$
$$P_3(x, y) \iff x = y, \quad P_4(x, y) \iff x \neq y.$$

To see this, suppose $R(u, v)$ is elementary in (A), fix two distinct elements $x \neq y$ in A and verify the following using trivial, "interchange" automorphisms as in (A):

Case 1, $R(x, x)$ and $R(x, y)$; in this case,

$$R(u, v) \iff P_1(u, v) \iff \mathbf{tt} \quad (u, v \in A).$$

This is because for each u there is an automorphism which exchanges u with x, and so $R(u, u)$, and if $u \neq v$, then there is an automorphism which exchanges u with x and v with v, and so $R(u, v)$

`Let me know of errors or better solutions.`

Case 2, $\neg R(x,x)$ and $\neg R(x,y)$; in this case, by the same argument,

$$R(u,v) \iff P_2(u,v) \iff \mathbf{ff}.$$

Case 3, $R(x,x)$ and $\neg R(x,y)$; in this case

$$R(u,v) \iff P_3(u,v) \iff u = v.$$

To see this, we consider for each u the automorphism which interchanges u with x and infer $R(u,u)$; and for any pair $u \neq v$ we consider the automorphism which interchanges u with x and v with y and infer $\neg R(u,v)$.

Case 4, $\neg R(x,x)$ and $R(x,y)$; in this case

$$R(u,v) \iff P_4(u,v) \iff u \neq v,$$

by a similar argument.

x2.39. Give an example which shows that the restriction on the Exists Elimination Rule (14) is necessary.

Solution. Trivially, $\vdash R(v) \to R(v)$; without the restriction, we could apply Rule (14) to this to get $\vdash \exists v R(v) \to R(v)$, which is clearly not satisfied by every assignment on every structure: just take $\mathbf{A} = (A, R^{\mathbf{A}})$ where $R^{\mathbf{A}}(x)$ is true for some x but not for every x.

x2.42. Prove Lemma 7C.9, that a theory is consistent if and only if all its finite subsets are consistent.

Solution. Like several facts we have proved, this is a simple consequence of the fact that *proofs are finite*: if $T_0 \subseteq T$ is inconsistent for a finite T_0, then there is a proof $\phi_0, \ldots, \phi_n$ of **ff** from T_0, which is also a proof of **ff** from T; and if T is inconsistent, then there is a proof $\phi_0, \ldots, \phi_n$ from T of **ff**, and then this same sequence is also a proof of **ff** from the finite set $T \cap \{\phi_0, \ldots, \phi_n\}$,