Math 114L, Spring 2021, Solutions to HW #6

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x2.21. Determine whether the following relations are arithmetical, i.e., elementary on the structure of arithmetic $\mathbf{N} = (\mathbb{N}, 0, S, +, \cdot)$.

1. $\mathrm{Prime}(x) \iff x$ is a prime number.
2. $\mathrm{TP}(x) \iff$ there are infinitely many twin primes y such that $x \leq y$.
3. $\mathrm{Exp}(x, w) \iff 2^x = w$, where 2^x is defined as usual,
$$2^0 = 1, \quad 2^{x+1} = 2 \cdot 2^x.$$
4. $\mathrm{Quot}(x, y, w) \iff \mathrm{quot}(x, y) = w$.
5. $\mathrm{Rem}(x, y, w) \iff \mathrm{rem}(x, y) = w$.
6. $x \perp y \iff x$ and y are coprime (i.e., no number other than 1 divides both x and y).

Solution. All these relations are arithmetical, and by trivial proofs, expect for the exponential for which we gave a proof in Section 4. For example, using Theorem 3J.1:

$$\begin{aligned}
\mathrm{Prime}(x) &\iff x \geq 2 \wedge (\forall y)(\forall z)[y \cdot z = x \to [y = 1 \vee z = 1]] \\
\mathrm{TP}(x) &\iff (\forall u)[x \leq u \to (\exists y)[u \leq y \wedge \mathrm{Prime}(y) \wedge \mathrm{Prime}(y + 2)]] \\
\mathrm{Quot}(x, y, w) &\iff (\exists q)(\exists r)[x = yw + r \wedge r < y]
\end{aligned}$$

x2.27. Prove (4) and (5) of Proposition 5A.1.

Solution. Proposition 5A.1, (4): Assume that y is a variable which does not occur in ϕ, $\phi\{x :\equiv y\}$ is the result of replacing x by y in all its free occurrences and prove

$$\exists x\phi \asymp \exists y\phi\{x :\equiv y\}$$
$$\forall x\phi \asymp \forall y\phi\{x :\equiv y\}$$

For the first of these, after we use the definitions of $\asymp$ and $\leftrightarrow$ and apply the Tarski condition for $\exists x$, it comes down that we need to show the following: for every structure $\mathbf{A}$, assignment π, formula ϕ, variable y which does not occur in ϕ and $t \in A$,

$$\mathbf{A}, \pi\{x := t\} \models \phi \iff \mathbf{A}, \pi\{y := t\} \models \phi\{x :\equiv y\}.$$

This is practically obvious when we understand what it says (as is (4), in fact), and it can be verified by a trivial (if tedious) structural induction on formulas.

The second case for $\forall$ can be proved in the same way or by using the equivalence $\forall x\phi \asymp \neg\exists x\neg\phi$.

(5) Distribution law for $\exists$ over $\vee$:

$$\exists x(\phi_1 \vee \cdots \vee \phi_n) \asymp \exists x\phi_1 \vee \cdots \vee \exists x\phi_n$$

This is proved by induction on $n \geq 2$.

Basis. $\exists x(\phi \vee \phi_2) \asymp \exists x\phi_1 \vee \exists x\phi_2$. By the definitions, we need to show that for every $\mathbf{A}, \pi$ and ϕ_1, ϕ_2,

$$\mathbf{A}, \pi \models \exists x(\phi_1 \vee \phi_2) \iff \mathbf{A}, \pi \models \exists x\phi_1 \text{ or } \mathbf{A}, \pi \models \exists x\phi_2$$

and for this, we compute, using the Tarski conditions:

$$\begin{aligned}
&\mathbf{A}, \pi \models \exists x(\phi_1 \vee \phi_2) \iff \text{for some } t \in A, \mathbf{A}, \pi\{x := t\} \models \phi \vee \phi_2 \\
&\quad \iff \text{for some } t \in A\Big(\mathbf{A}, \pi\{x := t\} \models \phi_1 \text{ or } \mathbf{A}, \pi\{x := t\} \models \phi_2\Big) \\
&\quad\quad \iff \text{for some } t \in A, \mathbf{A}, \pi\{x := t\} \models \phi_1 \\
&\quad\quad\quad \text{or for some } t \in A,\ \mathbf{A}, \pi\{x := t\} \models \phi_2 \\
&\quad\quad\quad\quad \iff \mathbf{A}, \pi \models \exists x\phi_1 \text{ or } \mathbf{A}, \pi \models \exists x\phi_2.
\end{aligned}$$

We skip the Induction Step which uses the basis and is simpler.

x2.28. Prove (6) of Proposition 5A.1 and infer Corollary 5A.2.
Solution. There are four logical equivalences to be proved and the proofs are all very similar, so we prove the first as an example. By the definition

$$\begin{aligned}
&\exists x\phi \wedge \psi \asymp \exists x(\phi \wedge \psi) \\
&\quad\quad \iff \text{for all } \mathbf{A} \text{ and } \pi, \mathbf{A}, \pi \models \Big(\exists x\phi \wedge \psi \leftrightarrow \exists x(\phi \wedge \psi)\Big),
\end{aligned}$$

so we fix (and hide) $\mathbf{A}$ and compute for arbitrary π using the definitions and the Tarski conditions (and skipping some easy steps):

$$\begin{aligned}
&\pi \models \Big(\exists x\phi \wedge \psi \leftrightarrow \exists x(\phi \wedge \psi)\Big) \\
&\quad \iff \Big(\pi \models \exists x\phi \wedge \psi \iff \pi \models \exists x(\phi \wedge \psi)\Big) \\
&\quad \iff \Big([\pi \models \exists x\phi \text{ and } \pi \models \psi] \iff [\pi \models \exists x(\phi \wedge \psi)]\Big) \\
&\Big([(\text{exists some } t)\pi\{x := t\} \models \phi \text{ and } \pi \models \psi] \\
&\quad \iff [(\text{exists some } t)\pi\{x := t\} \models (\phi \wedge \psi)]\Big) \\
&\quad \iff \Big([(\text{exists some } t)\pi\{x := t\} \models \phi \wedge \psi] \\
&\quad\quad \iff [(\text{exists some } t)\pi\{x := t\} \models (\phi \wedge \psi)]\Big),
\end{aligned}$$

where in the last, crucial step we used the hypothesis that x does not occur free in ψ.

Corollary 5A.2 follows easily from these four logical equivalences by structural induction on formulas.

`Let me know of errors or better solutions.`

x2.30. For the empty signature τ (for which the τ-structures are just sets) decide whether the following properties of τ-structures are basic elementary, elementary or neither; and if your answer is positive to one of these, find a theory which axiomatizes the given property:

1. A has exactly 3 elements.
2. A is finite.
3. A is infinite.

Solution. 1. The class of structures with three elements is basic elementary, axiomatized by the single sentence

$$\exists x \exists y \exists z \Big(x \neq y \wedge y \neq z \wedge z \neq x \wedge \forall u [u = x \vee u = y \vee u = z] \Big).$$

Part of 3. The class of infinite structures is elementary as it consists precisely of the models of $\{\phi_n \mid n = 2, 3, \dots\}$ where

$$\phi_n \equiv \exists v_1 \exists v_2 \cdots \exists v_n \bigwedge\!\!\!\bigwedge_{1 \leq i < j \leq n} (v_i \neq v_j).$$

The class of infinite structures is not basic elementary and the class of finite structures is not even elementary in the empty signature; these facts will follow from basic theorems that we will prove in Section 7.

x2.31. For the signature $\tau = (\leq)$ with just one binary relation symbol, decide whether the following properties of τ-structures are basic elementary or elementary, and if your answer is positive, define the relevant theory:

1. $(P, \leq)$ is an infinite partial ordering.
2. $(P, \leq)$ is a finite partial ordering.
3. $(P, \leq)$ is an infinite linear ordering.
4. $(P, \leq)$ is a finite linear ordering.

Solution. The properties PO, LO of partial and linear orderings are basic elementary, cf. 6B.2, and we use this in our answers.

1. The class of infinite posets is elementary by Problem x2.30. It is not basic elementary.

2. The class of finite posets is not elementary.

3. The class of infinite linear orderings is elementary (by the same argument we give for posets) but not basic elementary.

4. The class of finite linear orderings is not elementary.

x2.29. (1) Let $\mathbf{L} = ([0, 1), 0, \leq)$, where $[0, 1)$ is the half-open interval of real numbers,

$$[0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$$

and 0 is a constant which names the number 0. Prove that $\mathbf{L}$ admits effective elimination of quantifiers. Infer that it is a decidable structure, i.e., there is an effective procedure which decides whether $\mathbf{L} \models \chi$, for an arbitrary sentence χ.

(2) Let $\mathbf{L}' = ([0,1), \leq)$ be the same linear ordering as in (1), but in the language without a name for 0. Does $\mathbf{L}'$ admit elimination of quantifiers?

(3) Is the structure $\mathbf{L}'$ decidable?

Solution. (1) To prove that $\mathbf{L}$ admits effective quantifier elimination, we can copy almost word-for-word the proof of Theorem 5B.4. The only changes that are needed are:

(a) Allow $x = 0, x > 0$ among the positive literals; $x < 0$ is replaced by **ff**.

(b) Allow $z_i = 0$ in Case 1, in which case you replace x by 0 and eliminate the quantifier this way.

(c) In Case 4, replace $x < u_1 \wedge \cdots \wedge x < u_l$ by $0 < u_1 \wedge \cdots \wedge 0 < u_l$ rather than treating the case symmetrically to Case 3.

The decidability of the structure follows because the only quantifier free sentences are **tt**, **ff** and propositional combinations of $0 = 0, 0 > 0$ which are clearly equivalent to one of **tt**, **ff**.

(2) $\mathbf{L}'$ does not admit elimination of quantifiers, because the formula $\forall y[x \leq y]$ with just x free (which defines 0) is not $\mathbf{L}'$-equivalent to any quantifier free formula ϕ in which only x may occur free. To see this, notice that every quantifier free formula with just x free is a propositional combination of

$$x = x,\ x > x,\ \mathbf{tt}, \mathbf{ff},$$

and the first two of these are respectively equivalent to **tt** and **ff**; so every quantifier free formula with one free variable is $\mathbf{L}'$-equivalent to either **tt** or **ff**, and cannot be equivalent to $\forall y[x \leq y]$, which is true of 0 but false of every other element of $[0, 1)$.

(3) The structure $\mathbf{L}'$ is decidable, because it is a reduct of the decidable structure $\mathbf{L}$. This means that every sentence θ of $\mathbf{L}'$ is also a sentence of $\mathbf{L}$ and with the same truth value, and so the decision procedure for $\mathbf{L}$ can be used to decide it.

Let me know of errors or better solutions.