Math 114L, Spring 2021, Solutions to HW #4

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x2.2. Determine the free and bound occurrences of variables in the following (misspelled) formula of $\mathsf{LPCI}(\leq)$:

$$\phi \equiv \forall y(x \leq y) \wedge \forall x \exists y(x \leq y \wedge \neg(y \leq x))$$

Which are the free variables of ϕ and which are its bound variables?

Solution. The only free occurrence of a variable in ϕ is the first occurrence of x, boxed in this copy:

$$\phi \equiv \forall y(\boxed{x} \leq y) \wedge \forall x \exists y(x \leq y \wedge \neg(y \leq x))$$

The only free variable of ϕ is x while the bound variables of ϕ are x and y. (So x is both a free and a bound variable of ϕ.)

x2.3. Consider the following two sentences in the language of posets:

$$\phi \equiv \exists \mathrm{v}_1 \exists \mathrm{v}_2 \forall \mathrm{v}_2[\mathrm{v}_1 \leq \mathrm{v}_2], \quad \psi \equiv \exists \mathrm{v}_1 \forall \mathrm{v}_2 \exists \mathrm{v}_2[\mathrm{v}_1 \leq \mathrm{v}_2].$$

What do they mean, and do they have the same truth value in every poset?

Solution. ϕ says that the poset has a least element; ψ says that the poset has a member less-equal to it, and is always true.

x2.5. Write out the correctly spelled form of $(\exists! x)\phi$ in (2-2).

Solution. Assuming that $x \equiv \mathrm{v}_i$, choose a variable v_j which does not occur in ϕ and set $(\exists! \mathrm{v}_i)\phi :\equiv \exists \mathrm{v}_j \forall \mathrm{v}_i((\phi \rightarrow \mathrm{v}_i = \mathrm{v}_j) \wedge (\mathrm{v}_i = \mathrm{v}_j \rightarrow \phi))$.

x2.6. Give an example of a term α, variables v_1, v_2 and terms t_1, t_2 such that $\alpha\{v_1 :\equiv t_1\}\{v_2 :\equiv t_2\} \not\equiv \alpha\{v_1 :\equiv t_1, v_2 :\equiv t_2\}$.

Solution. Take $\alpha \equiv f(v_1, v_2)$ with two distinct variables and let c be a constant; then

$$f(v_1, v_2)\{v_1 :\equiv v_2,\}\{v_2 :\equiv c\} \equiv f(v_2, v_2)\{v_2 :\equiv c\} \equiv f(c, c),$$
$$f(v_1, v_2)\{v_1 :\equiv v_2, v_2 :\equiv c\} \equiv f(v_2, c).$$

x2.16. Determine whether the following relations are elementary on a fixed, symmetric graph $\mathbf{G} = (G, E)$, and if your answer is positive, find a formula which defines them.

(1) $P(x, y) \iff d(x, y) \leq 2$.
(2) $P(x, y) \iff d(x, y) = 2$.
(3) $P(x, y) \iff d(x, y) < \infty$.
(4) $P(x, y, z) \iff d(x, y) \leq d(x, z)$
(5) $P(x) \iff$ every y can be joined to x.

Solution. Recall that (by convention), $d(x,x) = 0$.

The first two of the relations in the problem are obviously elementary on every symmetric graph,

$$\begin{aligned} d(x,y) \leq 2 &\iff x = y \vee E(x,y) \vee (\exists z)[E(x,z) \wedge E(z,y)], \\ d(x,y) = 2 &\iff x \neq y \wedge \neg E(x,y) \wedge (\exists z)[E(x,z) \wedge E(z,y)]. \end{aligned}$$

(3): this relation is not elementary on some graphs. We will construct graphs with this property further on.

(4): this is not also not elementary on some graphs, like (3). We will eventually prove this.

(5): this relation is elementary on every symmetric graph, which is a surprize because it looks complex, a bit like (3) and (4). To prove this, we consider two cases on any given graph:

Case 1, $\mathbf{G}$ *is connected.* In this case every node can be connected with any given x so that $P(x)$ is true of all x and is defined by the formula $x = x$.

Case 2, $\mathbf{G}$ *is not connected*, so that there are two points y, z such that no path connects y with z. In this case, $P(x)$ fails for every x; because if every point could be connected by a path to x, then there would be a path from y to z, going first from y to x and then from x to z. So $P(x)$ is always false and can be defined by the formula $x \neq x$.

The result looks surprizing because the natural way to understand the question is *whether there is a single extended formula* $\phi(x)$ *which defines the relation* P; and this indeed fails on some graphs, as we will see further on.

x2.18. Prove that every two structures

$$\mathbf{N}_1 = (\mathbb{N}_1, 0_1, S_1, +_1, \cdot_1), \quad \mathbf{N}_2 = (\mathbb{N}_2, 0_2, S_2, +_2, \cdot_2)$$

which satisfy the Peano axioms in 1D are isomorphic. HINT: Let

$$\begin{aligned} X = \Big\{ t \in \mathbb{N}_1 \mid \ &\text{there exists a function } f : A \to B \text{ such that} \\ &0_1 \in A, f(0_1) = 0_2 \in \mathbb{N}_2, \text{ and for all } t \in \mathbb{N}_1, \\ &\qquad S_1(t) \in A \Longrightarrow \Big[t \in A \ \&\ f(t) \in \mathbb{N}_2 \ \&\ f(S_1(t)) = S_2(f(t)) \Big] \Big\}. \end{aligned}$$

Use the Induction Axiom on $\mathbf{N}_1$ to prove that $X = \mathbb{N}_1$ and there is an injection $\sigma : \mathbb{N}_1 \rightarrowtail \mathbb{N}_2$ such that

$$\sigma(0_1) = 0_2 \text{ and for all } t \in \mathbb{N}_1, \sigma(S_1(t)) = S_2(\sigma(t));$$

and then use the Induction Axiom on $\mathbf{N}_2$ to prove that $\sigma[N_1] = N_2$.

Solution. The hint amounts to a full proof for those who know a bit of set theory, which, in any case, is needed to prove Dedekind's characterization of $\mathbf{N}$.

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Let me know of errors or better solutions.
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