Math 114L, Spring 2021, Solutions to HW #3

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x1.6. Construct the truth table for the formula $(p \rightarrow q) \wedge \neg(q \rightarrow p)$.
Solution.

p	q	$(p \rightarrow q)$	$(q \rightarrow p)$	$\neg(q \rightarrow p)$	$(p \rightarrow q) \wedge \neg(q \rightarrow p)$
0	0	1	1	0	0
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	1	0	0

x1.15. Suppose $R(i, j)$ is a relation defined for $i, j \leq n$, choose a doubly indexed sequence of distinct propositional variables $\{p_{ij}\}_{i,j \leq n}$, and consider the assignment

$$v(p_{ij}) = \begin{cases} 1, & \text{if } R(i, j), \\ 0, & \text{otherwise.} \end{cases}$$

The variables $\{p_{ij}\}$ can be used to express various properties about the relation R. Recall for example that

$$R \text{ is symmetric} \iff (\text{for all } i, j \leq n)[R(i, j) \iff R(j, i)];$$

now easily,

$$R \text{ is symmetric} \iff v \models \bigwedge_{i \leq n} \bigwedge_{j \leq n} [p_{ij} \leftrightarrow p_{ji}].$$

Find propositional formulas which express the following properties of R:

(a) R is the graph of a function, i.e., $(R(i, j)\ \&\ R(i, k)) \Longrightarrow j = k$.
(b) R is the graph of a one-to-one function.
(c) R is the graph of a surjection—a function from $\{0, \ldots, n\}$ onto $\{0, \ldots, n\}$.

Solution. We will use the following extended version of the finite conjunction and disjunction constructs: for any finite sequence $\phi_0, \ldots, \phi_n$ of formulas and any non-empty set $I \subseteq \{0, 1, \ldots, n\}$,

$$\bigwedge_{i \in I} \phi_i :\equiv \bigwedge_{j \leq k} \phi_{f(j)}, \quad \bigvee_{i \in I} \phi_i :\equiv \bigvee_{j \leq k} \phi_{f(j)}$$

where $f : \{0, \ldots, k\} \rightarrow \{0, \ldots, n\}$ enumerates I in increasing order; and in case $I = \emptyset$,

$$\bigwedge_{i \in \emptyset} \phi_i :\equiv \phi_0 \vee \neg\phi_0, \quad \bigvee_{i \in \emptyset} \phi_i :\equiv \phi_0 \wedge \neg\phi_0.$$

Fr example,

$$\bigwedge_{i \in \{2,5\}} \phi_i :\equiv \phi_2 \& \phi_5, \quad \bigvee_{i \in \{0,5,7\}} \phi_i :\equiv \phi_0 \vee \phi_5 \vee \phi_7.$$

This makes it easier to manipulate finite conjunctions and disjunctions without even needing to worry whether they are empty. It is useful here to read

$$\bigwedge\!\!\!\bigwedge_{i\in I}\phi_i \text{ as ``for all } i \in I, \phi_i\text{''}, \quad \bigvee\!\!\!\bigvee_{i\in I}\phi_i \text{ as ``there exists } i \in I \text{ such that } \phi_i\text{''},$$

where the set I will be defined by various conditions.

(a) $\chi_a :\equiv \neg\Big(\bigvee\!\!\!\bigvee_{i,j,k\leq n, j\neq k}[p_{ij} \wedge p_{ik}]\Big)$.

(b) $\chi_b :\equiv \chi_a \wedge \neg\Big(\bigvee\!\!\!\bigvee_{i,j,k\leq n, i\neq j}[p_{ik} \wedge p_{j,k}]\Big)$.

(c) $\chi_b :\equiv \chi_a \wedge \bigwedge\!\!\!\bigwedge_{j\leq n}\bigvee\!\!\!\bigvee_{i\leq n} p_{ij}$.

x1.17. Prove that the "repetition" clause (D2) in the definition of deduction is not needed: i.e., *if $T \vdash \chi$, then there is a deduction of χ from T without repetitions.* (The clause was included in the definition so that we can more easily combine deductions without restriction.)

Solution. A deduction $\chi_0, \dots, \chi_k$ from T as defined by (D1) – (D4) in 3A is also a deduction in which (D2) (repetition) is not allowed as a justification for including some formula χ_n; because if $\chi \equiv \chi_n$ occurs earlier in the sequence and m is least such that $\chi \equiv \chi_m$, then χ is either an assumption (in T) or an axiom or follows from some χ_i, χ_j *with* $i, j < m < n$ by Modus Ponens—and then it can be re-inserted in the deduction as χ_n with the same justification.

x1.19. Prove the ($\neg$)-introduction rule in Lemma 3A.4: that

$$\text{if } T, \chi \vdash \phi \text{ and } T, \chi \vdash \neg\phi, \text{ then } T \vdash \neg\chi.$$

Solution. The hypothesis and the Deduction Theorem 3A.3 give us deductions (from T) of $\chi \to \phi$ and $\chi \to \neg\phi$. To get a deduction of $\neg\chi$ from T we put these two deductions together and then follow them with an instance of Axiom (3) and two applications of Modus Ponens as follows:

$$\dots, \chi \to \phi, \dots, \chi \to \neg\phi, (\chi \to \phi) \to [(\chi \to \neg\phi) \to \neg\chi],$$
$$(\chi \to \neg\phi) \to \neg\chi, \neg\chi.$$

x1.22. Prove that if T is not consistent, then $T \vdash \chi$ for every χ.

Solution. Assume that $T \vdash \phi$ and $T \vdash \neg\phi$, which with Axiom (1) give deductions from T of $\neg\chi \to \phi$ and $\neg\chi \to \neg\phi$. Now use Axiom (3) with two applications of Modus Ponens to get from T a proof of $\neg\neg\chi$, from which χ follows by Axiom (4) and another Modus Ponens.

x1.24. Prove that a set of formulas T is consistent if and only if every finite subset of T is consistent.

`Let me know of errors or better solutions.`

Solution. The key ideas are that *provability is monotone* and *proofs are finite*, and it is easier to prove the contrapositive claim, i.e.,

$$T \text{ is inconsistent} \quad \Longleftrightarrow \quad \text{some finite subset of } T \text{ is inconsistent.}$$

In the direction ($\Leftarrow$), if $T_0 \vdash (\chi \wedge \neg\chi)$ for some finite $T_0 \subset T$, then also $T \vdash (\chi \wedge \neg\chi)$ by monotonicity; and in the direction ($\Rightarrow$), if $T \vdash (\chi \wedge \neg\chi)$, then there is a finite deduction from T

$$\phi_0, \phi_1, \ldots, \phi_n \equiv (\chi \wedge \neg\chi),$$

and if we let $T_0 = \{\phi_i \mid i \leq n \ \&\ \phi_i \in T\}$, then T_0 is a finite subset of T and the same sequence of formulas is a deduction from T_0, so $T_0 \vdash (\chi \wedge \neg\chi)$.

x1.27. Suppose T is an infinite set of formulas. Prove that if every finite subset $T_0 \subset T$ of T is satisfiable, then T is satisfiable.

Solution. If every finite subset T_0 of T is satisfiable, then every finite $T_0 \subset T$ is consistent, since it cannot be that $v \models (\chi \wedge \neg\chi)$; and so T is consistent by Problem x1.24, and hence T is satisfiable by the Completeness Theorem 3B.5. *Note*: The Compactness Theorem is useful in various applications (some of which we will meet later) and its formulation does not involve formal deductions; we proved it using formal deductions, of course, but it can also be proved directly from the semantics of PL using a bit of set theory.