Math 114L, Spring 2021, Solutions to HW #2

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x1.8. Prove Lemma 2A.1.

Solution. This is a trivial—if messy to write up—exercise in using the method of *proof by structural induction.*

We let S be the set of all formulas ϕ such that

$$\begin{aligned}&\text{if } (\vec{p}, q, \vec{r}) \text{ is any sequence of distinct (formal) variables}\\ &\qquad\text{which includes all the variables which occur in } \phi\\ &\qquad\qquad\text{and } q \text{ does not occur in } \phi,\\ &\qquad\qquad\qquad\text{then } F^{\phi}_{\vec{p},q,\vec{r}}(\vec{x}, y, \vec{z}) = F^{\phi}_{\vec{p},\vec{r}}(\vec{x}, \vec{z}),\end{aligned}$$

and we need to show that

$$S \text{ is propositionally closed,}$$

as this is defined on page 2. We need to check:

(1) Every propositional variable is in S. With the notation we have set up, this means that for each i,

$$F^{p_i}_{\vec{p},q,\vec{r}}(\vec{x}, y, \vec{z}) = F^{p_i}\phi_{\vec{q},\vec{r}}(\vec{x}, \vec{z}),$$

which is true, because

$$F^{p_i}_{\vec{p},q,\vec{r}}(\vec{x}, y, \vec{z}) = F^{p_i}\phi_{\vec{q},\vec{r}}(\vec{x}, \vec{z}) = x_i,$$

and the corresponding equation for every q_j.

(2) If $\phi \in S$, then $(\neg\phi) \in S$. Compute:

$$\begin{aligned}F^{\neg\phi}_{\vec{p},q,\vec{r}}(\vec{x}, y, \vec{z}) &= 1 - F^{\phi}_{\vec{p},q,\vec{r}}(\vec{x}, y, \vec{z}) \quad \text{(by def)}\\ &= 1 - F^{\phi}_{\vec{p},\vec{r}}(\vec{x}, y, \vec{z}) \quad \text{(by ind.hyp)}\\ &= F^{\neg\phi}_{\vec{p},\vec{r}}(\vec{x}, y, \vec{z}) \quad \text{(by def).}\end{aligned}$$

The proof of (3) is practically identical.

x1.10. Prove equation (2-4) in the proof of the Replacement Theorem 2D.1, i.e., the following: for every formula ϕ whose variables are in the list $p_1, \dots, p_k$, if $\vec{q} \equiv q_1, \dots, q_l$ is a sequence of distinct variables which include all the variables in $\psi_1, \dots, \psi_k$, $\vec{y} \equiv y_1, \dots, y_l$ and

$$\chi \equiv \phi\{p_1 :\equiv \psi_1, \dots, p_k :\equiv \psi_k\},$$

then

$$(*) \qquad F^{\chi}_{\vec{q}}(\vec{y}) = F^{\phi}_{\vec{p}}(F^{\psi_1}_{\vec{q}}(\vec{y}), \dots, F^{\psi_k}_{\vec{q}}(\vec{y})).$$

Hint: Use structural induction on ϕ.

Solution. By structural induction on ϕ, letting $\vec{x} \equiv x_1, \ldots, x_k$. We consider two of the cases, the others being similar.

Case "variable", $\phi \equiv p_i$. Now $F^{\phi}_{\vec{p}}(\vec{x}) = x_i$, $\chi \equiv \psi_i$, $F^{\chi}_{\vec{q}}(\vec{y}) = F^{\psi_i}_{\vec{q}}(\vec{y})$, and so

$$F^{\phi}_{\vec{p}}(F^{\psi_1}_{\vec{q}}(\vec{y}), \ldots, F^{\psi_k}_{\vec{q}}(\vec{y})) = F^{\psi_i}_{\vec{q}}(\vec{y}) = F^{\chi}_{\vec{q}}(\vec{y}).$$

Case "conjunction", $\phi \equiv \phi_1 \wedge \phi_2$. With the natural notation and appealing to the induction hypothesis in the third line,

$$\begin{aligned} &F^{\phi}_{\vec{p}}(F^{\psi_1}_{\vec{q}}(\vec{y}), \ldots, F^{\psi_k}_{\vec{q}}(\vec{y})) \\ &\quad = \max\Big(F^{\phi_1}_{\vec{p}}(F^{\psi_1}_{\vec{q}}(\vec{y}), \ldots, F^{\psi_k}_{\vec{q}}(\vec{y})), F^{\phi_2}_{\vec{p}}(F^{\psi_1}_{\vec{q}}(\vec{y}), \ldots, F^{\psi_k}_{\vec{q}}(\vec{y}))\Big) \\ &\qquad\qquad = \max\Big(F^{\chi_1}_{\vec{q}}(\vec{y}), F^{\chi_2}_{\vec{q}}(\vec{y})\Big) = F^{\chi}_{\vec{q}}(\vec{y}). \end{aligned}$$

x1.11. Prove that the formula (2) in Theorem 2D.3 is a tautology.

Solution. We use the Tarski conditions, Theorem 2C.1, the point of the problem being that they "translate" formulas to English:

$$\begin{aligned} &v \models (\phi \to \psi) \to ((\phi \to (\psi \to \chi)) \to (\phi \to \chi)) \\ &\quad \iff v \not\models (\phi \to \psi) \text{ or } v \models (\phi \to (\psi \to \chi)) \to (\phi \to \chi) \\ &\quad \iff v \not\models (\phi \to \psi) \text{ or } v \not\models (\phi \to (\psi \to \chi)) \text{ or } v \models (\phi \to \chi) \\ &\quad \iff v \not\models (\phi \to \psi) \text{ or } v \not\models (\phi \to (\psi \to \chi)) \text{ or } v \not\models \phi \text{ or } v \models \chi. \end{aligned}$$

If $v \not\models \phi$ or $v \models \chi$ we are done; so

$$\text{assume } v \models \phi \text{ and } v \not\models \chi$$

and continue the computation from above,

$$\begin{aligned} \ldots &\iff (v \models \phi \text{ and } v \models \neg\psi) \text{ or } v \not\models (\phi \to (\psi \to \chi)) \\ &\iff v \models \neg\psi \text{ or } (v \models \phi \text{ and } v \not\models \psi \to \chi) \\ &\iff v \models \neg\psi \text{ or } (v \models \psi \text{ and } v \models \neg\chi) \\ &\iff v \models \neg\psi \text{ or } v \models \psi \end{aligned}$$

and the claim on the last line is true.

x1.14. Prove that

$$\text{if } T, \phi \models \psi, \text{ then } T \models \phi \to \psi.$$

Solution. In plain English, the claim is that for every assignment v,

$$\text{if } v \models T \cup \{\phi\}, \text{ then } \Big(\text{if } v \models T, \text{ then } \big(\text{if } v \models \phi, \text{ then } v \models \psi\big)\Big),$$

and this is clearly true.

`Let me know of errors or better solutions.`

x1.18. Prove the ($\wedge$)-introduction rule in Lemma 3A.4: that

$$\text{if } T \vdash \phi \text{ and } T \vdash \psi, \text{ then } T \vdash (\phi \wedge \psi).$$

Solution. We are given a proof of ϕ and a proof of ψ (from T), we put them together and we continue as follows:

$$\ldots, \phi, \ldots, \psi, \phi \to (\psi \to (\phi \wedge \psi)) \quad \text{Axiom (5)},$$
$$\psi \to (\phi \wedge \psi) \quad \text{(Modus Ponens)}, \phi \wedge \psi \quad \text{(Modus Ponens)}$$

x1.20. Prove the ($\vee$)-elimination rule in Lemma 3A.5, that

$$\text{if } T, \phi \vdash \chi \text{ and } T, \psi \vdash \chi, \text{ then } T, \phi \vee \psi \vdash \chi.$$

Solution. The Deduction Theorem applied to the hypotheses gives us proofs (from T) of $\phi \to \chi$ and $\psi \to \chi$. We view these as proofs from $T, \phi \vee \psi$, we put them together and we continue as follows:

$$\ldots, \phi \to \chi, \ldots, \psi \to \chi,$$
$$(\phi \to \chi) \to ((\psi \to \chi) \to ((\phi \vee \psi) \to \chi)) \quad \text{(Axiom (8)}$$
$$\phi \vee \psi \to \chi \quad \text{(two applications of Modus Ponens)}$$
$$\phi \vee \psi \quad \text{(using the hypothesis)}, \quad \chi \quad \text{(Modus Ponens)}$$

x1.21. Prove the ($\neg$)-elimination rule in Lemma 3A.5, that

$$\text{if } T \vdash \neg\neg\phi, \text{ then } T \vdash \phi.$$

Solution. We are given a deduction from T of $\neg\neg\phi$ and we continue as follows:

$$\ldots, \neg\neg\phi, \neg\neg\phi \to \phi, \quad \text{(Axiom (4)}, \phi \quad \text{(Modus Ponens)}$$

x1.23. Prove that for every formula χ,

$$\vdash \chi \vee \neg\chi.$$

HINT: Prove $\neg\neg(\chi \vee \neg\chi)$ and then use Axiom (4), or use the natural introduction and elimination rules in Lemmas 3A.4, 3A.5.

Solution. One way to do this is to prove the following *Contrapositive* (CP) inference rule, which has many applications:

(CP) $\qquad$ if $T \vdash \phi \to \psi$, then $T \vdash \neg\psi \to \neg\phi$

Proof of (CP), skipping the T which plays no role.

1. Assume $\vdash \phi \to \psi$, call it HYP.
2. Using the axioms, easily, $\phi \to \psi, \phi \to \neg\psi \vdash \neg\phi$.
3. Using HYP and 2, $\phi \to \neg\psi \vdash \neg\phi$.
4. From the axioms again, $\neg\psi \vdash \phi \to \neg\psi$

`Let me know of errors or better solutions.`

5. From 4 and 3, $\neg\psi \vdash \neg\phi$.
6. And by the Deduction Theorem, from 5, $\vdash \neg\psi \rightarrow \neg\phi$.

The CP rule can also be stated in the form

$$\text{(CP)} \qquad \text{if } T, \phi \vdash \psi, \text{ then } T, \neg\psi \vdash \neg\phi$$

which is equivalent to it using the Deduction Theorem. We use it in this form to show that the hint suffices.

1. By the $\vee$-intro rule, $\chi \vdash \chi \vee \neg\chi$ and $\neg\chi \vdash \chi \vee \neg\chi$.
2. By the CP rule, from 1, $\neg(\chi \vee \neg\chi) \vdash \neg\chi$ and $\neg(\chi \vee \neg\chi) \vdash \neg\neg\chi$
3. Now from 2, the $\vee$-elim rule gives $\vdash \neg\neg(\chi \vee \neg\chi)$.

x1.7. Let $\downarrow$ be the *Sheffer stroke*, the binary connective defined by the truth table

p	q	$p \downarrow q$
0	0	1
0	1	0
1	0	0
1	1	0

We read $(\phi \downarrow \psi)$ as "*neither* ϕ *nor* ψ". Define the $\downarrow$-formulas using only this connective (rather than $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$), and prove that every n-ary bit function can be defined by a $\downarrow$-formula with n propositional variables.

Solution. The classical connectives $\neg, \wedge, \vee, \rightarrow$ can be be introduced as abbreviations of formulas using the Sheffer stroke $\downarrow$ successively by

$$\neg\phi :\equiv (\phi \downarrow \phi), \quad (\phi \wedge \psi) :\equiv ((\neg\phi) \downarrow (\neg\psi)),$$
$$(\phi \vee \psi) :\equiv (\neg((\neg\phi) \wedge ((\neg\psi)))), \quad (\phi \rightarrow \psi) :\equiv ((\neg\phi) \vee \psi)$$

and then every formula can be translated to a $\downarrow$-formula which has the same truth table and hence defines the same bit function. Notice that these translations are quite complex: e.g., even using misspellings,

$$p \wedge q :\equiv (p \downarrow p) \downarrow (q \downarrow q)$$

and the $\downarrow$-formula for $p \vee q$ is worse.

x1.25. Prove *Peirce's Law*, the formula

$$(((p \rightarrow q) \rightarrow p) \rightarrow p)$$

Note: It is trivial to check that Peirce's Law is valid, i.e., every assignment v satisfies it: just take cases on whether $v \models p$ or $v \models \neg p$. The challenge is to give a proof and check which axioms or which natural introduction and elimination rules are needed.

Solution. It is enough to show that

$$(*) \qquad (p \rightarrow q) \rightarrow p, \neg p \vdash p;$$

Let me know of errors or better solutions.

because, trivially

$$(p \to q) \to p, \neg p \vdash \neg p,$$

and so by $(\neg)$-intro,

$$(p \to q) \to p \vdash \neg\neg p;$$

and then by $(\neg)$-elim,

$$(p \to q) \to p \vdash p,$$

from which the Deduction Theorem ($(\to)$-intro) gives Peirce's law.

To prove $(*)$, we argue as follows:

$$\neg p, p, \neg q \vdash p \text{ and } \neg p, p, \neg q \vdash \neg p \quad \text{(trivially)},$$

so by $(\neg)$-intro, $\neg p, p \vdash \neg\neg q$; then by $(\neg)$-elim, $\neg p, p \vdash q$; and then by the Deduction Theorem again,

$$\neg p \vdash p \to q$$

which gives trivially

$$\neg p, (p \to q) \to p \vdash p \to q;$$

and then by Modus Ponens, $\neg p, (p \to q) \to p \vdash p$, i.e., $(*)$.

`Let me know of errors or better solutions.`