Math 114L, Spring 2021, Solutions to HW #1

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x1.2. A sequence $\alpha_0, \dots, \alpha_k$ of strings is a *propositional formula construction* if for each $n \leq k$, either α_n is a propositional variable, or $\alpha_n \equiv (\neg \alpha_i)$ for some $i < n$, or $\alpha_n \equiv (\alpha_i \bullet \alpha_j)$ for some binary connective $\bullet$ and some $i, j < n$.

Prove that a string of symbols ϕ is a propositional formula if and only if it occurs in some propositional formula construction.

Solution. We prove that every formula ϕ occurs in some (propositional formula) construction by structural induction on formulas.

(1) Each propositional variable $\mathbf{A}_i$ occurs in a construction; because as a sequence of length 1, $\mathbf{A}_i$ is a construction.

(2) If ϕ occurs in a construction, then $(\neg\phi)$ occurs in a construction; because if $\alpha_0, \dots, \alpha_k$ is a construction in which ϕ occurs, say $\phi \equiv \alpha_i$, then the sequence

$$\alpha_0, \dots, \alpha_k, (\neg \alpha_i)$$

is also a construction and ϕ occurs in it.

(3) For any of the three binary connectives, if ϕ and ψ occur in constructions, then $(\phi \bullet \psi)$ occurs in a construction; because if

$$\alpha_0, \dots, \alpha_k \text{ and } \beta_0, \dots, \beta_l$$

are constructions such that $\alpha \equiv \alpha_i$ and $\beta \equiv \beta_j$ for some i, j, then the sequence

$$\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l, (\alpha_i \bullet \beta_j)$$

is also a construction and $(\phi \bullet \psi)$ occurs in it.

For the other direction, it is enough to prove that if

$$\alpha_0, \dots, \alpha_k$$

is a propositional formula construction, then every α_n is a formula. This is almost immediate from the definitions, by (complete, finite) induction on $n \leq k$. For example, at the basis, α_0 must be a propositional variable $\mathbf{A}_i$, since no formulas precede it that could be used to justify its inclusion in the construction. At the induction step, to take just one case, if $\alpha_n \equiv (\neg \alpha_i)$ for some $i < n$, then α_i is a formula by the induction hypothesis, and so its negation α_n is also a formula.

x1.3. Prove (2) of Lemma 1A.2, that for every non-empty, proper, initial part α of a formula ϕ, the number of left parentheses "(" in α is greater than the number of right parentheses ")" in α.

Solution. To save typing, let us say that a string α has property P if *every non-empty, proper initial part of α has more left parens than right* *parens*. To prove by structural induction that all formulas have property P, it is enough to check the following:

(1) Each variable $\mathbf{A}_i$ has property P; because as a string (with one element), $\mathbf{A}_i$ has no non-empty, proper part.

For this proof only, we use the notation α^- to name any non-empty, proper initial part of a string α, so that

$$\text{if } \alpha \equiv (\wedge(\mathbf{A}_2), \text{ then } \alpha^- \text{ could be any one of } (\quad (\wedge \quad (\wedge(\quad (\wedge(\mathbf{A}_2.$$

(2) If a formula ϕ has property P, then $(\neg\phi)$ has property P. The non-empty, proper initial parts of $(\neg\phi)$ are all strings of the form

$$(\quad (\neg \quad (\neg\phi^- \quad (\neg\phi$$

and they all have property P using the induction hypothesis and (for the last case) that ϕ is a formula—so that ϕ is balanced and so $(\phi$ has one more left than right paren.

(3) If ϕ, ψ have property P and $\bullet$ is any of the binary connectives, then $(\phi \bullet \psi)$ has property P. The non-empty, proper initial parts of $(\phi \bullet \psi)$ are exactly all strings in one of the following forms:

$$(\quad (\phi^- \quad (\phi \quad (\phi\bullet \quad (\phi\bullet\psi^- \quad (\phi\bullet\psi$$

and the assumption that ϕ, ψ have property P implies again (by inspection) that in all these strings there are more '(' than ')'.

x1.4. Prove Theorem 1A.3. Hint: For the most interesting part (3) of the uniqueness claim, suppose

$$(\psi_1 \bullet_1 \chi_1) \equiv (\psi_2 \bullet_2 \chi_2)$$

with ψ_1 shorter than ψ_2 and derive a contradiction using Lemma 1A.2.

Solution. We must show that for every formula ϕ exactly one of the following is true, and then with uniquely determined "parts":

(1) ϕ is a propositional variable $\mathbf{A}_i$.
(2) There is a formula ψ such that $\phi \equiv (\neg\psi)$.
(3) There are formulas ψ, χ and a binary connective $\bullet$ such that $\phi \equiv (\psi \bullet \chi)$.

The proof is by structural induction on formulas, so we need to check the following three claims:

(a) Each $\mathbf{A}_i$ satisfies (1) and does not satisfy (2) or (3). This is obvious, since any string which satisfies (2) or (3) has larger length than 1.

(b) If ψ is a formula, then $(\neg\psi)$ satisfies (2) and does not satisfy (1) or (3). This is immediate for (1) and it is true for (3) because if

$$(\neg\psi) \equiv (\psi_1 \bullet \chi_1)$$

with some formulas ψ_1, χ_1, then the first symbol in ψ_1 would be $\neg$; while (by an easy structural induction), no formula starts with $\neg$.

(c) If $\phi \equiv (\psi \bullet \chi)$ with some formulas ψ, χ and a binary connective $\bullet$, then it satisfies (3) with these ψ, χ and no others and it does not satisfy (1) or (2).

The second part of the claim follows from what we have proved in (a) and (b). To check the first claim, it is enough to verify the following:

$$\text{if } (\psi_1 \bullet_1 \chi_1) \equiv (\psi_2 \bullet_2 \chi_2), \text{ then } \psi_1 \equiv \psi_2, \bullet_1 \equiv \bullet_2, \chi_1 \equiv \chi_2.$$

The conclusion of this implication is clearly true if ψ_1 and ψ_2 have the same length; but if ψ_1 is shorter than ψ_2 and

$$(\psi_1 \bullet_1 \chi_1) \equiv (\psi_2 \bullet_2 \chi_2),$$

then ψ_1 is a non-empty, proper initial part of ψ_2 and it is balanced by (1) of Lemma 1A.2, contradicting Problem x1.3.

x1.5. For each of the following equations between strings, determine whether there are formulas $\phi, \psi, \chi, \phi', \psi', \chi'$ which make them true:
(a) $((\phi \wedge \psi) \vee \chi) \equiv (\phi' \vee (\psi' \wedge \chi'))$.
(b) $((\phi \wedge \psi) \vee \chi) \equiv (\phi' \wedge (\psi' \vee \chi'))$.
You must prove your answers.

Solution. (b) is not possible, because the main connective of the formula on the left is $\vee$ while the main connective of the formula on the right is $\wedge$. (a) is true if we take arbitrary ϕ, ψ, ψ', χ' and set

$$\phi' \equiv (\phi \wedge \psi), \quad \chi \equiv (\psi' \wedge \chi').$$