You may refer to and use every result in the Notes and the slides of the lectures for all three Parts of the class and every problem in the nine homework assignments whose solutions have been posted, **except** for Problem 3.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the "architecture" of your argument.

All problems are worth the same, but they vary greatly in difficulty: make sure you do all the easy ones.

Problem 1. Prove that for any signature τ, every $\mathsf{LPCI}(\tau)$-term t satisfies exactly one of the following three conditions.

(a) $t \equiv v$ for a uniquely determined variable v.
(b) $t \equiv c$ for a uniquely determined constant c.
(c) $t \equiv f(t_1, \dots, t_n)$ for a uniquely determined function symbol f and uniquely determined terms $t_1, \dots, t_n$.

Solution. As explained in Section 2B (page 5) of Part 2 of the Notes, for any signature τ,

t is an $\mathsf{LPCI}(\tau)$-term if it is a member of every set S of strings which is closed under $\mathsf{LPCI}(\tau)$-term-formation, i.e.,

(1) every variable v_i is a member of S (as a string of length 1);
(2) every τ-constant is a member of S (as a string of length 1);
(3) If f is a function symbol in τ with arity n and $t_1, \dots, t_n$ are all members of S, then the string $f(t_1, \dots, t_n)$ is a member of S.

So to prove the *Parsing Lemma* for $\mathsf{LPCI}(\tau)$ terms claimed by this problem, we need to show that the set of strings S which satisfy (a) – (c) is closed is closed under $\mathsf{LPCI}(\tau)$-term formation. This is quite simple and we will skip writing it down.

Problem 2. For the signature $\tau = (E)$ with just one binary relation symbol, decide whether the following properties of τ-structures (graphs) are basic elementary or elementary and prove your answer:

(1) (G, E) is a symmetric graph.
(2) (G, E) is symmetric and connected.

Solution. A structure (G, E) is a symmetric graph if it satisfies (the universal closures of) the formulas

$$\neg E(v, v), \quad E(u, v) \to E(v, u).$$

so the class of symmetric graphs is basic elementary.

On the other hand, *the class of connected graphs is not elementary.* To prove this by contradiction, suppose it were, so

$$(G, E) \text{ is a connected graph} \iff (A, E) \models T$$

for some T, choose two fresh constants a, b and consider the theory

$$T^* = T \cup \{d(a, b) > 1, d(a, b) > 2, d(a, b) > 3, \dots\}$$

where $d(x, y)$ is the *distance* from x to y, the length of the smallest path from x to y. Now every finite subset T_0 of T^* has a model, e.g.,

$$\mathbf{G}(n) = (\mathbb{N}, a^n, b^n, E)$$

where $E(i, j) \iff |i - j| = 1$ and $a^n = 0, b^n = n$, so $d(0, b^n) = n$ for some n (which depends on T_0). Now the Compactness Theorem gives us a model $(A, a^{\mathbf{A}}, b^{\mathbf{A}}, E^{\mathbf{A}})$ of T^* which is not connected, because $d(a^{\mathbf{A}}, b^{\mathbf{A}}) = \infty$.

Problem 3. (For this problem, you can only use results earlier in the LPCI Notes than 7C.4.)

Prove the $(\neg)$-Introduction Rule in Lemma 7C.4 of the Notes for LPCI, that if χ is a sentence, then for any theory T,

$$\text{if } T, \chi \vdash \psi \text{ and } T, \chi \vdash \neg\psi, \text{ then } T \vdash \neg\chi.$$

Solution. The hypothesis and the Deduction Theorem give us that

$$T \vdash \chi \to \psi \text{ and } T \vdash \chi \to \neg\psi;$$

we put these proofs together to get from T

$T : \dots, \chi \to \psi, \dots, \chi \to \neg\psi,$
$(\chi \to \psi) \to ((\chi \to \neg\psi) \to \neg\chi)$ (Axiom (3) of the Hilbert system),
(Modus Ponens twice) $\neg\chi$.

Problem 4. Let T be a $(0, +)$-theory such that

if $\mathbf{A} = (A, 0^{\mathbf{A}}, +^{\mathbf{A}}) \models T$,
then $\mathbf{A}$ *has an expansion* $\mathbf{B} = (A, 0^{\mathbf{A}}, +^{\mathbf{A}}, S^{\mathbf{B}}, \cdot^{\mathbf{B}})$ *such that* $\mathbf{B} \models \mathsf{PA}$.

Determine whether for every $(0, +)$-sentence χ,

$$(*) \qquad \mathsf{PA} \vdash \chi \Longrightarrow T \vdash \chi$$

and prove your answer.

SOLUTION. The claim is true. To prove it, suppose that χ is an arbitrary $(0, +)$-sentence such that $\mathsf{PA} \vdash \chi$, so by the Soundness Theorem,

$$\mathbf{B} \models \mathsf{PA} \Longrightarrow \mathbf{B} \models \chi \quad (\mathbf{B} \text{ any } \tau_{\mathsf{PA}}\text{-structure}).$$

Now let $\mathbf{A}$ be any $(0, +)$-structure such that $\mathbf{A} \models T$; by the hypothesis of the problem, $\mathbf{A}$ has an expansion $\mathbf{B}$ such that $\mathbf{B} \models \mathsf{PA}$; so $\mathbf{B} \models \chi$; so $\mathbf{A} \models \chi$ by Compositionality; and since $\mathbf{A}$ was an arbitrary model of T, $T \models \chi$; and so $T \vdash \chi$ by the Completeness Theorem.

Problem 5. Let $\mathbf{N}_+ = (\mathbb{N}, 0, +)$ be the reduct of $\mathbf{N}$ to the vocabulary $(0, +)$;

let $\mathbf{N}^*_+ = (\mathbb{N}^*, \overline{0}, \overline{+})$ be the corresponding reduct of a non-standard model of true arithmetic, as these were defined in the last Section 7E of the LPCI Notes;

and let $\mathbf{E}_+ = (E, 0, +^E)$ where E is the set of even numbers and $+^E$ is the restriction of addition in $\mathbf{N}$ to the even numbers.

Prove that these three structures are elementarily equivalent, i.e., for every $\mathsf{LPCI}(0, +)$-sentence χ,

$$\mathbf{N}_+ \models \chi \iff \mathbf{N}^*_+ \models \chi \iff \mathbf{E}_+ \models \chi.$$

SOLUTION. We need to prove two equivalences in this problem, and what makes it interesting is that they hold for entirely different reasons.

(1) By Compositionality and the basic property of $\mathbf{N}^*$, for any $\mathsf{LPCI}(0, +)$-sentence χ,

$$\mathbf{N}_+ \models \chi \iff \mathbf{N} \models \chi \iff \mathbf{N}^* \models \chi \iff \mathbf{N}^*_+ \models \chi.$$

(2) The "doubling" map $\sigma(n) = 2n$ is (easily) an isomorphism of the structure $\mathbf{N}_+ = (\mathbb{N}, 0, +)$ with $\mathbf{E}_+ = (E, 0, +^E)$ and hence, for every $\mathsf{LPCI}(0, +)$-sentence χ,

$$\mathbf{N}_+ \models \chi \iff \mathbf{E}_+ \models \chi.$$

Now (1) and (2) taken together show that

$$\mathsf{E}_+ \models \chi \iff \mathsf{N}_+ \models \chi \iff \mathsf{N}_+^* \models \chi$$

as required.

Problem 6. Construct a model of Raphael Robinson's τ_{PA}-theory Q defined in 6B.5 of the LPCI Notes which is not elementarily equivalent with **N**.

SOLUTION. Let $A = \mathbb{N} \cup \{a\}$, where a is any object not in $\mathbb{N}$ (that we think of as being infinitely large). We interpret 0 by 0, and the functions $S, +, \cdot$ as usual on $\mathbb{N}$, and for a we set (for any $n \in \mathbb{N}$):

1. $S(a) = a$.
2. $a + n = n + a = a + a = a$.
3. $a \cdot 0 = 0$; $a \cdot (n + 1) = a \cdot a = a$.

We now must check all the axioms of the Robinson system, but this is quite trivial. For example: S is clearly one-to-one and never takes on the value 0; the Robinson axiom holds because every $x \in A$ is either 0 or the successor of something—that something being a, if $x = a$; and for addition, in the cases when one of the arguments is a:

$$a + 0 = a; a + S(n) = a = S(a) = S(a + n);$$
$$n + S(a) = n + a = a = S(a) = S(n + a);$$
$$a + S(a) = a + a = a = S(a) = S(a + a).$$

The computation is similar for multiplication.

This structure is not elementarily equivalent with **N** because it satisfies the sentence $(\exists x)[S(x) = x]$ which, of course, fails in **N**.

Problem 7. Call (for this problem only) a τ-structure $\mathbf{A} = (A, \dots)$ **nice** if for each $x \in A$, there is a closed term t_x (no variables) such that

$$\mathrm{value}^{\mathbf{A}}(t_x) = x;$$

for example, **N** is nice because $n = \mathrm{value}^{\mathbf{N}}(\Delta(n))$ (as in (2-2) of the Notes on LPCI).

Prove that if **A** is nice, then for all $x \in A$ and all assignments π into A,

$$\mathbf{A}, \pi\{v := x\} \models \phi \iff \mathbf{A}, \pi \models (\exists v)[\phi \wedge v = t_x]$$

Solution. This is a simple generalization of Lemma 2.1 in TG of the third part of the Notes and it can be proved by (basically) copying the proof of that lemma and replacing $\Delta(n)$ by t_x throughout.

Problem 8. **True** or **False**: *A unary relation $R(n)$ is arithmetical if and only if there is a number f such that*

$$R(n) \iff D(f, n) \in \mathrm{Truth}(\mathbf{N}),$$

where

$$D(f, n) = \#(\exists \mathrm{v}_0) * f * \#(\wedge \mathrm{v}_0 =) * \delta(n) * \langle \mathrm{sc}[)] \rangle$$

is the arithmetical function defined in the proof of Lemma 2.2 of Part 3 of the Notes. (To get some credit you must give the correct answer; and to get full credit, you must prove it.)

Solution. By Lemma 2.1 and the definition of $D(f, n)$ in Lemma 2.2 of Part 3 of the Notes reproduced here, for every, unary arithmetical relation $R(n)$, there is a formula ϕ with just v_0 free such that

$$R(n) \iff \mathbf{N} \models \exists \mathrm{v}_0(\phi \wedge \mathrm{v}_0 = \Delta(n)) \iff D(\#(\phi), n) \in \mathrm{Truth}(\mathbf{N}),$$

which gives $R(n) \iff D(f, n) \in \mathrm{Truth}(\mathbf{N})$ with $f = \#(\phi)$.

For the converse, we assume that a relation $R(n)$ satisfies

$$\begin{aligned} R(n) &\iff D(f, n) \in \mathrm{Truth}(\mathbf{N}) \\ &\iff \#(\exists \mathrm{v}_0) * f * \#(\wedge \mathrm{v}_0 =) * \delta(n) * \langle \mathrm{sc}[)] \rangle \in \mathrm{Truth}(\mathbf{N}) \end{aligned}$$

with some f, and we must prove that it is arithmetical.

We fix f and n and assume that $D(f, n) \in \mathrm{Truth}(\mathbf{N})$.

By (1) of Lemma 1.2 of Part 3 of the Notes where the concatenation operation $u * v$ on (codes of) strings is defined,

if u and v are not both codes of strings, then $u * v = 0$.

This means that f is the code of a string, since every member of $\mathrm{Truth}(\mathbf{N})$ is the code of a string and 0 is not the code of a string. More than that, every member of $\mathrm{Truth}(\mathbf{N})$ is the code of a formula; so $D(f, n)$ is the code of a formula; and by the Parsing Lemma for formulas, f must be the code of a formula ϕ, uniquely determined by the formula whose code is $D(f, n)$. And finally, no variable other than v_0 can be free in ϕ since $D(f, n)$ is the code of a sentence; so

$$R(n) \iff \mathbf{N} \models \exists \mathrm{v}_0(\phi \wedge \mathrm{v}_0 = \Delta(n)),$$

which by the Lemma 2.1 quoted in the beginning of this solution implies that $R(n)$ is arithmetical.

Problem 9. Suppose $A(n,x)$ is an arithmetical relation. For each of the following claims, decide whether it is true or false and prove your answer.

(1) The relation

$$B(n) \iff_{\text{df}} \text{there is a finite sequence } (x_0, x_1, \dots, x_k) \text{ such that } A(n,x_0), A(n,x_1), \dots, A(n,x_k) \text{ are all true}$$

is arithmetical.

(2) The relation

$$B(n) \iff_{\text{df}} \text{there is an infinite sequence } x_0 < x_1 < \dots, \text{ such that } A(n,x_0), A(n,x_1), \dots, \text{ are all true}$$

is arithmetical.

Solution. Both are true. The first uses the tuple coding to quantify over finite sequences; for the second we use the equivalence

$$B(n) \iff (\forall x)(\exists y > x)A(n,y)$$

Problem 10. Let γ_{PA} be the Gödel sentence for PA used in the Second Proof of the First Incompleteness Theorem 2.7 of the third Part of the Notes (with $T = \mathsf{PA}$). Let

$$T_1 = \mathsf{PA} \cup \{\gamma_{\mathsf{PA}}\}, \quad T_2 = \mathsf{PA} \cup \{\neg\gamma_{\mathsf{PA}}\}.$$

(1) Is T_1 consistent?

(2) Is T_1 sound for $\mathbf{N}$?

(3) Is T_2 consistent?

(4) Is T_2 sound for $\mathbf{N}$?

You must prove your answers.

Solution. (1) – (3) are true and (4) is false.

(2) is true, because one of the two basic properties of γ_{PA} is that it is true in $\mathbf{N}$, so $\mathbf{N} \models T_1$; and this implies (1), since a theory which has a model is consistent.

(3). T_2 is consistent, because the second of the basic properties of γ_{PA} is that $\mathsf{PA} \nvdash \gamma_{\mathsf{PA}}$.

(4) is false because by the first basic property of γ_{PA} again, $\mathbf{N} \not\models \gamma_{\mathsf{PA}}$.