Math 114C, Winter 2019, Solutions to HW #7

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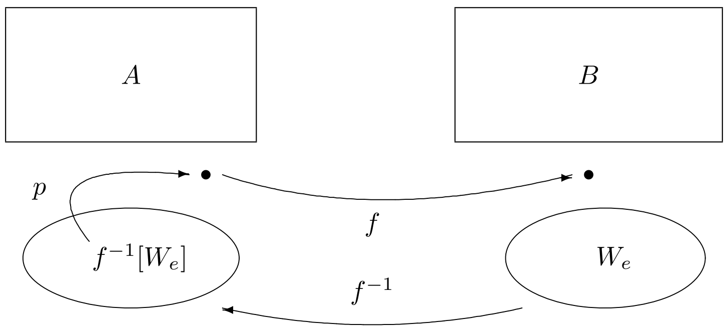
x4C.1. Prove that if A is creative, B is r.e. and $A \leq_1 B$, then B is also creative.

Solution. The hypothesis gives us one-to-one, recursive functions f and p such that

$$x \in A \iff f(x) \in B,$$
$$W_e \subseteq B^c \implies p(e) \in B^c \setminus W_e.$$

From the figure it is obvious that the productive function for B^c which we need



is the function

$$p_B(e) = f(p(g(e))),$$

where g has the property that

$$W_{g(e)} = f^{-1}[W_e];$$

and a g with this property is the function $g(e) = S^1_1(\widehat{h}, e)$, where $\widehat{h}$ is a code of $h(e,x) = \{e\}(f(x))$.

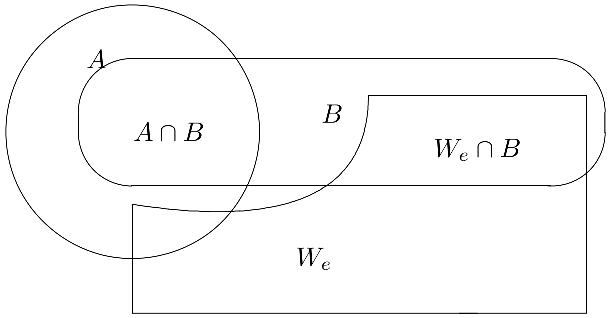
x4C.2. Prove that if A is simple and B is r.e., infinite, then the intersection $A \cap B$ is infinite.

Solution. If the intersection $A \cap B$ were finite, then there would exist some k such that $x \geq k \Longrightarrow x \notin (A \cap B)$; and in this case, the infinite, r.e. set $\{x \in B \mid x \geq k\}$ would be a subset of the complement A^c of A, which is absurd, since A is simple.

x4C.3. Prove that if A and B are simple sets, then their intersection $A \cap B$ is also simple.

Solution. The set $A \cap B$ is r.e., and its complement is infinite (since it contains the complement of A), so that it is enough to show that

$$W_e \text{ infinite} \Longrightarrow W_e \cap A \cap B \neq \emptyset.$$



Towards a contradiction, let W_e be infinite such that

$$W_e \subseteq (A \cap B)^c = A^c \cup B^c,$$

and let

$$C = W_e \cap B.$$

C is r.e., and

$$\begin{aligned} t \in C &\implies t \in W_e \ \&\ t \in B \\ &\implies [t \in A^c \vee t \in B^c] \ \&\ t \in B \\ &\implies t \in A^c, \end{aligned}$$

i.e., $C \subseteq A^c$ and so (since A is simple), C is finite. It follows that

$$W_e \cap B^c = W_e \setminus C$$

is infinite, r.e. (as the difference of an infinite, r.e. and a finite set) and $W_e \cap B^c \subseteq B^c$, which is absurd, since B is simple.

x4D.1. Prove that for some z, $W_z = \{z, z+1, \dots\} = \{x \mid x \geq z\}$.
Solution. By the 2nd Recursion Theorem, there exists some z such that

$$\varphi_z(x) = \begin{cases} 1, & \text{if } z \leq t, \\ \uparrow, & \text{otherwise,} \end{cases}$$

and obviously,

$$W_z = \{t \mid \varphi_z(t)\downarrow\} = \{z, z+1, \dots\}.$$

x4D.2. Prove that for some z, $\varphi_z(t) = t \cdot z$.
Solution. The function

$$f(z, t) = z \cdot t$$

is recursive, and so there exists some z such that

$$\varphi_z(t) = f(z, t) = z \cdot t.$$

`Let me know of errors or better solutions.`

x5A.1. Classify in the arithmetical hierarchy the set

$$A = \{e \mid W_e \subseteq \{0,1\}\}.$$

Solution. A is Π^0_1-complete, so in the class $\Pi^0_1 \setminus \Sigma^0_1$. Proof:

$$x \in A \iff (\forall y)[y \in W_e \Longrightarrow y \leq 1],$$

so A is Π^0_1. To show the Π^0_1-completeness, we define for each recursive relation $P(x,y)$ the partial function

$$g(x,t) = \mu y[\neg P(x,y)],$$

with values depending only on x, so that if $\widehat{g}$ is a code of it and

$$f(x) = S^1_1(\widehat{g}, x),$$

then

$$(\forall y)P(x,y) \implies (\forall t)g(x,t)\uparrow \Longrightarrow W_{f(x)} = \emptyset,$$
$$\neg(\forall y)P(x,y) \implies (\forall t)g(x,t)\downarrow \Longrightarrow W_{f(x)} = \mathbb{N};$$

more specifically,

$$(\forall y)P(x,y) \iff W_{f(x)} \subseteq \{0,1\} \iff f(x) \in A,$$

and A is Π^0_1-complete.

x5A.2. Classify in the arithmetical hierarchy the set

$$B = \{e \mid W_e \text{ is finite and non-empty}\}.$$

Solution. The proof is exactly as for (2) of 5A.6, with a small change in the definition of the function g,

$$g(x,u) = \mu y[u = 0 \vee (\forall i \leq u)\neg Q(x,i,(y)_i)].$$

x5A.3. Classify in the arithmetical hierarchy the set

$$C = \{x \mid \text{there exist infinitely many twin primes } \geq x\},$$

where y is a twin prime number if both y and $y+2$ are prime.

Solution. If there exist infinitely many twin primes, then $C = \mathbb{N}$; and if not, then $C = \emptyset$, so that whatever the correct answer to the classical, open problem, C is recursive.

`Let me know of errors or better solutions.`