Math 114C, Winter 2019, Solutions to HW #5

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x4A.2. Let $R(\vec{x}, w)$ be a semirecursive relation such that for every $\vec{x}$ there exist at least two numbers $w_1 \neq w_2$ such that $R(\vec{x}, w_1)$ and $R(\vec{x}, w_2)$. Prove that there exist two, total recursive functions $f(\vec{x})$, $g(\vec{x})$ such that for all $\vec{x}$,

$$R(\vec{x}, f(\vec{x})) \ \& \ R(\vec{x}, g(\vec{x})) \ \& \ f(\vec{x}) \neq g(\vec{x}).$$

Solution. By the Σ^0_1-Selection Lemma 4A.7 and the hypothesis, there exists a total function $f(\vec{x})$, such that for all $\vec{x}$,

$$R(\vec{x}, f(\vec{x}));$$

and since the relation $R(\vec{x}, w) \ \& \ w \neq f(\vec{x})$ is semirecursive, by the same Lemma and the hypothesis, it follows that there exists a total function $g(\vec{x})$ such that for all $\vec{x}$,

$$R(\vec{x}, g(\vec{x})) \ \& \ g(\vec{x}) \neq f(\vec{x}),$$

as the Problem requires.

x4A.3. Let $R(\vec{x}, w)$ be a semirecursive relation such that for every $\vec{x}$, there exists at least one w such that $R(\vec{x}, w)$.

(1) Prove that there exists a total recursive function $f(n, \vec{x})$, such that

$$(1) \qquad R(\vec{x}, w) \iff (\exists n)[w = f(n, \vec{x})].$$

(2) Prove that if, in addition, for every $\vec{x}$, there exist infinitely many w such that $R(\vec{x}, w)$, then there exists a total, recursive $f(n, \vec{x})$ which satisfies (1) and is "1-1 in n", i.e., for all $\vec{x}, m, n$,

$$m \neq n \Longrightarrow f(m, \vec{x}) \neq f(n, \vec{x}).$$

Solution. By the hypothesis,

$$R(\vec{x}, w) \iff (\exists y)P(\vec{x}, w, y),$$

where $P(\vec{x}, w, y)$ is a recursive relation, and by the Σ^0_1-Selection Lemma 4A.7 and the hypothesis, there exists a total function $g(\vec{x})$, such that for all $\vec{x}$,

$$R(\vec{x}, g(\vec{x})).$$

(1) We set

$$f(n, \vec{x}) = \begin{cases} (n)_0, \text{if } P(\vec{x}, (n)_0, (n)_1), \\ g(\vec{x}), & \text{otherwise.} \end{cases}$$

(2) We define the required function $f(n, \vec{x})$ by complete recursion 1B.16, as follows:

$$f(n, \vec{x}) = \Big(\mu n[P(\vec{x}, (n)_0, (n)_1) \ \& \ (\forall i < n)[(n)_0 \neq f(i, \vec{x})]\Big)_0.$$

The proof (from the hypothesis) that $f(n, \vec{x})$ is total and has the required property is not difficult.

x4B.1. Prove that there is a recursive, partial function $f(e)$, such that

$$W_e \neq \emptyset \Longrightarrow [f(e)\downarrow \ \& \ f(e) \in W_e].$$

Solution. The relation

$$R(e, y) \iff [y \in W_e]$$

is Σ^0_1, so, by the Σ^0_1-Selection Lemma (4A.7), there is a recursive partial function $f(e)$ such that

$$\begin{aligned} f(e)\downarrow &\iff (\exists y)R(e, y) &&\iff W_e \neq \emptyset, \\ f(e)\downarrow &\implies R(e, f(e)) &&\implies f(e) \in W_e; \end{aligned}$$

this is the one we seek.

x4B.2. For each of the following propositions decide whether it is true for arbitrary, recursively enumerable sets A and B. Prove your positive answers and give counterexamples for the negative ones.

(1) $A \cap B$ is r.e.
(2) $A \cup B$ is r.e.
(3) $A \setminus B = \{x \in A \mid x \notin B\}$ is r.e.

Solution. (1) is true, because the relation

$$x \in A \cap B \iff x \in A \ \&\ x \in B$$

is semirecursive (as a conjunction of semirecursive relations), and the corresponding observation for disjunction shows that $A \cup B$ is also r.e. (3) is not true in general, because it implies (with $A = \emptyset$) that the complement of every r.e. set is r.e., which is not true for K.

x4B.3. Prove that every infinite r.e. set has an infinite, recursive subset.

Solution. By the hypothesis, there exists a (total) recursive $f : \mathbb{N} \to \mathbb{N}$ which enumerates the given A,

$$A = \{f(0, f(1), \dots\}.$$

We define g by the recursion,

$$\begin{aligned} g(0) &= f(0) \\ g(n+1) &= f(\mu m[f(m) > g(n)]), \end{aligned}$$

and we observe that g is a total function (since A is infinite, and, for every n, it must have members $> g(n)$) and (by its definition), it is increasing,

$$g(0) < g(1) < \cdots \ ;$$

so the set $B = \{g(0), g(1), \dots\}$ enumerated by g is recursive, infinite, and (obviously) a subset of A.

x4B.4. Are the following claims true or false? Give proofs or counterexamples.

(1) There is a (total) recursive function $u_1(e, m)$ such that for all e, m,

$$W_{u(e,m)} = W_e \cup W_m.$$

(2) There is a (total) recursive function $u_2(e, m)$ such that for all e, m,

$$W_{u(e,m)} = W_e \cap W_m.$$

(3) There is a (total) recursive function $u_3(e,m)$ such that for all e,m,

$$W_{u(e,m)} = W_e \setminus W_m.$$

Solution. (1) Let

$$f_1(e,m,x) = \begin{cases} 1, & \text{if } \varphi_e(x)\downarrow \ \& \ \varphi_m(x)\downarrow, \\ \uparrow, & \text{otherwise;} \end{cases}$$

the graph of f_1 is Σ^0_1 with the computation

$$f_1(e,m,x) = w \iff w = 1 \ \& \ \varphi_e(x)\downarrow \ \& \ \varphi_m(x)\downarrow,$$

so f_1 is recursive. If $\widehat{f_1}$ a code of it, then

$$\{\widehat{f_1}\}(e,m,x)\downarrow \iff x \in W_e \cap W_m,$$

so,

$$\{S^2_1(\widehat{f_1},e,m)\}(x) = \{\widehat{f_1}\}(e,m,x)\downarrow \iff x \in W_e \cap W_m,$$

and

$$u_1(e,m) = S^2_1(\widehat{f_1},e,m)$$

is the function we seek, which moreover is primitive recursive.

(2) Exactly the same construction, with a partial function $f_2(e,m,x)$, such that

$$f_2(e,m,x) = w \iff w = 1 \ \& \ [x \in W_e \vee x \in W_m].$$

(3) This is not true, by (c) of Problem 1.

x4B.5. Does there exist a total, recursive function $f(e,m)$ such that for all e,m,

$$W_{f(e,m)} = \{x + y \mid x \in W_e \text{ and } y \in W_m\}?$$

You must prove your answer.

Solution. It is true: we set

$$R(e,m,t) \iff (\exists x,y)[t = x + y \ \& \ x \in W_e \ \& \ y \in W_m]$$

and we notice that R is a semirecursive relation, so that for some recursive partial function $g(e,m,t)$,

$$g(e,m,t)\downarrow \iff R(e,m,t) \iff (\exists x,y)[t = x + y \ \& \ x \in W_e \ \& \ y \in W_m].$$

If $\hat{g}$ is any code of g, then

$$g(e,m,t) = \{S^2_1(\hat{g},e,m)\}(t),$$

and with $f(e,m) = S^2_1(\hat{g},e,m)$ we have

$$t \in W_{f(e,m)} \iff g(e,m,t)\downarrow \iff (\exists x,y)[t = x + y \ \& \ x \in W_e \ \& \ y \in W_m],$$

which is what we wanted.

x4B.6. Let $f : \mathbb{N} \to \mathbb{N}$ be a (total) recursive function, $A \subseteq \mathbb{N}$ and let

$$f[A] = \{f(x) \mid x \in A\}$$
$$f^{-1}[A] = \{x \mid f(x) \in A\}$$

be the *image* and the *inverse image* of A by f. For each one of the following claims decide whether it is true or not, prove your positive answers and give counterexamples for the negative ones.

(1) If A is r.e., is $f[A]$ also r.e.?
(2) If A is r.e., is $f^{-1}[A]$ also r.e.?
(3) If A is recursive, is $f[A]$ also recursive?
(4) If A is recursive, is $f^{-1}[A]$ also recursive?

Solution. The proofs for the positive answers use the fact that the graph

$$G_f(x, w) \iff f(x) = w$$

of a total recursive function is a recursive relation, and the closure properties of the class of semirecursive relations.

(1): yes, because

$$x \in f[A] \iff (\exists y)[y \in A \ \& \ x = f(y)].$$

(2): yes, because

$$x \in f^{-1}[A] \iff f(x) \in A.$$

(3): not necessarily, because every r.e. (non-empty) set is a recursive image of the recursive set $\mathbb{N}$, by definition, but not every r.e. set is recursive.

(4): yes, because

$$x \in f^{-1}[A] \iff f(x) \in A,$$

and the set of recursive relations is closed under recursive substitutions.

x4B.7. The *closure* $\overline{A}$ of a set $A \subseteq \mathbb{N}$ under a partial function $f : \mathbb{N} \rightharpoonup \mathbb{N}$ is the smallest set B such that $B \supseteq A$ and B is closed for f, i.e.,

$$[x \in B \ \& \ f(x)\downarrow] \Longrightarrow f(x) \in B.$$

(1) Prove that if A is r.e. and $f(x)$ is recursive, then the closure $\overline{A}$ of A under f is also r.e.

(2) Prove that there exists a primitive recursive function $u(e, m)$, such that for all e and m, the set $W_{u(e,m)}$ is the closure $\overline{W_e}$ of W_e under the recursive partial function φ_m with code m.

Solution. We need to find a "construction" of $\overline{A}$, and for this we define first by recursion the sets

$$A_0 = A, \qquad A_{n+1} = f[A_n] = \{y \mid (\exists x)[x \in A_n \ \& \ f(x) = y\},$$

and we consider their union,

$$\overline{A} = \bigcup_n A_n.$$

By its definition, $\overline{A}$ is closed under f, because, if $x \in \overline{A}$ and $y = f(x)\downarrow$, then for some n, $x \in A_n$, and so, $y = f(x) \in A_{n+1} \subseteq \overline{A}$. On the other hand, if B contains A and is closed under f, then by an easy induction, $A_n \subseteq B$ for every n, so $\overline{A} = \bigcup_n A_n \subseteq B$. It follows that

$$\overline{A} = \bigcup_n A_n,$$

and it is enough to show the required propositions for $\bigcup_n A_n$.

We notice first that each A_n is r.e., since $A_0 = A$ is by the hypothesis, and inductively,

$$t \in A_{n+1} \iff (\exists x)[x \in A_n \ \&\ t = f(x)].$$

For (2) (with $A = W_e$) which implies (1), the basic observation is that

$$x \in A_n$$
$$\iff (\exists u_0, u_1, \dots, u_n)[u_0 \in W_e \ \&\ u_1 = f(u_0) \ \&\ \cdots \ \&\ u_n = f(u_{n-1}) = x],$$

which is obvious — or it is easily shown by induction on n. We set

$$R(e,x) \iff (\exists u)(\exists n)[(u)_0 \in W_e \ \&\ (\forall i < n)[(u)_{i+1} = f((u)_i)] \ \&\ (u)_n = x].$$

This relation is Σ^0_1, and from the observation it follows that (with $A = W_e$),

$$x \in \overline{A} \iff R(e,x).$$

Let $h(e,x)$ be a recursive partial function such that

$$R(e,x) \iff h(e,x)\downarrow;$$

if $\widehat{h}$ is a code of $h(x)$, then

$$u(e) = S^1_1(\widehat{h}, e)$$

is the function we seek.

`Let me know of errors or better solutions.`