Math 114C, Winter 2019, Solutions to HW #2

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x1C.2. Prove that for all partial functions $f, g, h : A \rightharpoonup B$,

$$f \sqsubseteq f, \quad [f \sqsubseteq g \ \& \ g \sqsubseteq h] \Longrightarrow f \sqsubseteq h, \quad [f \sqsubseteq g \ \& \ g \sqsubseteq f] \Longrightarrow f = g.$$

Solution. By the definition, in order to show $f \sqsubseteq f$ we have to show that

$$f(\vec{x}) = w \Longrightarrow f(\vec{x}) = w,$$

which is obvious. For the second part, we have to show that if

$$f(\vec{x}) = w \Longrightarrow g(\vec{x}) = w \text{ and } g(\vec{x}) = w \Longrightarrow h(\vec{x}) = w,$$

then

$$f(\vec{x}) = w \Longrightarrow h(\vec{x}) = w,$$

and this is again obvious. Finally, we have to show that if

$$f(\vec{x}) = w \Longrightarrow g(\vec{x}) = w \text{ and } g(\vec{x}) = w \Longrightarrow f(\vec{x}) = w,$$

then $f = g$, that is, for every $\vec{x}$, $f(\vec{x}) = g(\vec{x})$. We consider two cases: if both values diverge (that is $f(\vec{x})\uparrow$ and $g(\vec{x})\uparrow$,), then $f(\vec{x}) = g(\vec{x})$, by the definitions; and if the one value converges, (for example) $f(\vec{x}) = w$, then $g(\vec{x}) = w = f(\vec{x})$, by the assumption.

x1C.8. (1) Prove that there exists exactly one partial function $p(x, y)$ which satisfies the equations

$$\begin{aligned} p(0, y) &= y, \\ p(x + 1, 0) &= 2x + 1, \\ p(x + 1, y + 1) &= 3p(x, y) + p(y, x) + 2, \end{aligned}$$

and compute the value $p(3, 2)$ for this p.

(2) Prove that the unique solution of this system is primitive recursive (or μ-recursive, if this is easier).

Solution. (1) First for the uniqueness, we accept that the partial functions p_1 and p_2 satisfy the system, and we show by induction on $\min(x, y)$ that

$$p_1(x, y) = p_2(x, y).$$

This holds trivially at the Basis, because, if $\min(x, y) = 0$, then either $x = 0$ and

$$p_1(0, y) = y = p_2(0, y),$$

or $x = z + 1 > 0$ and $y = 0$, in which case

$$p_1(z + 1, 0) = 2z + 1 = p_2(z + 1, 0).$$

At the Induction Step,

$$\begin{aligned} p_1(z + 1, w + 1) &= 3p_1(z, w) + p_1(w, z) + 2 \\ &= 3p_2(z, w) + p_2(w, z) + 2 \qquad \text{(Ind. Hypothesis)} \\ &= p_2(z + 1, w + 1). \end{aligned}$$

For the existence of solution, we show by appealing to the Basic Recursion Lemma 1A.2 with $W = (\mathbb{N} \rightharpoonup \mathbb{N})$ (as in the corresponding proof for the Ackermann function, Lemma 1A.7) that for every n there exists (exactly one) function p_n with domain the pairs (x, y) such that $x, y \leq n$, which satisfies the following equations:

$$\begin{aligned} p_n(0, y) &= y, && (y \leq n), \\ p_n(x+1, 0) &= 2x + 1, && (x + 1 \leq n), \\ p_n(x+1, y+1) &= 3p_n(x, y) + p_n(y, x) + 2, && (x + 1, y + 1 \leq n). \end{aligned}$$

It follows (from these equations, by induction on $\min(x, y)$) that

$$\text{if } n < m \text{ and } x, y \leq n, \text{ then } p_n(x, y) = p_m(x, y).$$

Finally we define the function $p(x, y)$ that we are looking for as the "union" of all of them,

$$p(x, y) = w \iff \text{ for some } n, p_n(x, y) = w.$$

(2) The function $p(x, y)$ is primitive recursive indeed, but this is not obvious. We outline two, different proofs.

(b1) We set

$$p_1(x, y) = p(x, y), \qquad p_2(x, y) = p(y, x),$$

and we verify easily (from the equations which $p(x, y)$ satisfies) that these functions satisfy the following system of "mutual recursion":

$$\begin{aligned} p_1(0, y) &= y \\ p_1(x+1, y) &= \begin{cases} 2x + 1 & \text{if } y = 0, \\ 3p_1(x, y-1) + p_2(x, y-1) + 2 & \text{otherwise,} \end{cases} \\ p_2(0, y) &= \begin{cases} 0 & \text{if } y = 0, \\ 2(y-1) + 1 & \text{otherwise,} \end{cases} \\ p_2(x+1, y) &= \begin{cases} x + 1 & \text{if } y = 0, \\ 3p_2(x, y-1) + p_1(x, y-1) + 2 & \text{otherwise.} \end{cases} \end{aligned}$$

Unfortunately, this is not precisely mutual primitive recursion, which we justified in Proposition 1B.15; it is **mutual nested recursion**, for which we can prove that it defines primitive recursive functions by the method of proof of 1B.15 and appeal to x1B.19*. It follows that the functions $p_1(x, y)$ and $p_2(x, y)$ are primitive recursive, which is what we want, since $p(x, y) = p_1(x, y)$.

(b2) The second proof is based on the generalization of definition by "complete primitive recursion" which we proved in Proposition 1B.16 so that it applies to definitions of functions of two variables. It is basically easier than the first proof (because it does not need nested recursion), and we will outline it fairly concisely.

We set

$$h(w, n) = \begin{cases} (n)_1, & \text{if } (n)_0 = 0, \\ 2((n)_0 - 1) + 1, & \text{if } (n)_0 > 0 \ \& \ (n)_1 = 0, \\ 3(w)_{\langle (n)_0 - 1, (n)_1 - 1 \rangle} + (w)_{\langle (n)_1 - 1, (n)_0 - 1 \rangle} + 2, & \text{otherwise,} \end{cases}$$

`Let me know of errors or better solutions.`

and we define by primitive recursion the following function $f(n)$:

$$f(0) = 1,$$
$$f(n+1) = f(n) * h(f(n), n),$$

where $u * v$ is the concatenation function, as in Proposition 1B.14. It is obvious from the definition that for every n, the value $f(n)$ is a sequence code, $\mathrm{lh}(f(n)) = n$, and

$$(*) \qquad \text{if } i < n \le m, \text{ then } (f(n))_i = (f(m))_i.$$

We will show by induction on $\max(x, y)$ that for all x, y, the function

$$p(x,y) = \Big(f(\langle x,y\rangle + \langle y,x\rangle + 1)\Big)_{\langle x,y\rangle}$$

satisfies the required equations, and so is primitive recursive. For the simple cases (and without needing the induction hypothesis),

$$\begin{aligned}
p(0,y) &= \Big(f(\langle 0,y\rangle + \langle y,0\rangle + 1)\Big)_{\langle 0,y\rangle} \\
&= \Big(f(\langle 0,y\rangle + 1)\Big)_{\langle 0,y\rangle} && \text{(by (*))} \\
&= h(f(\langle 0,y\rangle, \langle 0,y\rangle) = y, \\
p(x+1,0) &= \Big(f(\langle x+1,0\rangle + \langle 0,x+1\rangle + 1)\Big)_{\langle x+1,0\rangle} \\
&= \Big(f(\langle x+1,0\rangle + 1)\Big)_{\langle x+1,0\rangle} && \text{(by (*))} \\
&= h(f(\langle x+1,0\rangle), \langle x+1,0\rangle) = 2x+1.
\end{aligned}$$

Finally, and by appealing to the induction hypothesis:

$$\begin{aligned}
p(x+1,y+1) &= \Big(f(\langle x+1,y+1\rangle + \langle y+1,x+1\rangle + 1)\Big)_{\langle x+1,y+1\rangle} \\
&= h(f(\langle x+1,y+1\rangle + \langle y+1,x+1\rangle), \langle x+1,y+1\rangle) \\
&= 3\Big(f(\langle x+1,y+1\rangle + \langle y+1,x+1\rangle)\Big)_{\langle x,y\rangle} \\
&\qquad + \Big(f(\langle x+1,y+1\rangle + \langle y+1,x+1\rangle)\Big)_{\langle y,x\rangle} + 2 \\
&= 3\Big(f(\langle x,y\rangle + \langle y,x\rangle + 1)\Big)_{\langle x,y\rangle} + \Big(f(\langle x,y\rangle + \langle y,x\rangle + 1)\Big)_{\langle y,x\rangle} + 2 \\
&= 3\Big(f(\langle x,y\rangle + 1)\Big)_{\langle x,y\rangle} + \Big(f(\langle y,x\rangle + 1)\Big)_{\langle y,x\rangle} + 2 \\
&= 3p(x,y) + p(y,x) + 2.
\end{aligned}$$

The last equation is from the Ind. Hyp., and we also used the obvious (for the classical coding)

$$\langle x,y\rangle + \langle y,x\rangle + 1 \le \langle x+1,y+1\rangle + \langle y+1,x+1\rangle.$$

From the theory up to this point, I know no easier proof that this function is μ-recursive.

`Let me know of errors or better solutions.`

x1C.9. For partial functions $g(\vec{x})$, $h(w, y, \vec{x})$, prove that the recursive equation

$$(1) \qquad p(y, \vec{x}) = \text{if } (y = 0) \text{ then } g(\vec{x}) \text{ else } h(p(y \dot{-} 1, \vec{x}), y \dot{-} 1, \vec{x})$$

has exactly one solution, namely the partial function f which is defined by the primitive recursion

$$\begin{aligned} f(0, \vec{x}) &= g(\vec{x}), \\ f(y + 1, \vec{x}) &= h(f(y, \vec{x}), y, \vec{x})). \end{aligned}$$

Solution. First we show that the function $f(y, \vec{x})$ which is defined by the primitive recursion

$$\begin{aligned} f(0, \vec{x}) &= g(\vec{x}) \\ f(y + 1, \vec{x}) &= h(f(y, \vec{x}), y, \vec{x}) \end{aligned}$$

satisfies the equation (1).

We consider cases.

For $y = 0$, the left-hand side of (1) is $f(0, \vec{x}) = g(\vec{x})$ (by the definition of f), and its right-hand side is

$$\text{if } (0 = 0) \text{ then } g(\vec{x}) \text{ else } h(f(0 \dot{-} 1, \vec{x}), 0 \dot{-} 1, \vec{x}) = g(\vec{x}),$$

that is, they are equal.

For a successor $y + 1$, the left-hand side of (1) is

$$f(y + 1, \vec{x}) = h(f(y, \vec{x}), y, \vec{x}),$$

again by the definition of f, and its right-hand side is

$$\begin{aligned} &\text{if } (y + 1 = 0) \text{ then } g(\vec{x}) \text{ else } h(f((y + 1) \dot{-} 1, \vec{x}), (y + 1) \dot{-} 1, \vec{x}) \\ &\qquad = \text{if } (y + 1 = 0) \text{ then } g(\vec{x}) \text{ else } h(f(y, \vec{x}), y, \vec{x}) \\ &\qquad\qquad = h(f(y, \vec{x}), y, \vec{x}) \end{aligned}$$

so that we have again equality.

For the uniqueness, let $p(y, \vec{x})$ be such that for all $y, \vec{x}$,

$$(\text{A}) \qquad p(y, \vec{x}) = \text{if } (y = 0) \text{ then } g(\vec{x}) \text{ else } h(p(y \dot{-} 1, \vec{x}), y \dot{-} 1, \vec{x});$$

we show by induction on y, that for all $\vec{x}$, $p(y, \vec{x}) = f(y, \vec{x})$.

At the basis this is obvious, since (by (A)),

$$p(0, \vec{x}) = \text{if } (0 = 0) \text{ then } g(\vec{x}) \text{ else } h(p(0 \dot{-} 1, \vec{x}), 0 \dot{-} 1, \vec{x}) = g(\vec{x}) = f(0, \vec{x}).$$

At the induction step, we compute:

$$\begin{aligned} p(y + 1, \vec{x}) &= \text{if } (y + 1 = 0) \text{ then } g(\vec{x}) \text{ else } h(p((y + 1) \dot{-} 1, \vec{x}), (y + 1) \dot{-} 1, \vec{x}) \\ &= \text{if } (y + 1 = 0) \text{ then } g(\vec{x}) \text{ else } h(p(y, \vec{x}), y, \vec{x}) \\ &= h(p(y, \vec{x}), y, \vec{x}) \\ &= h(f(y, \vec{x}), y, \vec{x}) \quad \text{(ind. hyp.)} \\ &= f(y + 1, \vec{x}). \end{aligned}$$

`Let me know of errors or better solutions.`

x1C.10. Prove that the following recursive equation has a unique solution, the function $\overline{p}(x,y) = \gcd(x,y)$:

$$(2) \qquad p(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ p(y,x), & \text{otherwise, if } x < y, \\ y, & \text{otherwise, if } y \mid x, \\ p(y, \mathrm{rem}(x,y)), & \text{otherwise.} \end{cases}$$

Solution. The basic observation is that for all natural numbers $x, y, d > 0$,

$$d \mid x \text{ and } d \mid y \iff d \mid y \text{ and } d \mid \mathrm{rem}(x,y),$$

which follows immediately from the basic equation of division,

$$x = y\,\mathrm{quot}(x,y) + \mathrm{rem}(x,y).$$

So the common divisors of x and y are the same as the common divisors of y and $\mathrm{rem}(x,y)$, and so these two finite sets have the same greatest member. If we now set

$$g(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \text{the least } d \text{ which divides } x \text{ and } y, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \gcd(x,y), & \text{otherwise,} \end{cases},$$

then by simple inspection we see that the function $g(x,y)$ satisfies the given equation. To show that it is the unique function which satisfies it, we show by an easy (complete) induction on $\max(x,y)$ that if $f(x,y)$ satisfies the equation, then for all x, y, $f(x,y) = g(x,y)$.

x2A.1. Prove that a word E is a term if and only if there is a term derivation $D = (A_0, \dots, A_n)$ with $E \equiv A_n$.
Solution. First we notice that *the set of words which occur in term derivations is closed under term formation* and so it contains every term; and for the converse we prove by induction on n that *if $(A_0, \dots, A_n)$ is a term derivation, then every A_i is a term.*

x2A.2. Prove parts (2) and (3) of Lemma 2A.5.
Solution. (2) Suppose Σ satisfies the (two) hypotheses of the Lemma and contains all individual constants, let

$$\Sigma' = \text{ the set of all strings which contain at least one variable,}$$

and check (easily) that *the union $\Sigma \cup \Sigma'$ is closed under term formation*; by the definition of terms then, every term A is either in Σ or in Σ'—and A is not in Σ' if it is closed, so it must be in Σ.

The argument is similar for (3).

x2A.3. Prove the Lemma of Unique Readability of Terms 2A.6.
Solution. Part (a) is shown by induction on terms, where the basis $A \equiv \mathsf{v}_i$ is obvious. At the induction step, if, for example $A \equiv \mathsf{f}_i^n(A_1, \dots, A_n)$, and the

word $w \equiv w_1 \cdots w_i$ is a proper initial segment of A, then $w \equiv \mathsf{f}_i^n$, or $w \equiv \mathsf{f}_i^n($, or $w \equiv \mathsf{f}_i^n(A_1, \dots, A_j$ for some j, or w is an initial segment of $\mathsf{f}_i^n(A_1, \dots, A_j$ for some (least) j, and in all these cases the desired result follows from the induction hypothesis.

Part (b) is obtained easily again by induction on terms.

x2A.4. Prove that if A_1, B_1, A_2, B_2 are terms of $\mathsf{T}(\mathbf{M})$ and

$$A_1 = B_1 \equiv A_2 = B_2$$

then $A_1 \equiv A_2$ and $B_1 \equiv B_2$.

Solution. The identity symbol "=" occurs exactly once in an equation, so the assumed identity of the two equations can only be true if $A_1 \equiv A_2$ and $B_1 \equiv B_2$.

x1B.24. (1) Prove that every section $A_n(x)$ of the Ackermann function is primitive recursive.

(2) Prove that for every primitive recursive function $f(x_1, \dots, x_n)$, there exists some m such that

$$f(x_1, \dots, x_n) < A_m(\max(x_1, \dots, x_n)) \quad (x_1, \dots, x_n \in \mathbb{N}). \tag{3}$$

(3) Prove that the Ackermann function $A(n, x)$ is not primitive recursive. Hint: Call a function $f(\vec{x})$ *A-bounded* if it satisfies (3) with some m, and prove that the collection of all A-bounded functions is primitively closed. Problems x1A.11 – x1A.14 provide the necessary Lemmas.

Solution. The basic functions S, P_i^n and C_q^n are obviously A-bounded, so it suffices to show that the set of A-bounded functions is closed under composition and primitive recursion.

Composition. Let $f(\vec{x}) = h(g_1(\vec{x}), \dots, g_m(\vec{x})$ with A-bounded $h, g_1, \dots, g_m$. With the obvious interpretation of the notation, we compute:

$$\begin{aligned} f(\vec{x}) &< A_{n_h}(\max(g_1(\vec{x}), \dots, g_m(\vec{x})) \\ &< A_{n_h}(\max(A_{n_1}(\vec{x}), \dots, A_{n_m}(\vec{x}))) && \text{(x1A.12)} \\ &\le A_k(A_m(\max(\vec{x}))) && (k = \max(n_h, n_1, \dots, n_m), \\ & && \quad \text{x1A.13, x1A.12)} \\ &< A_{k+2}(\max(\vec{x})) && (x1A.14). \end{aligned}$$

Primitive recursion. Let

$$\begin{aligned} f(0, \vec{x}) &= g(\vec{x}) \\ f(y+1, \vec{x}) &= h(f(y, \vec{x}), y, \vec{x}) \end{aligned}$$

and, from the hypothesis and Problem x1A.13, let $k \ge 2$ be such that

$$\begin{aligned} g(\vec{x}) &< A_k(\max(\vec{x})) \\ h(w, y, \vec{x}) &< A_k(\max(w, y, \vec{x})). \end{aligned}$$

First we show by induction on y (and obvious Basis) that

$$f(y, \vec{x}) < A_{k+1}(\max(\vec{x}) + y).$$

For the Induction Step:

$$f(y+1, \vec{x}) < A_k(\max(f(y, \vec{x}), y, \vec{x}))$$

$$\begin{aligned}
&< A_k(\max(A_{k+1}(\max(\vec{x}) + y), y, \vec{x}) && \text{(ind. hyp.)}\\
&< A_k(A_{k+1}(\max(\vec{x}) + y)) && \text{(x1A.12)}\\
&= A_{k+1}(\max(\vec{x}) + y + 1).
\end{aligned}$$

From this inequality and the obvious

$$z + y < 3 + 2 \cdot \max(z, y) = A_2(\max(z, y)) \le A_k(\max(z, y))$$

it follows that

$$\begin{aligned}
f(y, \vec{x}) &< A_{k+1}(A_{k+1}(\max(y, \vec{x})))\\
&< A_{k+3}(\max(y, \vec{x})) \qquad \text{(x1A.14)}.
\end{aligned}$$

x1C.12. Prove that there is no partial function $g(y, \vec{x})$ such that equation (38) has exactly two solutions.

Solution. Towards a contradiction (without the parameters $\vec{x}$), let $f(y)$ and $h(y)$ be two different solutions, and let y be the least number for which $f(y) \neq h(y)$. At first we observe that (if $y > 0$),

$$g(y-1) = 0 \text{ or } g(y-1)\uparrow,$$

otherwise $f(y) = f(y-1) = h(y-1) = h(y)$, which contradicts the choice of y. We show by induction that for every t,

$$(*) \qquad g(y+t)\downarrow \ \& \ g(y+t) \neq 0.$$

For the basis, $g(y)\downarrow$ (otherwise $f(y) = h(y) =\uparrow$), and $g(y) \neq 0$ (otherwise $f(y) = h(y) = y$); and for the induction step, the induction hypothesis implies

$$f(y+t+1) = f(y) \neq h(y) = h(y+t+1),$$

which implies as in the basis $g(y+t+1)\downarrow \ \& \ g(y+t+1) \neq 0$.

Finally we observe (by considering cases $z < y$ and $z \ge y$) that if (*) holds, then the recursive equation (38) has infinitely many solutions, all the functions of the form

$$f_q(z) = \begin{cases} f(z), & \text{if } z < y,\\ q, & \text{if } z \ge y,\end{cases}$$

for some q; and this is absurd.