Math 114C, Winter 2019, Solutions to HW #1

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x1A.4. Prove that for any pair of real numbers $\alpha \geq 0$, $\beta > 0$, there exists exactly one natural number q, such that for some (real) r,

$$\alpha = \beta q + r, \quad 0 \leq r < \beta.$$

It follows that r is also unique, since $r = \alpha - \beta q$. The numbers q and r are the *quotient* and the *remainder* of the division of α by β, and we denote them by

$$\mathrm{quot}(\alpha, \beta) = q, \quad \mathrm{rem}(\alpha, \beta) = r.$$

It is also convenient to set

$$\mathrm{quot}(\alpha, 0) = 0, \quad \mathrm{rem}(\alpha, 0) = \alpha,$$

so that these functions are defined for all α, β and always satisfy the equation $\alpha = \beta \cdot \mathrm{quot}(\alpha, \beta) + \mathrm{rem}(\alpha, \beta)$.

Solution. We show first the existence of q, and then its uniqueness.

If $\alpha = 0$, then $\alpha = \beta \cdot 0 + 0$, so the number we want is $q = 0$.

For $\alpha > 0$ we use the "Archimedean property" of the real numbers: since $\beta > 0$, there exists some natural number n such that $\alpha < n\beta$, so that the set

$$A = \{n \mid \alpha < n\beta\}$$

is not empty, and has a least element n_0. Obviously $n_0 > 0$ (because $\alpha > 0$); we set

$$q = n_0 - 1 \geq 0 \qquad r = \alpha - q\beta,$$

and observe that

$$\alpha \geq q\beta,$$

from the choice of n_0 (because if $\alpha < (n_0 - 1)\beta$, then $n_0 - 1 \in A$ and n_0 is not least in A); therefore

$$r \geq 0.$$

We must also show that $r < \beta$, and for this we argue by contradiction: if $r \geq \beta$, then

$$\alpha = q\beta + r \geq q\beta + \beta = (q + 1)\beta = n_0\beta,$$

which contradicts the characteristic property of n_0, that $\alpha < n_0\beta$.

For the uniqueness of the quotient and the remainder, let

$$\alpha = q_1\beta + r_1 \qquad 0 \leq r_1 < \beta$$
$$\alpha = q_2\beta + r_2 \qquad 0 \leq r_2 < \beta,$$

and (towards a contradiction) $q_1 \neq q_2$. By abstracting the first equation from the second we get

$$r_1 - r_2 = (q_2 - q_1)\beta;$$

but $|r_1 - r_2| < \beta$ (because $0 \leq r_1 < \beta, 0 \leq r_2 < \beta$), while

$$|(q_2 - q_1)\beta| = |q_2 - q_1|\beta \geq \beta,$$

(because $q_2 - q_1$ is a positive or negative integer), which is absurd.

x1A.6. The Fibonacci sequence is defined by the recursion

$$a_0 = 0, \quad a_1 = 1, \quad a_{n+2} = a_n + a_{n+1}. \tag{1}$$

(1) Compute the value a_9.
(2) Prove that for every n,

$$a_{n+2} \geq \lambda^n, \text{ where } \lambda = \frac{1+\sqrt{5}}{2}.$$

The basic observation is that λ is one of the roots of the second degree equation

$$x^2 = x + 1. \tag{2}$$

Prove also that if $\rho = \frac{1-\sqrt{5}}{2}$ is the other root of (2), then, for every n

$$a_n = \frac{\lambda^n - \rho^n}{\sqrt{5}}.$$

Solution. (1) From the recursive equation that defines it, the Fibonacci sequence starts with the numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

so that $a_9 = 34$.

(2) We show by complete induction that for every n, $a_{n+2} \geq \lambda^n$.

Basis: For $n = 0$, $a_2 = 1 \geq 1 = \lambda^0$. The proposition is also true for $n = 1$:

$$a_3 = a_2 + a_1 = 2 > \frac{1+\sqrt{5}}{2} = \lambda^1.$$

Induction Step. The induction hypothesis (for the complete induction) is that the proposition is true for all $k \leq n$, and we can also assume that $n \geq 2$, since we verified its truth for $n = 1$. We compute:

$$\begin{aligned} a_{n+2} &= a_n + a_{n+1} \\ &\geq \lambda^{n-2} + \lambda^{n-1} \quad \text{(ind. hyp. since } n \geq 2, \text{ so that } n-2 \geq 0), \\ &= \lambda^{n-2}(\lambda + 1) \\ &= \lambda^{n-2}\lambda^2 = \lambda^n. \end{aligned}$$

For the third part, use complete induction again, after verifying separately the cases $n = 0, 1$, which are trivial.

Induction Step: with the induction hypothesis, we compute for $n = k+2 \geq 2$, so that $k \geq 0$:

$$\begin{aligned} a_{k+2} &= a_k + a_{k+1} \\ &= \frac{1}{\sqrt{5}}[\lambda^k - \rho^k + \lambda^{k+1} - \rho^{k+1}] \\ &= \frac{1}{\sqrt{5}}[\lambda^k(1+\lambda) - \rho^k(1+\rho)] \\ &= \frac{1}{\sqrt{5}}[\lambda^{k+2} - \rho^{k+2}]. \end{aligned}$$

x1A.9. For the Ackermann sections, show that

$$A_1(x) = x + 2, \quad A_2(x) = 2x + 3.$$

`Let me know of errors or better solutions.`

Solution. For the first of the two given equations, by induction on x: for the basis $A_1(0) = A_0(1) = 1 + 1 = 0 + 2$, and at the induction step

$$\begin{aligned} A_1(x+1) &= A_0(A_1(x)) \\ &= A_1(x) + 1 \\ &= x + 2 + 1 && \text{(ind. hyp.)} \\ &= (x+1) + 2. \end{aligned}$$

The second is shown similarly.

x1A.11. Prove that for every n and every x, $A_n(x) \geq 1$.

Solution. By induction on n, we show that for every x, $A_n(x) \geq 1$. BASIS, $n = 0$: $A_0(x) = x + 1 \geq 1$. INDUCTION STEP: $A_{n+1}(0) = A_n(1) \geq 1$ from the induction hypothesis, and $A_{n+1}(x+1) = A_n(A_{n+1}(x)) \geq 1$, again from the induction hypothesis.

x1A.12. Prove that every section A_n of the Ackermann function is *strictly increasing*, that is

$$x < y \Longrightarrow A_n(x) < A_n(y).$$

Infer that for all n, x, $A_n(x) \geq x$. HINT: Prove by double induction that $A_n(x) < A_n(x+1)$.

Solution. Here it is useful to state separately the two parts of the hint:

Lemma. For every function $f(x)$ on the natural numbers,

$$(\forall x)[f(x) < f(x+1)] \Longrightarrow (\forall x)(\forall y)[x < y \Longrightarrow f(x) < f(y)].$$

This holds because if $x < y$, then for some m, $x + m + 1 = y$, and

$$f(x) < f(x+1) < f(x+2) < \cdots < f(x+m+1) = f(y).$$

(Or, more elegantly, we can do an induction on m, where $y = x + m + 1$.)

For the Problem, it suffices now to show that for every n,

$$(\forall x)[A_n(x) < A_n(x+1)],$$

and we do this by induction on n, where the basis for $A_0(x) = x + 1$ is obvious. In the induction step, we show by induction on x that

$$A_{n+1}(x) < A_{n+1}(x+1).$$

For the BASIS, $x = 0$, we compute:

$$\begin{aligned} A_{n+1}(1) &= A_n(A_{n+1}(0)) \\ &= A_n(A_n(1)) \\ &> A_n(1) && \text{(ind. hyp., and Lemma, } A_n(1) > A_n(0) \geq 1.) \\ &= A_{n+1}(0) \end{aligned}$$

For the INDUCTION STEP, we compute:

$$\begin{aligned} A_{n+1}(x+1) &= A_n(A_{n+1}(x)) \\ &< A_n(A_{n+1}(x+1)) && \text{(ind. hyp., for } n \text{ and } x\text{, and Lemma)} \\ &= A_{n+1}(x+2). \end{aligned}$$

x1B.4. Prove that for every $n \geq 1$, the n-place functions

$$\min_n(x_1, \dots, x_n) = \text{the least of } x_1, \dots, x_n$$
$$\max_n(x_1, \dots, x_n) = \text{the greatest of } x_1, \dots, x_n$$

are primitive recursive.

Solution. By induction on $n \geq 2$, we show first that every $\min_n$ is primitive recursive; we have already done this for the Basis $n = 2$, and for the Induction Step we simply observe that

$$\min_{n+1}(x_1, \dots, x_n, x_{n+1}) = \min_2(\min_n(x_, \dots, x_n), x_{n+1}).$$

The same technique works for $\max_n$.

x1B.10. Prove Corollary 1B.7.

Solution. From (5) in Proposition 1B.6, the relation

$$Q_1(z, \vec{x}) \iff (\exists i \leq z) P(i, \vec{x})$$

is primitive recursive; and then so is

$$Q(\vec{x}) \iff Q_1(f(\vec{x}), \vec{x}) \iff (\exists i \leq f(\vec{x})) P(i, \vec{x})$$

using closure of the class of primitive recursive relations under primitive recursive substitutions.

x1B.16. Prove that the following two functions (*restriction* and *concatenation*) are primitive recursive:

$$u \restriction i = \begin{cases} \langle u_0, \dots, u_{i-1} \rangle & \text{if } u = \langle u_0, \dots, u_{n-1} \rangle \text{ with } i \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

$$u * v = \begin{cases} \langle u_0, \dots, u_{n-1}, v_0, \dots, v_{m-1} \rangle, & \text{if } u = \langle u_0, \dots, u_{n-1} \rangle, \\ & \quad\; v = \langle v_0, \dots, v_{m-1} \rangle, \\ 0, & \text{otherwise.} \end{cases}$$

Solution. For the first function (*restriction*) we use the primitive recursion

$$u \restriction 0 = \langle \Lambda \rangle$$
$$u \restriction (i+1) = \text{append}(u \restriction i, (u)_i).$$

For *concatenation*, by primitive recursion again, we set

$$f(0, u, v) = u$$
$$f(i+1, u, v) = \text{append}(f(i, u, v), (v)_i)$$
$$u * v = f(\text{lh}(v), u, v).$$

x1B.18. Prove that if the relation $H(w, \vec{x})$ is primitive recursive and $P(y, \vec{x})$ satisfies the equivalence

$$P(y, \vec{x}) \iff H(\langle \chi_P(0, \vec{x}), \dots, \chi_P(y \dot{-} 1, \vec{x}) \rangle, y, \vec{x}),$$

then $P(y, \vec{x})$ is also primitive recursive.

`Let me know of errors or better solutions.`

Solution. From the hypothesis, the characteristic function of $P(y, \vec{x})$ satisfies the equation

$$\chi_P(y, \vec{x}) = \chi_H(\langle \chi_P(0, \vec{x}), \dots, \chi_P(y \dot{-} 1, \vec{x}) \rangle, \vec{x}),$$

and therefore is primitive recursive from 1B.16.

x1A.10. Find a "closed" formula for $A_3(x)$, like those for A_1 and A_2 in the preceding problem.

Solution. In general, in order to find an explicit expression of a recursively defined function, we start with computations for small values of the variable and examine if they can be expressed by some simple, common formula. For the case at hand, the given recursive definition is

$$\begin{aligned} A_3(0) &= A_2(1) = 3 + 2 \cdot 1 = 5 \\ A_3(x+1) &= 3 + 2A_3(x), \end{aligned}$$

and from it we compute:

$$\begin{aligned} A_3(1) &= 3 + 2[3 + 2] \\ &= 3 + 3 \cdot 2 + 2^2 \\ A_3(2) &= 3 + 2[3 + 3 \cdot 2 + 2^2] \\ &= 3 + 3 \cdot 2 + 3 \cdot 2^2 + 2^3 \\ A_3(3) &= 3 + 2[3 + 3 \cdot 2 + 3 \cdot 2^2 + 2^3] \\ &= 3 + 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + 2^4. \end{aligned}$$

At this point, with a little imagination we guess the general formula

$$\begin{aligned} A_3(x) &= 3[1 + 2^1 + 2^2 + \cdots + 2^x] + 2^{x+1} \\ &= 3[2^{x+1} - 1] + 2^{x+1} \\ &= 4 \cdot 2^{x+1} - 3 = 2^{x+3} - 3. \end{aligned}$$

This gives the correct value $2^3 - 3 = 5$ for $x = 0$, and, inductively,

$$\begin{aligned} A_3(x+1) &= 3 + 2A_3(x) \\ &= 3 + 2[2^{x+3} - 3] \qquad \text{(ind. hyp.)} \\ &= 3 + 2^{x+4} - 6 \\ &= 2^{x+4} - 3. \end{aligned}$$

x1B.19. Prove that for any three functions $g(x)$, $h(w, x, y)$ and $\tau(x, y)$, there exists exactly one function $f(x, y)$ which satisfies the equations

$$f(0, y) = g(y), \qquad f(x+1, y) = h(f(x, \tau(x, y)), x, y);$$

and if the given functions are primitive recursive, then $f(x, y)$ is also primitive recursive.

Solution. First we show by induction that for every n, there exists a unique function

$$f_n : \{x \mid x < n\} \times \mathbb{N} \to \mathbb{N}$$

which satisfies the given equations for all the pairs (x, y) with $x < n$. The proposition is trivial at the basis $n = 0$, since there is no $x < 0$. For the induction step, we have f_n from the induction hypothesis and we set

$$f_{n+1} = \begin{cases} f_n(x, y), & \text{if } x < n, \\ h(f_n(x \dot{-} 1, \tau(x \dot{-} 1, y)), x \dot{-} 1, y), & \text{if } x = n; \end{cases}$$

it is obvious that this f_{n+1} satisfies the given equations for $x < n + 1$ (and all y), and if some f' also satisfies the equations for $x < n + 1$, easily (by finite induction on $x < n + 1$), $f'(x, y) = f_{n+1}(x, y)$.

Finally, we set

$$f(x, y) = f_{x+1}(x, y)$$

and verify that this f satisfies the system for all the values of x and y; and (by induction on x, again), it is the unique "solution" of the system.

To show that the function $f(x, y)$ is primitive recursive, we define first the auxiliary function $\tau^*(i, x, y)$ by the primitive recursion:

$$\begin{aligned} \tau^*(0, x, y) &= y \\ \tau^*(i + 1, x, y) &= \tau(x \dot{-} (i + 1), \tau^*(i, x, y)). \end{aligned}$$

This function "is discovered" if we do some computations of values of $f(x, y)$, which show that the value $f(x, y)$ depends on the "initial value"

$$g(\tau^*(x, x, y)).$$

Finally, we set

$$\varphi(i, x, y) = f(i, \tau^*(x \dot{-} i, x, y)),$$

and it suffices to show that the function $\varphi(i, x, y)$ is primitive recursive, since, obviously,

$$f(x, y) = \varphi(x, x, y).$$

Directly from the definition

$$\begin{aligned} \varphi(0, x, y) &= f(0, \tau^*(x, x, y)) \\ &= g(\tau^*(x, x, y)), \end{aligned}$$

so that it suffices to find some primitive recursive function ψ such that

$$(*) \qquad \varphi(i + 1, x, y) = \psi(\varphi(i, x, y), x, y),$$

and with this aim we compute, for $i + 1 \le x$, so that $x \dot{-} (i + 1) = x - i - 1$:

$$\begin{aligned} \varphi(i + 1, x, y) &= f(i + 1, \tau^*(x - i - 1, x, y)) \quad \text{(because } i < x\text{)} \\ &= h(f(i, \tau(i, \tau^*(x - i - 1, x, y))), i, \tau^*(x - i - 1, x, y)). \end{aligned}$$

Now, from the definition of τ^*, if $i + 1 \le x$ and we set

$$j = x - i - 1 \quad \text{so that} \quad i = x - j - 1, \quad \text{and} \quad j + 1 = x - i,$$

it follows that

$$\begin{aligned} \tau(i, \tau^*(x - i - 1, x, y)) &= \tau(x - (j + 1), \tau^*(j, x, y)) \\ &= \tau^*(j + 1, x, y) = \tau^*(x \dot{-} i, x, y); \end{aligned}$$

and with this value, we continue the computation,

$$\varphi(i + 1, x, y) = h(f(i, \tau(i, \tau^*(x - i - 1, x, y))), i, \tau^*(x - i - 1, x, y))$$

$$\begin{aligned}
&= h(f(i, \tau^*(x \dot{-} i, x, y)), i, \tau^*(x \dot{-} (i+1), x, y)) \\
&= h(\varphi(i, x, y), i, \tau^*(x - i - 1, x, y)).
\end{aligned}$$

From this equation comes the primitive recursive ψ which satisfies the equation (*), and the proof is at last complete.