Please show **all** your work! Answers without supporting work will not be given credit.

Name:____

1. **Integration** Using one or more of the methods discussed, solve the following integrals:

- $\int \sin(x^2)x^3dx$

  Seeing the composition of functions $x \to x^2 \to \sin(x^2)$, we begin with the substitution $t = x^2$. This gives us $\frac{dt}{dx} = 2x$ or $dx = \frac{dt}{2x}$. Substituting in we get $\int \sin(t)x^3\frac{dt}{2x}$ which simplifies to $\frac{1}{2}\int \sin(t)x^2dt$. We do not want to have any x left in our integral since we are now integrating with respect to t, but luckily $x^2 = t$ so we have $\frac{1}{2}\int \sin(t)tdt$ which can now be integrated by parts. First we set $u = t$ and $dv = \sin(t)$ and we get $du = dt$ and $v = -\cos(t)$

$$\frac{1}{2}\int \sin(t)tdt = \frac{1}{2}\left(-t\cos(t) - \int -\cos(t)dt\right) = \frac{1}{2}\left(-t\cos(t) + \sin(t)\right) + C.$$

  We started out with the variable x so we should answer with it,

$$\frac{1}{2}\left(-x^2\cos(x^2) + \sin(x^2)\right) + C.$$

- $\int \frac{x^5+1}{x(x+2)}dx$

  Since the degree of the numerator is greater than or equal to the degree of the denominator we must first use polynomial long division. We find that

$$\frac{x^5+1}{x(x+2)} = x^3 - 2x^2 + 4x - 8 + \frac{16x+1}{x(x+2)}.$$

  Then we do partial fraction decomposition on the last: $\frac{16x+1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$, which gives us $A = \frac{1}{2}$ and $B = \frac{31}{2}$. So in total,

$$\int \frac{x^5+1}{x(x+2)}dx = \int x^3 - 2x^2 + 4x - 8 + \frac{1}{2x} + \frac{31}{2(x+2)}dx.$$

  Solving this gives $x^4/4 - (2x^3)/3 + 2x^2 - 8x + (\ln(x))/2 + \frac{31}{2}\ln(x+2) + C$.

- $\int \sin(x)\cos(x)dx$

  Substitute $u = \sin(x)$, then we get $\int udu = u^2/2 + C = \sin^2(x)/2 + C$.

- $\int \frac{\sec^2(x)}{\tan^2(x) - \tan(x)}dx$

  The composition of functions here is $x \to \tan(x) \to \frac{1}{\tan^2(x) - \tan(x)}$. So we do the subsitution $u = \tan(x)$ and recall $du = \sec^2(x)dx$ so

$$\int \frac{\sec^2(x)}{\tan^2(x) - \tan(x)}dx = \int \frac{1}{u^2 - u}du$$

which is done by partial fractions $\dfrac{1}{u^2-u} = \dfrac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$ and so we get $A = -1$ and $B = 1$. Now we have $\int \dfrac{1}{u^2-u}du = \int \dfrac{-1}{u} + \dfrac{1}{u-1}du = -\ln(u) + \ln(u-1) + C$ and substituting back in gives us

$$\ln\left(\frac{\tan(x)-1}{\tan(x)}\right) + C.$$

- $\int \dfrac{e^x}{(e^x-1)(e^x+3)}dx$

  Do a substitution $u = e^x$ and we get $\int \dfrac{e^x}{(e^x-1)(e^x+3)}dx = \int \dfrac{du}{(u-1)(u+3)}dx$ which is a partial fraction problem. The answer is

  $$1/4(\ln(e^x-1) - \ln(e^x+3)) + C.$$

- $\int (\ln(x))^3\, dx$

  Do repeated integration by parts, starting with $u = (\ln(x))^3$ and $dv = dx$. This gives $du = \frac{3(\ln(x))^2}{x}dx$ and $v = x$. and so $\int (\ln(x))^3\, dx = x\,(\ln(x))^3 - \int 3\,(\ln(x))^2\, dx$. Repeating gives us

  $$x(\ln(x))^3 - 3x(\ln(x))^2 + 6x\ln(x) - 6x + C.$$

- $\int \sqrt{4-\sqrt{x}}dx$

  Do a substitution $u = 4-\sqrt{x}$, this gives us $du = -\frac{dx}{2\sqrt{x}}$, or rather $dx = -2\sqrt{x}du$ so our integral becomes $\int \sqrt{4-\sqrt{x}}dx = -2\int \sqrt{u}\sqrt{x}du$. As before we do not want to have an x in our integral, so we go back to our substitution rule and solve for $\sqrt{x}$, $u = 4-\sqrt{x}$ leads to $\sqrt{x} = (4-u)$. Now we have $-2\int \sqrt{u}\sqrt{x}du = -2\int \sqrt{u}(4-u)du = -2\int(4\sqrt{u} - u\sqrt{u})du$, where all we did is distribute. This can be solved using power rule:

  $$-2(4(2/3)u^{3/2} - (2/5)u^{5/2}) + C = -2(4(2/3)(4-\sqrt{x})^{3/2} - (2/5)(4-\sqrt{x})^{5/2}) + C$$

The End.