Please show **all** your work! Answers without supporting work will not be given credit.

Name:____

1. **Logistics**
   - Office Hours: MS 3957, Tuesday at 10a and Thursdays at 11a
   - SMC: MS 3974 Monday-Thursday 9a-3p (I'll be there Thursdays at 10a)

2. **Plug and Play**

   Using your differentiation skills, decide which of the following (possibly more than one) are solutions to the differential equation given:

   (a) $\frac{dy}{dt} = ty$

   i. $y = e^{2t}t$

   ii. $y = e^{t^2}$

   iii. $y = e^{\frac{t^2}{2}} + \pi$

   iv. $y = \pi e^{\frac{t^2}{2}}$ Only this one is a solution.

   (b) $f'(x) = f''(x) + x$

   i. $f(x) = x^2/2 + x$ This is a solution.

   ii. $f(x) = e^x$

   iii. $f(x) = 10$

   iv. $f(x) = e^x + 10 + x^2/2 + x$ This is a solution.

3. **Come Out and Play (Keep 'em Separated)**

   One nice class of differential equations are the separable ones, i.e. the ones where you can move all the y stuff to one side and all the t stuff to other side. For example, with $\frac{dy}{dt} = \frac{y}{t}$, we can separate this into $\frac{dy}{y} = \frac{dt}{t}$ by multiplying both sides by dt [1] and dividing both sides by y. If your differential equation is in Newton notation like $f'(x) = f(x)x$ it's useful to convert it to Leibniz notation $\frac{df}{dx} = f(x)x$ so you can separate it as $\frac{df}{f(x)} = xdx$. Once separated, integrate both sides.

   Use separation of variables to solve the following:

   (a) $\frac{dy}{dt} = \frac{y}{t}$

   Separating gives $\frac{dy}{y} = \frac{dt}{t}$ and so we integrate to get $\ln|y| = \ln|t| + C$ and then we isolate y to get $y = Bt$.

   (b) $\frac{df}{dx} = f(x)x$

   Separating gives $\frac{df}{f} = xdx$ and so we integrate to get $\ln|f| = x^2/2 + C$ and then we isolate f to get $f = Be^{x^2/2}$.

   (c) $\frac{dy}{dt} = \frac{t^2+e^t}{y}$

   Separating gives $ydy = (t^2 + e^t)dt$ and so we integrate to get $y^2/2 = t^3/3 + e^t + C$ and then we isolate y to get $y = \pm\sqrt{2(t^3/3 + e^t + C)}$.

[1] We won't be too worried about whether we can do this; for now, think of this as a mnemonic aid.

4. **My Chemical Reaction**

   Consider a chemical reaction involving three reactants X, Y and Z, combining to form a substance S. We will assume that the rate at which the molecules of S are formed is proportional to the product of the concentrations of X, Y and Z at time t. Let $x(t)$, $y(t)$, $z(t)$, and $s(t)$ be the concentrations of chemicals X, Y, Z, and S present at time t. Then we have the rate law

   $$\frac{ds}{dt} = kx(t)y(t)z(t).$$

   Assume that for every new molecule of S, one molecule of each X, Y, and Z is used, and so we have the relations $x(t) = x(0) - s(t)$, $y(t) = y(0) - s(t)$, and $z(t) = z(0) - s(t)$. Solve the resulting separable equation.

   In class we simplified to the case of two reactants and $x(0) = 1$ and $y(0) = 2$.

   Thus the differential equation is $\frac{ds}{dt} = k(1-s)(2-s)$ and so we separate variables to get $\frac{ds}{(1-s)(2-s)} = kdt$ and do partial fractions to get $\int \frac{1}{1-s} - \frac{1}{2-s} ds = kt + C$. Using logarithm rules and simplifying we get $s(t) = \dfrac{Be^{kt} - 2}{Be^{kt} - 1}$.

5. **Sherlock, Doctor Who, and You**

   In forensic medicine, determining the time of death of a victim can be critical in convicting the murderer. Mathematics can aid this process by using Newton's law of cooling. This law states that the rate at which the temperature, T, of a body changes in time, t, is proportional to the difference between the body's temperature, T, and the ambient temperature, A. Mathematically, this statement corresponds to the following differential equation: $\frac{dT}{dt} = k(A - T)$ where k is a positive constant proportionate to the thermal conductivity of the body. A large k means the body readily conducts heat and quickly adjusts to the ambient temperature. A small k means the body is well insulated and slowly adjusts to the ambient temperature.

   (a) Find the general solution to the differential equation.
   Separate variables to get $\frac{dT}{A-T} = kdt$ and from there we get $T(t) = A - Be^{-kt}$.

   (b) Mister Burns was found dead in a wheatfield at 10:04 AM. The wheatfield temperature was 65F and Mister Burns' body temperature was 70F. After 2 minutes, his body's temperature dropped to 69F. When did he die?
   Define $t = 0$ to be 10:04a where we measure time in minutes. We start by finding B, $T(0) = 70 = 65 - Be^0$ so $B = -5$. And thus $T(t) = 65 + 5e^{-kt}$. When $t = 2$ we have that $T(2) = 69 = 65 + 5e^{-k(2)}$ and solving this for k gives us $k = \ln(4/5)/(-2)$. Now we know k and B we can try to solve for when $T(t) = 98$, the temperature of a living person. We get $t = -17$ and so he died at 9:47 AM.

The End.