Homework 6 Solutions

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Section 11.4, Problem 1: For the boundary-value problem

$$\begin{aligned} & y'' = -(y')^2 - y + \log x, \\ & 1 \le x \le 2, \\ & y(1) = 0, \quad y(2) = \log 2, \end{aligned} \tag{1}$$

write the nonlinear system and formulas for Newton's method.

Solution: We divide $[1,2]$ into $N+1$ subintervals whose endpoints are $x_i = 1 + ih$, for $i = 0, 1, \ldots, N+1$, and consider the discretization of the boundary-value problem in (1):

$$y''(x_i) = -(y(x_i)')^2 - y(x_i) + \log x_i. \tag{2}$$

Replacing $y''(x_i)$ and $y'(x_i)$ by appropriate centered difference formulas, equation (2) becomes:

$$\begin{aligned} \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_i) \quad = \quad & -\left(\frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6} y'''(\eta_i) \right)^2 \\ & - y(x_i) + \log x_i, \end{aligned}$$

for some ξ_i and η_i in the interval (x_{i-1}, x_{i+1}).
The difference method results when the error terms are deleted and the boundary conditions are employed:

$$w_0 = 0, \quad w_{N+1} = \log 2,$$

and

$$-\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - \left(\frac{w_{i+1} - w_{i-1}}{2h} \right)^2 - w_i + \log x_i = 0, \tag{3}$$

for each $i = 1, 2, \ldots, N$. Multiplying (3) by h^2, we obtain

$$-w_{i+1} + 2w_i - w_{i-1} - \left(\frac{w_{i+1} - w_{i-1}}{2} \right)^2 - h^2 w_i + h^2 \log x_i = 0,$$

which can be written as:

$$-w_{i+1} + 2w_i - w_{i-1} - \left(\frac{w_{i+1} - w_{i-1}}{2} \right)^2 - h^2 w_i + h^2 \log x_i = 0,$$

or

$$-w_{i-1} + 2w_i - w_{i+1} - \frac{1}{4} \big(w_{i-1}^2 - 2w_{i-1} w_{i+1} + w_{i+1}^2 \big) - h^2 w_i + h^2 \log x_i = 0.$$

Thus, the $N \times N$ nonlinear system is:

$$
\begin{aligned}
-0 + 2w_1 - w_2 - \frac{1}{4}\big(0^2 - 2 \cdot 0 \cdot w_2 + w_2^2\big) - h^2 w_1 + h^2 \log x_1 &= 0, \\
-w_1 + 2w_2 - w_3 - \frac{1}{4}\big(w_1^2 - 2w_1 w_3 + w_3^2\big) - h^2 w_2 + h^2 \log x_2 &= 0, \\
-w_2 + 2w_3 - w_4 - \frac{1}{4}\big(w_2^2 - 2w_2 w_4 + w_4^2\big) - h^2 w_3 + h^2 \log x_3 &= 0, \\
&\vdots \quad \vdots \\
-w_{N-2} + 2w_{N-1} - w_N - \frac{1}{4}\big(w_{N-2}^2 - 2w_{N-2} w_N + w_N^2\big) - h^2 w_{N-1} + h^2 \log x_{N-1} &= 0, \\
-w_{N-1} + 2w_N - \log 2 - \frac{1}{4}\big(w_{N-1}^2 - 2w_{N-1} \log 2 + (\log 2)^2\big) - h^2 w_N + h^2 \log x_N &= 0,
\end{aligned}
$$

where we designate the left-hand side of the first equation as $F_1(w_1, \ldots, w_N)$, the second equation as $F_2(w_1, \ldots, w_N)$, ..., the last equation as $F_N(w_1, \ldots, w_N)$. Also, we designate $\vec{F} = (F_1, \ldots, F_N)^T$ and $\vec{w} = (w_1, \ldots, w_N)^T$.

We use Newton's method for nonlinear systems to approximate the solution to the system $\vec{F}(\vec{w}) = 0$ above. A sequence of iterates $\vec{w}^{(k)} = (w_1^{(k)}, w_2^{(k)}, \ldots, w_N^{(k)})^T$ is generated that converges to the solution of this system. The Jacobian matrix J for this system is

$$
J(w_1, \ldots, w_N) = \begin{bmatrix}
\frac{\partial F_1}{\partial w_1} & \frac{\partial F_1}{\partial w_2} & \frac{\partial F_1}{\partial w_3} & \cdots & \frac{\partial F_1}{\partial w_N} \\
\frac{\partial F_2}{\partial w_1} & \frac{\partial F_2}{\partial w_2} & \frac{\partial F_2}{\partial w_3} & \cdots & \frac{\partial F_2}{\partial w_N} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial F_{N-1}}{\partial w_1} & \frac{\partial F_{N-1}}{\partial w_2} & \frac{\partial F_{N-1}}{\partial w_3} & \cdots & \frac{\partial F_{N-1}}{\partial w_N} \\
\frac{\partial F_N}{\partial w_1} & \frac{\partial F_N}{\partial w_2} & \frac{\partial F_N}{\partial w_3} & \cdots & \frac{\partial F_N}{\partial w_N}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
2 - h^2 & -1 + \frac{1}{2} \cdot 0 - \frac{1}{2} w_2 & 0 & \cdots & 0 \\
-1 - \frac{1}{2} w_1 + \frac{1}{2} w_3 & 2 - h^2 & -1 + \frac{1}{2} w_1 - \frac{1}{2} w_3 & \ddots & \vdots \\
0 & & & & \vdots \\
\vdots & & \ddots & & \vdots \\
\vdots & & & & 0 \\
\vdots & \ddots & -1 - \frac{1}{2} w_{N-2} + \frac{1}{2} w_N & 2 - h^2 & -1 + \frac{1}{2} w_{N-2} - \frac{1}{2} w_N \\
0 & 0 & 0 & \cdots & 2 - h^2
\end{bmatrix}.
$$

We can now use the Netwon's method for nonlinear systems

$$
\vec{w}^{(k)} = \vec{w}^{(k-1)} - J^{-1}\big(\vec{w}^{(k-1)}\big) \vec{F}\big(\vec{w}^{(k-1)}\big).
$$

Section 7.1, Problem 1: Find $||\mathbf{x}||_\infty$ and $||\mathbf{x}||_2$ for the following vectors:
a) $\mathbf{x} = (3, -4, 0, \frac{3}{2})^T$;
c) $\mathbf{x} = (\sin k, \cos k, 2^k)^T$ for a fixed positive integer k.

Solution: The L_∞ and L_2 norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ are defined by

$$\begin{aligned} ||\mathbf{x}||_\infty &= \max_{1 \le i \le n} |x_i|, \\ ||\mathbf{x}||_2 &= \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

a) For $\mathbf{x} = (3, -4, 0, \frac{3}{2})^T$:

$$\begin{aligned} ||\mathbf{x}||_\infty &= \max\left\{ |3|, |-4|, |0|, \left|\frac{3}{2}\right| \right\} = 4, \\ ||\mathbf{x}||_2 &= \sqrt{3^2 + (-4)^2 + 0^2 + \left(\frac{3}{2}\right)^2} = 5.22015325. \end{aligned}$$

c) For $\mathbf{x} = (\sin k, \cos k, 2^k)^T$, k is a positive integer :

$$\begin{aligned} ||\mathbf{x}||_\infty &= \max\left\{ |\sin k|, |\cos k|, |2^k| \right\} = 2^k, \\ ||\mathbf{x}||_2 &= \sqrt{\sin^2 k + \cos^2 k + (2^k)^2} = \sqrt{1 + 4^k}. \end{aligned}$$

Section 7.1, Problem 2(a): Verify that the function $||\cdot||_1$, defined on $\mathbb{R}^n$ by

$$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i|,$$

is a norm on $\mathbb{R}^n$.

Solution:
(i) For all $\mathbf{x} \in \mathbb{R}^n$,

$$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i| \ge 0. \quad \checkmark$$

(ii) If $\mathbf{x} = \mathbf{0}$, then

$$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} 0 = 0. \quad \checkmark$$

If $||\mathbf{x}||_1 = 0$, we have $\sum_{i=1}^{n} |x_i| = 0$, and thus, $\mathbf{x} = \mathbf{0}$. ✓
(iii) For all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

$$||\alpha\mathbf{x}||_1 = \sum_{i=1}^{n} |\alpha x_i| = \sum_{i=1}^{n} |\alpha||x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha|\, ||\mathbf{x}||_1. \quad \checkmark$$

(iii) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$||\mathbf{x} + \mathbf{y}||_1 = \sum_{i=1}^{n} |x_i + y_i| \le \sum_{i=1}^{n} \big(|x_i| + |y_i|\big) = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = ||\mathbf{x}||_1 + ||\mathbf{y}||_1. \quad \checkmark$$

Thus, $||\cdot||_1$ is a norm on $\mathbb{R}^n$.

Section 7.1, Problem 2(c): Prove that for all $\mathbf{x} \in \mathbb{R}^n$, $||\mathbf{x}||_1 \geq ||\mathbf{x}||_2$.

Solution: Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, and note that

$$\big(|x_1| + |x_2| + \ldots + |x_n|\big)^2 \quad \geq \quad x_1^2 + x_2^2 + \ldots + x_n^2,$$

or

$$\left(\sum_{i=1}^{n} |x_i|\right)^2 \geq \sum_{i=1}^{n} x_i^2,$$

or

$$\sum_{i=1}^{n} |x_i| \geq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}},$$

which means that $||\mathbf{x}||_1 \geq ||\mathbf{x}||_2$.

Section 7.1, Problem 4(c): Find $||\cdot||_\infty$ for the following matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Solution: We have

$$||A||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$

Since

$$\begin{aligned} \sum_{j=1}^{n} |a_{1j}| &= |a_{11}| + |a_{12}| + |a_{13}| = |2| + |-1| + |0| = 3, \\ \sum_{j=1}^{n} |a_{2j}| &= |a_{21}| + |a_{22}| + |a_{23}| = |-1| + |2| + |-1| = 4, \\ \sum_{j=1}^{n} |a_{3j}| &= |a_{31}| + |a_{32}| + |a_{33}| = |0| + |-1| + |2| = 3, \end{aligned}$$

we have $||A||_\infty = \max\{3, 4, 3\} = 4$.

Section 7.1, Problem 7: Show by example that $||\cdot||_{\circledast}$, defined by $||A||_{\circledast} = \max_{1 \leq i,j \leq n} |a_{ij}|$, does not define a matrix norm.

Solution: A function $||\cdot||_{\circledast}$ is a matrix norm only if it satisfies definition 7.8 on page 424. Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then, $AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. We have $||A||_{\circledast} = 1$, $||B||_{\circledast} = 1$, and $||AB||_{\circledast} = 2$, and thus, $||AB||_{\circledast} \geq ||A||_{\circledast}||B||_{\circledast}$, which contradicts one of the conditions for being a norm.

Section 7.1, Problem 9(a): The Frobenius norm (which is not a natural norm) is defined for an $n \times n$ matrix A by

$$||A||_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Show that $|| \cdot ||_F$ is a matrix norm.

Solution: For all $n \times n$ matrices A and B and all real numbers α, we have:

(i)

$$||A||_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \geq 0. \quad \checkmark$$

(ii)

$$||A||_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} = 0 \text{ if and only if } A \text{ is a 0 matrix.} \quad \checkmark$$

(iii)

$$\begin{aligned} ||\alpha A||_F^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha\, a_{ij}|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha|^2\, |a_{ij}|^2 = |\alpha|^2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 = |\alpha|^2 ||A||_F^2. \\ &\Rightarrow \; ||\alpha A||_F = |\alpha|\, ||A||_F. \quad \checkmark \end{aligned}$$

(iv) Here, we will use Cauchy-Schwarz Inequality: $\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2 \right)^{\frac{1}{2}}$.

$$\begin{aligned} ||A+B||_F^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} + b_{ij}|^2 \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|)^2 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \Big(|a_{ij}|^2 + 2|a_{ij}||b_{ij}| + |b_{ij}|^2 \Big) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}||b_{ij}| + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 \leq \qquad \text{(Cauchy-Schwarz)} \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 + 2 \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 \Big)^{\frac{1}{2}} + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 \\ &= \left(\Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \Big)^{\frac{1}{2}} + \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 \Big)^{\frac{1}{2}} \right)^2 \\ &= \big(||A||_F + ||B||_F \big)^2. \\ &\Rightarrow \; ||A+B||_F \leq ||A||_F + ||B||_F. \quad \checkmark \end{aligned}$$

(v) Note that

$$AB = \begin{bmatrix} \sum_{k=1}^{n} a_{1k} b_{k1} & \sum_{k=1}^{n} a_{1k} b_{k2} & \cdots & \cdots & \sum_{k=1}^{n} a_{1k} b_{kn} \\ \sum_{k=1}^{n} a_{2k} b_{k1} & \sum_{k=1}^{n} a_{2k} b_{k2} & \cdots & \cdots & \sum_{k=1}^{n} a_{2k} b_{kn} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \sum_{k=1}^{n} a_{nk} b_{k1} & \sum_{k=1}^{n} a_{nk} b_{k2} & \cdots & \sum_{k=1}^{n} a_{nk} b_{k,n-1} & \sum_{k=1}^{n} a_{nk} b_{kn} \end{bmatrix}.$$

$$
\begin{aligned}
||AB||_F^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}\Big|\sum_{k=1}^{n} a_{ik}b_{kj}\Big|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}|a_{ik}b_{kj}|\right)^2 \leq \qquad \text{(Cauchy-Schwarz)} \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}|a_{ik}|^2\sum_{k=1}^{n}|b_{kj}|^2\right)... \\
&\leq \Big(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\Big)\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2\Big) = ||A||_F^2||B||_F^2. \\
&\Rightarrow ||AB||_F \leq ||A||_F||B||_F. \quad \checkmark
\end{aligned}
$$

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Section 7.1, Problem 9(c): For any matrix A, show that $||A||_2 \le ||A||_F \le n^{1/2}||A||_2$.

Solution: The definitions of $||\cdot||_F$ and $||\cdot||_2$ norms are:

$$||A||_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

$$||A||_2 = \max_{||\mathbf{x}||_2=1} ||A\mathbf{x}||_2.$$

Note, that $A\mathbf{x}$ is a vector:

$$A\mathbf{x} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj}x_j \end{bmatrix}.$$

Thus, we have

$$||A\mathbf{x}||_2 = \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2\right)^{\frac{1}{2}}.$$

❶ We first show that $||A||_2 \le ||A||_F$.
For vector $\mathbf{x}$, such that $||\mathbf{x}||_2 = 1$, we have

$$\begin{aligned}
||A\mathbf{x}||_2^2 &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2 \le \qquad \text{(Cauchy-Schwarz)} \\
&\le \sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}}\right)^2 \\
&= \sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n} a_{ij}^2\right)\left(\sum_{j=1}^{n} x_j^2\right)\right) \\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)\cdot ||\mathbf{x}||_2^2 \\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)\cdot 1 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 \\
&= ||A||_F^2.
\end{aligned}$$

We showed that, $||A\mathbf{x}||_2 \le ||A||_F$ for all $\mathbf{x}$, such that $||\mathbf{x}||_2 = 1$.
Thus, $\max_{||\mathbf{x}||_2=1} ||A\mathbf{x}||_2 \le ||A||_F$, or $||A||_2 \le ||A||_F$.

❷ We now show that $||A||_F \le n^{1/2}||A||_2$.
Let $x_i = \dfrac{1}{\sqrt{n}}$ for all $1 \le i \le n$. Then,

$$||A||_2 = \max_{||\mathbf{x}||_2=1} ||A\mathbf{x}||_2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = \frac{1}{n}||A||_F.$$

Thus, $||A||_F \le n^{1/2}||A||_2$.

Section 7.3, Problem 2(c): Find the first two iterations of the Jacobi method for the following linear system, using $\mathbf{x}^{(0)} = \mathbf{0}$:

$$\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2, \\ x_1 + 4x_2 - x_3 - x_4 &= -1, \\ -x_1 - x_2 + 5x_3 + x_4 &= 0, \\ x_1 - x_2 + x_3 + 3x_4 &= 1. \end{aligned}$$

Solution: The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{array}{lrcl} E_1: & 4x_1 + x_2 - x_3 + x_4 &=& -2, \\ E_2: & x_1 + 4x_2 - x_3 - x_4 &=& -1, \\ E_3: & -x_1 - x_2 + 5x_3 + x_4 &=& 0, \\ E_4: & x_1 - x_2 + x_3 + 3x_4 &=& 1 \end{array}$$

has the unique solution $\mathbf{x} = (-0.75342, 0.041096, -0.28082, 0.69178)$.
To convert $A\mathbf{x} = \mathbf{b}$ to the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, solve equation E_1 for x_1, E_2 for x_2, E_3 for x_3, E_4 for x_4, to obtain

$$\begin{array}{lclllll} x_1 &=& & -\frac{1}{4}x_2 & +\frac{1}{4}x_3 & -\frac{1}{4}x_4 & -\frac{1}{2}, \\ x_2 &=& -\frac{1}{4}x_1 & & +\frac{1}{4}x_3 & +\frac{1}{4}x_4 & -\frac{1}{4}, \\ x_3 &=& \frac{1}{5}x_1 & +\frac{1}{5}x_2 & & -\frac{1}{5}x_4, & \\ x_4 &=& -\frac{1}{3}x_1 & +\frac{1}{3}x_2 & -\frac{1}{3}x_3 & & +\frac{1}{3}. \end{array}$$

Then $A\mathbf{x} = \mathbf{b}$ can be written in the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, with

$$T = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{5} & 0 & -\frac{1}{5} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 0 \\ \frac{1}{3} \end{bmatrix}.$$

For initial approximation, we let $\mathbf{x}^{(0)} = (0, 0, 0, 0)^T$. Then $\mathbf{x}^{(1)}$ is given by

$$\begin{array}{lclllllcl} x_1^{(1)} &=& & -\frac{1}{4}x_2^{(0)} & +\frac{1}{4}x_3^{(0)} & -\frac{1}{4}x_4^{(0)} & -\frac{1}{2} &=& -0.5, \\ x_2^{(1)} &=& -\frac{1}{4}x_1^{(0)} & & +\frac{1}{4}x_3^{(0)} & +\frac{1}{4}x_4^{(0)} & -\frac{1}{4} &=& -0.25, \\ x_3^{(1)} &=& \frac{1}{5}x_1^{(0)} & +\frac{1}{5}x_2^{(0)} & & -\frac{1}{5}x_4^{(0)} & &=& 0, \\ x_4^{(1)} &=& -\frac{1}{3}x_1^{(0)} & +\frac{1}{3}x_2^{(0)} & -\frac{1}{3}x_3^{(0)} & & +\frac{1}{3} &=& 1/3. \end{array}$$

The next iterate, $\mathbf{x}^{(2)}$, is given by

$$\begin{array}{lclllllcl} x_1^{(2)} &=& & -\frac{1}{4}x_2^{(1)} & +\frac{1}{4}x_3^{(1)} & -\frac{1}{4}x_4^{(1)} & -\frac{1}{2} &=& -0.52083, \\ x_2^{(2)} &=& -\frac{1}{4}x_1^{(1)} & & +\frac{1}{4}x_3^{(1)} & +\frac{1}{4}x_4^{(1)} & -\frac{1}{4} &=& -0.041667, \\ x_3^{(2)} &=& \frac{1}{5}x_1^{(1)} & +\frac{1}{5}x_2^{(1)} & & -\frac{1}{5}x_4^{(1)} & &=& -0.21667, \\ x_4^{(2)} &=& -\frac{1}{3}x_1^{(1)} & +\frac{1}{3}x_2^{(1)} & -\frac{1}{3}x_3^{(1)} & & +\frac{1}{3} &=& 0.41667. \end{array}$$

Section 7.3, Problem 4(c): Find the first two iterations of the Gauss-Seidel method for the following linear system, using $\mathbf{x}^{(0)} = \mathbf{0}$:

$$\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2, \\ x_1 + 4x_2 - x_3 - x_4 &= -1, \\ -x_1 - x_2 + 5x_3 + x_4 &= 0, \\ x_1 - x_2 + x_3 + 3x_4 &= 1. \end{aligned}$$

Solution: In section 7.3, Problem 2(c), we used Jacobi method to solve the linear system above. The following equations were used:

$$\begin{array}{lcllllll} x_1^{(k)} &=& & -\frac{1}{4}x_2^{(k-1)} & +\frac{1}{4}x_3^{(k-1)} & -\frac{1}{4}x_4^{(k-1)} & -\frac{1}{2}, \\ x_2^{(k)} &=& -\frac{1}{4}x_1^{(k-1)} & & +\frac{1}{4}x_3^{(k-1)} & +\frac{1}{4}x_4^{(k-1)} & -\frac{1}{4}, \\ x_3^{(k)} &=& \frac{1}{5}x_1^{(k-1)} & +\frac{1}{5}x_2^{(k-1)} & & -\frac{1}{5}x_4^{(k-1)} & , \\ x_4^{(k)} &=& -\frac{1}{3}x_1^{(k-1)} & +\frac{1}{3}x_2^{(k-1)} & -\frac{1}{3}x_3^{(k-1)} & & +\frac{1}{3}. \end{array}$$

However, since for $i > 1$, $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ have already been computed, these are probably better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$. Hence, Gauss-Seidel uses the most recently available approximations to $x_1, \ldots, x_{i-1}$ in a calculation of the next iterate:

$$\begin{array}{lcllllll} x_1^{(k)} &=& & -\frac{1}{4}x_2^{(k-1)} & +\frac{1}{4}x_3^{(k-1)} & -\frac{1}{4}x_4^{(k-1)} & -\frac{1}{2}, \\ x_2^{(k)} &=& -\frac{1}{4}x_1^{(k)} & & +\frac{1}{4}x_3^{(k-1)} & +\frac{1}{4}x_4^{(k-1)} & -\frac{1}{4}, \\ x_3^{(k)} &=& \frac{1}{5}x_1^{(k)} & +\frac{1}{5}x_2^{(k)} & & -\frac{1}{5}x_4^{(k-1)} & , \\ x_4^{(k)} &=& -\frac{1}{3}x_1^{(k)} & +\frac{1}{3}x_2^{(k)} & -\frac{1}{3}x_3^{(k)} & & +\frac{1}{3}. \end{array}$$

For initial approximation, we let $\mathbf{x}^{(0)} = (0, 0, 0, 0)^T$. Then $\mathbf{x}^{(1)}$ is given by

$$\begin{array}{lcllllllr} x_1^{(1)} &=& & -\frac{1}{4}x_2^{(0)} & +\frac{1}{4}x_3^{(0)} & -\frac{1}{4}x_4^{(0)} & -\frac{1}{2} &=& -0.5, \\ x_2^{(1)} &=& -\frac{1}{4}x_1^{(1)} & & +\frac{1}{4}x_3^{(0)} & +\frac{1}{4}x_4^{(0)} & -\frac{1}{4} &=& -0.125, \\ x_3^{(1)} &=& \frac{1}{5}x_1^{(1)} & +\frac{1}{5}x_2^{(1)} & & -\frac{1}{5}x_4^{(0)} & &=& -0.125, \\ x_4^{(1)} &=& -\frac{1}{3}x_1^{(1)} & +\frac{1}{3}x_2^{(1)} & -\frac{1}{3}x_3^{(1)} & & +\frac{1}{3} &=& 0.5. \end{array}$$

The next iterate, $\mathbf{x}^{(2)}$, is given by

$$\begin{array}{lcllllllr} x_1^{(2)} &=& & -\frac{1}{4}x_2^{(1)} & +\frac{1}{4}x_3^{(1)} & -\frac{1}{4}x_4^{(1)} & -\frac{1}{2} &=& -0.625, \\ x_2^{(2)} &=& -\frac{1}{4}x_1^{(2)} & & +\frac{1}{4}x_3^{(1)} & +\frac{1}{4}x_4^{(1)} & -\frac{1}{4} &=& 0, \\ x_3^{(2)} &=& \frac{1}{5}x_1^{(2)} & +\frac{1}{5}x_2^{(2)} & & -\frac{1}{5}x_4^{(1)} & &=& -0.225, \\ x_4^{(2)} &=& -\frac{1}{3}x_1^{(2)} & +\frac{1}{3}x_2^{(2)} & -\frac{1}{3}x_3^{(2)} & & +\frac{1}{3} &=& 0.61667. \end{array}$$

Comparing $\mathbf{x}^{(2)}$ to the exact solution $\mathbf{x} = (-0.75342, 0.041096, -0.28082, 0.69178)$, we see that Gauss-Seidel method gave more accurate results than Jacobi method.