Homework 5 Solutions

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Section 5.10, Problem 7: Investigate stability for the difference method

$$w_{i+1} = -4w_i + 5w_{i-1} + 2h\Big[f(t_i, w_i) + 2hf(t_{i-1}, w_{i-1})\Big],$$

for $i = 1, 2, \ldots, N-1$, with starting values w_0, w_1.

Solution: For a multistep method to be stable, it has to satisfy the root condition. A multistep method is said to satisfy the root condition if all roots λ_i of the characteristic polynomial $P(\lambda)$ are such that $|\lambda_i| \leq 1$, and if $|\lambda_i| = 1$, then λ_i is simple.
For the multistep method

$$w_{i+1} + 4w_i - 5w_{i-1} = 2h\Big[f(t_i, w_i) + 2hf(t_{i-1}, w_{i-1})\Big],$$

the characteristic polynomial is

$$P(\lambda) = -5 + 4\lambda + \lambda^2,$$

which has roots

$$\lambda_1 = 1, \ \lambda_2 = -5.$$

Since $|\lambda_2| > 1$, this multistep method does not satisfy the root condition, and therefore is unstable. ✓

Section 11.3, Problem 3(a): Write the discretization of the following boundary value problem

$$\begin{aligned} & y'' = -3y' + 2y + 2x + 3, \\ & 0 \leq x \leq 1, \\ & y(0) = 2, \quad y(1) = 1, \end{aligned}$$

in matrix-vector notation $A\mathbf{w} = \mathbf{b}$.

Solution: At the interior points x_i, for $i = 1, 2, \ldots, N$, the differential equation to be approximated is

$$y''(x_i) = -3y'(x_i) + 2y(x_i) + 2x_i + 3. \tag{1}$$

Since

$$\begin{aligned} y''(x_i) &= \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_i), \\ y'(x_i) &= \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6} y'''(\eta_i), \end{aligned}$$

we can write the numerical approximation to (1) as

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + 3\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) - 2w_i = 2x_i + 3. \tag{2}$$

Multiplying both sides of (2) by $-h^2$ gives

$$-(w_{i+1} - 2w_i + w_{i-1}) - \frac{3h}{2}(w_{i+1} - w_{i-1}) + 2h^2 w_i = -h^2(2x_i + 3).$$

Collecting w_{i-1}, w_i, and w_{i+1} terms, we obtain

$$-\left(1 - \frac{3h}{2}\right)w_{i-1} + (2 + 2h^2)w_i - \left(1 + \frac{3h}{2}\right)w_{i+1} = -h^2(2x_i + 3). \tag{3}$$

The resulting system of equations can be expressed in the tridiagonal $N \times N$ matrix form

$$A\mathbf{w} = \mathbf{b}, \qquad \text{where}$$

$$A = \begin{bmatrix} 2+2h^2 & -1-\frac{3h}{2} & 0 & \cdots & 0 \\ -1+\frac{3h}{2} & 2+2h^2 & -1-\frac{3h}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1-\frac{3h}{2} \\ 0 & \cdots & 0 & -1+\frac{3h}{2} & 2+2h^2 \end{bmatrix},$$

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -h^2(2x_1+3) + \left(1 - \frac{3h}{2}\right)w_0 \\ -h^2(2x_2+3) \\ \vdots \\ -h^2(2x_{N-1}+3) \\ -h^2(2x_N+3) + \left(1 + \frac{3h}{2}\right)w_{N+1} \end{bmatrix}. \quad \checkmark$$

In order to see that this system satisfies (3), look at a couple of rows of matrix A, for example, the second row:

$$-\left(1 - \frac{3h}{2}\right)w_1 + (2 + 2h^2)w_2 - \left(1 + \frac{3h}{2}\right)w_3 = -h^2(2x_2 + 3).$$

Also, first and last elements of $\mathbf{b}$ might be a little daunting. However, if we look at the first row (for example), we see that

$$(2 + 2h^2)w_1 - \left(1 + \frac{3h}{2}\right)w_2 = -h^2(2x_1 + 3) + \left(1 - \frac{3h}{2}\right)w_0,$$

or

$$-\left(1 - \frac{3h}{2}\right)w_0 + (2 + 2h^2)w_1 - \left(1 + \frac{3h}{2}\right)w_2 = -h^2(2x_1 + 3),$$

which satisfies equation (3).

Also, note that the book considers the general second-order boundary value problem:

$$y''(x) = p(x)y'(x) + q(x)y(x) + r(x).$$

For our problem, $p(x) = -3$, $q(x) = 2$, and $r(x) = 2x + 3$. Plugging these values into the formulas in the book, we can verify whether our calculations are correct.

Section 11.3, Problem 3(c): Write the discretization of the following boundary value problem

$$
\begin{aligned}
&y'' = -(x+1)y' + 2y + (1-x^2)e^{-x},\\
&0 \le x \le 1,\\
&y(0) = -1, \quad y(1) = 0,
\end{aligned}
$$

in matrix-vector notation $A\mathbf{w} = \mathbf{b}$.

Solution: At the interior points x_i, for $i = 1, 2, \ldots, N$, the differential equation to be approximated is

$$
y''(x_i) = -(x_i+1)y'(x_i) + 2y(x_i) + (1-x_i^2)e^{-x_i}. \tag{4}
$$

Since

$$
\begin{aligned}
y''(x_i) &= \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12}y^{(4)}(\xi_i),\\
y'(x_i) &= \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta_i),
\end{aligned}
$$

we can write the numerical approximation to (4) as

$$
\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + (x_i+1)\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) - 2w_i = (1-x_i^2)e^{-x_i}. \tag{5}
$$

Multiplying both sides of (5) by $-h^2$ gives

$$
-(w_{i+1} - 2w_i + w_{i-1}) - \frac{(x_i+1)h}{2}(w_{i+1} - w_{i-1}) + 2h^2 w_i = -h^2(1-x_i^2)e^{-x_i}.
$$

Collecting w_{i-1}, w_i, and w_{i+1} terms, we obtain

$$
-\left(1 - \frac{(x_i+1)h}{2}\right)w_{i-1} + (2+2h^2)w_i - \left(1 + \frac{(x_i+1)h}{2}\right)w_{i+1} = -h^2(1-x_i^2)e^{-x_i}. \tag{6}
$$

The resulting system of equations can be expressed in the tridiagonal $N \times N$ matrix form

$$
A\mathbf{w} = \mathbf{b}, \qquad \text{where}
$$

$$
A = \begin{bmatrix}
2+2h^2 & -1 - \frac{(x_1+1)h}{2} & 0 & \cdots & 0\\
-1 + \frac{(x_2+1)h}{2} & 2+2h^2 & -1 - \frac{(x_2+1)h}{2} & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & -1 - \frac{(x_{N-1}+1)h}{2}\\
0 & \cdots & 0 & -1 + \frac{(x_N+1)h}{2} & 2+2h^2
\end{bmatrix},
$$

$$
\mathbf{w} = \begin{bmatrix} w_1\\ w_2\\ \vdots\\ w_{N-1}\\ w_N \end{bmatrix}, \quad \text{and} \quad
\mathbf{b} = \begin{bmatrix}
-h^2(1-x_1^2)e^{-x_1} + \left(1 - \frac{(x_1+1)h}{2}\right)w_0\\
-h^2(1-x_2^2)e^{-x_2}\\
\vdots\\
-h^2(1-x_{N-1}^2)e^{-x_{N-1}}\\
-h^2(1-x_N^2)e^{-x_N} + \left(1 + \frac{(x_N+1)h}{2}\right)w_{N+1}
\end{bmatrix}. \quad \checkmark
$$

In order to see that this system satisfies (6), look at a couple of rows of matrix A, for example, the second row:

$$-\left(1-\frac{(x_2+1)h}{2}\right)w_1+(2+2h^2)w_2-\left(1+\frac{(x_2+1)h}{2}\right)w_3=-h^2(1-x_2^2)e^{-x_2}.$$

Also, first and last elements of $\mathbf{b}$ might be a little daunting. However, if we look at the first row (for example), we see that

$$(2+2h^2)w_1-\left(1+\frac{(x_1+1)h}{2}\right)w_2=-h^2(1-x_1^2)e^{-x_1}+\left(1-\frac{(x_1+1)h}{2}\right)w_0,$$

or

$$-\left(1-\frac{(x_1+1)h}{2}\right)w_0+(2+2h^2)w_1-\left(1+\frac{(x_1+1)h}{2}\right)w_2=-h^2(1-x_1^2)e^{-x_1},$$

which satisfies equation (6).

Also, note that the book considers the general second-order boundary value problem:

$$y''(x)=p(x)y'(x)+q(x)y(x)+r(x).$$

For our problem, $p(x)=-(x+1)$, $q(x)=2$, and $r(x)=(1-x^2)e^{-x}$. Plugging these values into the formulas in the book, we can verify whether our calculations are correct.