Final Review Problems

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These sample review problems do not necessarily represent the content, length, or depth of the material you will be tested on.

Among other things, it is also a good idea to go over the homework sets.

1 Fourier Series

Problem: Find the general continuous least squares trigonometric polynomial $S_n(x)$ for

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \leq 0, \\ 1, & \text{if } 0 < x < \pi. \end{cases}$$

Solution: The continuous least squares approximation $S_n(x)$ is in the form

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

where, for $k = 1, 2, \ldots$,

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx, \\
&= \frac{1}{\pi} \int_{-\pi}^{0} 0\, dx + \frac{1}{\pi} \int_{0}^{\pi} 1\, dx = 1, \quad \checkmark \\
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx, \\
&= \frac{1}{\pi} \int_{-\pi}^{0} 0 \cdot \cos kx\, dx + \frac{1}{\pi} \int_{0}^{\pi} 1 \cdot \cos kx\, dx, \\
&= \frac{1}{\pi} \frac{\sin kx}{k} \Big|_{x=0}^{x=\pi} = 0, \quad \checkmark \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{0} 0 \cdot \sin kx\, dx + \frac{1}{\pi} \int_{0}^{\pi} 1 \cdot \sin kx\, dx, \\
&= -\frac{1}{\pi} \frac{\cos kx}{k} \Big|_{x=0}^{x=\pi} = -\frac{1}{\pi} \left(\frac{(-1)^k - 1}{k} \right) = \frac{1}{\pi} \left(\frac{1 - (-1)^k}{k} \right). \quad \checkmark
\end{aligned}$$

Thus,

$$S_n(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{n} \left(\frac{1 - (-1)^k}{k} \right) \sin kx. \quad \checkmark$$

2 QR decomposition and Least Squares

The QR decomposition (also called the QR factorization) of a matrix is a decomposition of the matrix into an orthogonal matrix and a triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$A = QR,$$

where Q is an orthogonal matrix (i.e. $Q^T Q = I$) and R is an upper triangular matrix. If A is nonsingular, then this factorization is unique.

There are several methods for actually computing the QR decomposition. One of such method is the Gram-Schmidt process.

2.1 Gram-Schmidt process

Consider the GramSchmidt procedure, with the vectors to be considered in the process as columns of the 3x3 matrix A. That is,

$$A = \left[\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n\ \right].$$

Then,

$$\begin{aligned}
\tilde{\mathbf{q}}_1 &= \mathbf{a}_1, \qquad \mathbf{q}_1 = \frac{\tilde{\mathbf{q}}_1}{||\tilde{\mathbf{q}}_1||_2}, \\
\tilde{\mathbf{q}}_2 &= \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2)\mathbf{q}_1, \qquad \mathbf{q}_2 = \frac{\tilde{\mathbf{q}}_2}{||\tilde{\mathbf{q}}_2||_2}, \\
\tilde{\mathbf{q}}_3 &= \mathbf{a}_3 - (\mathbf{q}_1^T \mathbf{a}_3)\mathbf{q}_1 - (\mathbf{q}_2^T \mathbf{a}_3)\mathbf{q}_2, \qquad \mathbf{q}_3 = \frac{\tilde{\mathbf{q}}_3}{||\tilde{\mathbf{q}}_3||_2}.
\end{aligned}$$

2.2 QR Factorization

The resulting QR factorization is

$$A = \left[\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3\ \right] = \left[\ \mathbf{q}_1 \mid \mathbf{q}_2 \mid \mathbf{q}_3\ \right] \begin{bmatrix} ||\tilde{\mathbf{q}}_1||_2 & \mathbf{q}_1^T a_2 & \mathbf{q}_1^T \mathbf{a}_3 \\ 0 & ||\tilde{\mathbf{q}}_2||_2 & \mathbf{q}_2^T \mathbf{a}_3 \\ 0 & 0 & ||\tilde{\mathbf{q}}_3||_2 \end{bmatrix} = QR.$$

2.3 Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

with the vectors $\mathbf{a}_1 = (1,1,0)^T,\ \mathbf{a}_2 = (1,0,1)^T,\ \mathbf{a}_3 = (0,1,1)^T$.
Note that all the vectors considered above and below are column vectors. From now on, I will drop ^{T} notation for simplicity, but we have to remember that all the vectors are column vectors.

Performing the Gram-Schmidt procedure, we obtain:

$$
\begin{aligned}
\tilde{\mathbf{q}}_1 &= \mathbf{a}_1 = (1,1,0), \\
\mathbf{q}_1 &= \frac{\tilde{\mathbf{q}}_1}{||\tilde{\mathbf{q}}_1||_2} = \frac{1}{\sqrt{2}}(1,1,0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \\
\tilde{\mathbf{q}}_2 &= \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2)\mathbf{q}_1 = (1,0,1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \left(\frac{1}{2}, -\frac{1}{2}, 1\right), \\
\mathbf{q}_2 &= \frac{\tilde{\mathbf{q}}_2}{||\tilde{\mathbf{q}}_2||_2} = \frac{1}{\sqrt{3/2}}\left(\frac{1}{2}, -\frac{1}{2}, 1\right) = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \\
\tilde{\mathbf{q}}_3 &= \mathbf{a}_3 - (\mathbf{q}_1^T \mathbf{a}_3)\mathbf{q}_1 - (\mathbf{q}_2^T \mathbf{a}_3)\mathbf{q}_2 \\
&= (0,1,1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \\
\mathbf{q}_3 &= \frac{\tilde{\mathbf{q}}_3}{||\tilde{\mathbf{q}}_3||_2} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Q &= \left[\ \mathbf{q}_1 \mid \mathbf{q}_2 \mid \mathbf{q}_3\ \right] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \\
R &= \begin{bmatrix} ||\tilde{\mathbf{q}}_1||_2 & \mathbf{q}_1^T a_2 & \mathbf{q}_1^T \mathbf{a}_3 \\ 0 & ||\tilde{\mathbf{q}}_2||_2 & \mathbf{q}_2^T \mathbf{a}_3 \\ 0 & 0 & ||\tilde{\mathbf{q}}_3||_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}.
\end{aligned}
$$

Also, review homework problems on solving linear least squares problem with QR decomposition.

3 Vector and Matrix Norms

The L_∞ and L_2 norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ are defined by

$$\begin{aligned} ||\mathbf{x}||_\infty &= \max_{1 \le i \le n} |x_i|, \\ ||\mathbf{x}||_2 &= \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

For matrices, the definitions of $|| \cdot ||_\infty$, $|| \cdot ||_F$, and $|| \cdot ||_2$ norms are:

$$\begin{aligned} ||A||_\infty &= \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|, \\ ||A||_F &= \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}, \\ ||A||_2 &= \max_{||\mathbf{x}||_2 = 1} ||A\mathbf{x}||_2. \end{aligned}$$

Note, that $A\mathbf{x}$ is a vector:

$$A\mathbf{x} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix}.$$

Thus, we have

$$||A\mathbf{x}||_2 = \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right)^2 \right)^{\frac{1}{2}}.$$

4 Iterative methods for the solution of linear systems of equations.

4.1 Jacobi Method

A general $n \times n$ linear system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Jacobi method is written in the form $\mathbf{x^{(k)}} = T\mathbf{x^{(k-1)}} + \mathbf{c}$ by splitting A. Let D be the diagonal matrix whose diagonal entries are those of A, $-L$ be the strictly lower-triangular part of A, and $-U$ be the strictly upper-triangular part of A. Hence,

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \\ &= D - L - U. \end{aligned}$$

The equation $A\mathbf{x} = \mathbf{b}$, or

$$(D - L - U)\mathbf{x} = \mathbf{b}, \tag{1}$$

is then transformed into

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b},$$

and, if D^{-1} exists,

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This results in the matrix form of the Jacobi iterative technique:

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}.$$

Introducing $T_j = D^{-1}(L + U)$ and $c_j = D^{-1}\mathbf{b}$ (here, "j" stands for "Jacobi"), the Jacobi method has the form

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j.$$

4.2 Gauss-Seidel Method

Equation (1) can be written as

$$(D - L)\mathbf{x} = U\mathbf{x} + \mathbf{b},$$

which gives the Gauss-Seidel method:

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

or

$$\mathbf{x}^{(k)} = (D - L)^{-1}U\mathbf{x}^{(k-1)} + (D - L)^{-1}\mathbf{b}.$$

Introducing $T_g = (D - L)^{-1}U$ and $c_g = (D - L)^{-1}\mathbf{b}$ (here, "g" stands for "Gauss-Seidel"), the Gauss-Seidel method has the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g.$$

Please see relevant homework problems for examples.

4.3 Spectral Radius

The **spectral radius** $\rho(A)$ of matrix A is defined by

$$\rho(A) = \max |\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } A.$$

4.4 Diagonal Dominance

The $n \times n$ matrix A is said to be strictly diagonally dominant when

$$|a_{ii}| > \sum_{j=1,\, j\neq i} |a_{ij}|, \tag{2}$$

holds for each $i = 1, 2, \ldots, n$.

4.5 Condition Number

The **condition number** of the nonsingular matrix A relative to a norm $||\cdot||$ is

$$K(A) = ||A|| \cdot ||A^{-1}||.$$

In particular, the condition number of A relative to $||\cdot||_\infty$ is

$$K(A) = ||A||_\infty \cdot ||A^{-1}||_\infty,$$

where

$$||A||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|.$$

5 Consistency, Stability and Convergence of Numerical Methods

A one-step difference equation method with local truncation error $\tau_i(h)$ at the ith step is said to be **consistent** if

$$\lim_{h\to 0} \max_{1\le i\le N} |\tau_i(h)| = 0. \tag{3}$$

A one-step difference equation method is said to be **convergent** if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| = 0, \tag{4}$$

where y_i is the exact solution and w_i is the approximation obtained from the difference method. Recall that for Euler's method, we have

$$\max_{1\le i\le N} |w_i - y(t_i)| \le \frac{Mh}{2L}|e^{L(b-a)} - 1|, \tag{5}$$

and, therefore, Euler's method is convergent with the linear (first order) rate of convergence of $O(h)$.

A method is **stable** if its results depend continuously on the initial data.

5.1 Stability of Multistep Methods

Problem: To approximate the initial value problem

$$y' = f(t, y) \tag{6}$$

for $t > 0$, consider a multistep method

$$w_{i+1} = 2w_{i-1} - w_i + h\Big[\frac{5}{2}f(t_i, w_i) + \frac{1}{2}f(t_{i-1}, w_{i-1})\Big].$$

Is this method stable?

Solution: For a multistep method to be stable, it has to satisfy the root condition. A multistep method is said to satisfy the root condition if all roots λ_i of the characteristic polynomial $P(\lambda)$ (for a general form of $P(\lambda)$ see equation (5.57) in the book) are such that $|\lambda_i| \leq 1$, and if $|\lambda_i| = 1$, then λ_i is simple.
The characteristic polynomial of the following multistep method

$$w_{i+1} + w_i - 2w_{i-1} = h\Big[\frac{5}{2}f(t_i, w_i) + \frac{1}{2}f(t_{i-1}, w_{i-1})\Big].$$

is

$$P(\lambda) = -2 + \lambda + \lambda^2,$$

which has roots

$$\lambda_1 = 1,\ \lambda_2 = -2.$$

Thus this multistep method does not satisfy the root condition, and therefore is unstable. ✓
Note that a quick way of writing a characteristic polynomial is to associate the coefficient a_0 to the leftmost grid point in the method's stencil. In the example above, the leftmost grid point has an index $i - 1$, and therefore, $a_0 = -2$, $a_1 = 1$, $a_2 = 1$.

5.2 Local Truncation Errors and Consistency

Problem: To approximate the initial value problem

$$y' = f(t, y) \tag{7}$$

for $t > 0$, consider a multistep method

$$w_{i+1} = 2w_{i-1} - w_i + h\Big[\frac{5}{2}f(t_i, w_i) + \frac{1}{2}f(t_{i-1}, w_{i-1})\Big].$$

Find the local truncation error.

Solution: Expanding terms in Taylor's series around t_i, we obtain the following local truncation error $\tau(h)$:

$$\begin{aligned}
\tau_{i+1}(h) &= \frac{1}{h}\bigg(y(t_{i+1}) + y(t_i) - 2y(t_{i-1}) - h\Big[\frac{5}{2}f(t_i, y(t_i)) + \frac{1}{2}f(t_{i-1}, y(t_{i-1}))\Big]\bigg) = \\
&= \frac{1}{h}\Big[y(t_{i+1}) + y(t_i) - 2y(t_{i-1})\Big] - \frac{5}{2}y'(t_i) - \frac{1}{2}y'(t_{i-1}) \\
&= \frac{1}{h}\Big[\ y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(\xi_{i1}) \\
&\qquad +y(t_i) \\
&\qquad -2\big(y(t_i) - hy'(t_i) + \frac{h^2}{2}y''(t_i) - \frac{h^3}{6}y'''(\xi_{i2})\big)\Big] \\
&\qquad\qquad -\frac{5}{2}y'(t_i) \\
&\qquad\qquad -\frac{1}{2}\Big[y'(t_i) - hy''(t_i) + \frac{h^2}{2}y'''(\xi_{i3})\Big] \\
&= \frac{1}{4}h^2 y'''(\xi_i),
\end{aligned}$$

where $t_{i-1} < \xi_i < t_{i+1}$.

5.3 Regions of Absolute Stability

Problem: Show that the Backward Euler (or Implicit Euler) method

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1})$$

is A-stable.

Solution: The region R of absolute stability is $R = \{h\lambda \in \mathbb{C} \mid |Q(h\lambda)| < 1\}$, where $w_{i+1} = Q(h\lambda)w_i$. A numerical method is said to be A-stable if its region of stability R contains the entire left half-plane.

In other words, in order to show that the method is A-stable, we need to show that when it is applied to the scalar test equation $y' = \lambda y = f$, whose solutions tend to zero for $\lambda < 0$, all the solutions of the method also tend to zero for a fixed $h > 0$ as $i \to \infty$.

For the Backward Euler method, we have

$$\begin{aligned}
&w_{i+1} = w_i + h\lambda w_{i+1},\\
&w_{i+1} - h\lambda w_{i+1} = w_i,\\
&w_{i+1}(1 - h\lambda) = w_i,\\
&w_{i+1} = \frac{1}{1-h\lambda} w_i,\\
&w_{i+1} = \left(\frac{1}{1-h\lambda}\right)^{n+1} w_0.
\end{aligned}$$

Thus,

$$Q(h\lambda) = \frac{1}{1-h\lambda}.$$

Note that for $Re(h\lambda) < 0$, $|Q(h\lambda)| < 1$. Therefore, the region of absolute stability R for the Backward Euler method contains the entire left half-plane, and hence, the method is A-stable.

Note that the region of absolute stability contains the interval $(2, +\infty)$.

6 Boundary Value Problems

Given the second-order boundary-value problem

$$\begin{aligned}&y''(x) = p(x)y'(x) + q(x)y(x) + r(x),\\ &a \le x \le b,\\ &y(a) = \alpha, \quad y(b) = \beta,\end{aligned}$$

the differential equation to be approximated at the interior points x_i is

$$y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i). \tag{8}$$

It might be helpful to know how to derive the following approximations to the first and second derivatives of $y(x_i)$ (pages 656-657 in the book). Expanding $y(x_{i+1})$ and $y(x_{i-1})$ in Taylor polynomials about x_i, and doing some arithmetic manipulations (as in the book), we obtain the following formulas:

$$\begin{aligned}y''(x_i) &= \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{12}y''''(\xi_i),\\ y'(x_i) &= \frac{1}{2h}\left[y(x_{i+1}) - y(x_{i-1})\right] - \frac{h^2}{6}y'''(\eta_i).\end{aligned}$$

The approximation to (8) is therefore:

$$\left(\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2}\right) + p(x_i)\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) + q(x_i)w_i = -r(x_i). \tag{9}$$

Problem: Write the discretization of the following boundary value problem

$$\begin{aligned}&y'' = -\frac{4}{x}y' + \frac{2}{x^2}y - \frac{2}{x^2}\log x,\\ &1 \le x \le 2,\\ &y(1) = -\frac{1}{2}, \quad y(2) = \log 2,\end{aligned}$$

in matrix-vector notation $A\mathbf{w} = \mathbf{b}$.

Solution: At the interior points x_i, for $i = 1, 2, \ldots, N$, the differential equation to be approximated is

$$y''(x_i) = -\frac{4}{x_i}y'(x_i) + \frac{2}{x_i^2}y(x_i) - \frac{2}{x_i^2}\log x_i. \tag{10}$$

Since

$$\begin{aligned}y''(x_i) &= \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12}y^{(4)}(\xi_i),\\ y'(x_i) &= \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta_i),\end{aligned}$$

we can write the numerical approximation to (10) as

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + \frac{4}{x_i}\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) - \frac{2}{x_i^2}w_i = -\frac{2}{x_i^2}\log x_i. \tag{11}$$

Multiplying both sides of (11) by $-h^2$ gives

$$-(w_{i+1} - 2w_i + w_{i-1}) - \frac{2h}{x_i}(w_{i+1} - w_{i-1}) + \frac{2h^2}{x_i^2}w_i = \frac{2h^2}{x_i^2}\log x_i.$$

Collecting w_{i-1}, w_i, and w_{i+1} terms, we obtain

$$-\left(1-\frac{2h}{x_i}\right)w_{i-1}+\left(2+\frac{2h^2}{x_i^2}\right)w_i-\left(1+\frac{2h}{x_i}\right)w_{i+1}=\frac{2h^2}{x_i^2}\log x_i. \tag{12}$$

The resulting system of equations can be expressed in the tridiagonal $N\times N$ matrix form

$$A\mathbf{w}=\mathbf{b},\qquad \text{where}$$

$$A=\begin{bmatrix} 2+\frac{2h^2}{x_1^2} & -1-\frac{2h}{x_1} & 0 & \cdots & 0 \\ -1+\frac{2h}{x_2} & 2+\frac{2h^2}{x_2^2} & -1-\frac{2h}{x_2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1-\frac{2h}{x_{N-1}} \\ 0 & \cdots & 0 & -1+\frac{2h}{x_N} & 2+\frac{2h^2}{x_N^2} \end{bmatrix},$$

$$\mathbf{w}=\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix},\quad \text{and}\quad \mathbf{b}=\begin{bmatrix} \frac{2h^2}{x_1^2}\log x_1+\left(1-\frac{2h}{x_1}\right)w_0 \\ \frac{2h^2}{x_2^2}\log x_2 \\ \vdots \\ \frac{2h^2}{x_{N-1}^2}\log x_{N-1} \\ \frac{2h^2}{x_N^2}\log x_N+\left(1+\frac{2h}{x_N}\right)w_{N+1} \end{bmatrix}. \quad \checkmark$$

In order to see that this system satisfies (12), look at a couple of rows of matrix A, for example, the second row:

$$-\left(1-\frac{2h}{x_2}\right)w_1+\left(2+\frac{2h^2}{x_2^2}\right)w_2-\left(1+\frac{2h}{x_2}\right)w_3=\frac{2h^2}{x_2^2}\log x_2.$$

Also, first and last elements of $\mathbf{b}$ might be a little daunting. However, if we look at the first row (for example), we see that

$$\left(2+\frac{2h^2}{x_1^2}\right)w_1-\left(1+\frac{2h}{x_1}\right)w_2=\frac{2h^2}{x_1^2}\log x_1+\left(1-\frac{2h}{x_1}\right)w_0,$$

or

$$-\left(1-\frac{2h}{x_1}\right)w_0+\left(2+\frac{2h^2}{x_1^2}\right)w_1-\left(1+\frac{2h}{x_1}\right)w_2=\frac{2h^2}{x_1^2}\log x_1,$$

which satisfies equation (12).

Also, note that the book considers the general second-order boundary value problem:

$$y''(x)=p(x)y'(x)+q(x)y(x)+r(x).$$

For our problem, $p(x)=-\frac{4}{x}$, $q(x)=\frac{2}{x^2}$, and $r(x)=-\frac{2}{x^2}\log x$. Plugging these values into the formulas in the book, we can verify whether our calculations are correct.