Oblique Shock Problem: Solution of Conservation Laws on Non-Rectangular Grid

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Contents

1 Abstract

We consider the problem of an oblique shock, generated by a supersonic flow over a sharp wedge, and the subsequent reflections from a flat plate located underneath the wedge and the wedge surface itself. We solve the Euler equations written in general (ξ, η) coordinates for a two-dimensional compressible flow problem using different numerical methods. The flow parameters and geometry for the problem are given in the attached figure. [1] We solve system of conservation laws in two spatial dimensions on nonrectangular grid by employing the finite volume formulation for the discretization of equations.

2 Grid Generation

Consider the following equation for **grid generation**:

$$\xi_{xx} + \xi_{yy} = 0 \quad (1)$$
$$\eta_{xx} + \eta_{yy} = 0 \quad (2)$$

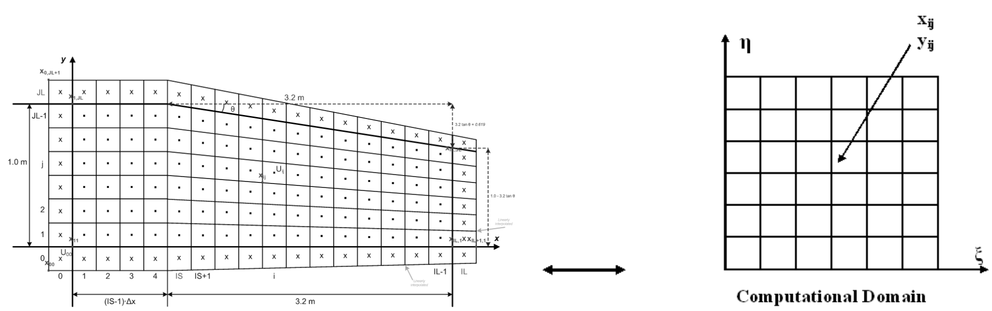

Figure 1: Transformation between physical and computational domains.

Summary [2]

• When the Poisson grid generators are used, the mapping is constructed by specifying the desired grid points (x, y) on the boundary of the physical domain with the interior point distribution determined through the solution of the equations (1) and (2) where (ξ, η) represent the coordinates in the computational domain.
• Equations (1) and (2) are then transformed to computational space by interchanging the roles of the independent and dependent variables.
• This yields a system of two elliptic equations.
• This system of equations is solved on a uniformly spaced grid in the computational plane. This provides the (x, y) coordinates for each point in physical space.

[1] The problem was originally stated as an article: *J.C.T. Wang and G.F. Widhopf, Journal of Computational Physics 84, 145-173, 1989.*

[2] As in J. Tannehill, D. Anderson, R. Pletcher, *Computational Fluid Mechanics and Heat Transfer*, Second Edition.

2.1 Derivation

Transform into computational coordinates to solve the Laplace equation numerically:

$$\begin{cases} x &= x(\xi,\eta) \\ y &= y(\xi,\eta) \end{cases} \quad \Leftrightarrow \quad \begin{cases} \xi &= \xi(x,y) \\ \eta &= \eta(x,y) \end{cases}$$

The Chain Rule gives: [3]

$$\begin{cases} \frac{\partial}{\partial x} &= \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial y} &= \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} \end{cases}$$

or, in matrix form

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}. \tag{3}$$

Similarly, the Chain Rule gives:

$$\begin{cases} \frac{\partial}{\partial \xi} &= x_\xi \frac{\partial}{\partial x} + y_\xi \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \eta} &= x_\eta \frac{\partial}{\partial x} + y_\eta \frac{\partial}{\partial y} \end{cases}$$

or, in matrix form

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix},$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}. \tag{4}$$

From (3) and (4), we observe that:

$$\begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix}^{-1} = \frac{1}{J} \begin{bmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{bmatrix},$$

where $J = x_\xi y_\eta - x_\eta y_\xi$, is the Jacobian of the transformation.
Therefore, the metric relations are:

$$\begin{cases} \xi_x = \frac{y_\eta}{J}, \\ \xi_y = -\frac{x_\eta}{J}, \\ \eta_x = -\frac{y_\xi}{J}, \\ \eta_y = \frac{x_\xi}{J}. \end{cases}$$

Using these, we can obtain expressions for ξ_{xx}, ξ_{yy}, η_{xx}, η_{yy}. Plugging these into (1) and (2), combining the resulting equations, we obtain the Laplace/Poisson equation in (ξ,η) plane:

$$\boxed{\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} \;=\; 0}$$

[3]Note: $f(x,y) \;=\; f(\xi,\eta) \;=\; f(\xi(x,y),\eta(x,y))$, then

$$\frac{\partial f}{\partial x} \;=\; \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} \;+\; \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} \;=\; \xi_x \frac{\partial f}{\partial \xi} \;+\; \eta_x \frac{\partial f}{\partial \eta}.$$

$$\boxed{\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} \;=\; 0}$$

where

$$\begin{aligned}
\alpha &= x_\eta^2 + y_\eta^2, \\
\beta &= x_\xi x_\eta + y_\xi y_\eta, \\
\gamma &= x_\xi^2 + y_\xi^2.
\end{aligned}$$

The equations are nonlinear, coupled.
The boundary conditions are specified:

$$\begin{array}{l}
x(\xi_1,\eta),\;\; y(\xi_1,\eta) \\
x(\xi_2,\eta),\;\; y(\xi_2,\eta) \\
x(\xi,\eta_1),\;\; y(\xi,\eta_1) \\
x(\xi,\eta_2),\;\; y(\xi,\eta_2)
\end{array}$$

The equations are solved to get interior solutions:

$$\begin{cases} x(\xi,\eta) = x_{ij} \\ y(\xi,\eta) = y_{ij}. \end{cases}$$

for $\;i = 2, 3, \ldots, IL-1;\;\; j = 2, 3, \ldots, JL-1.$
Note that the first and last column ($i = 0$ and $i = IL+1$) and the first and last row ($j = 0$ and $j = JL+1$) represent the **ghost points**.
Note: $\;\xi_1 = 1,\ \xi_2 = IL,\;\ \eta_1 = 1,\ \eta_2 = JL.$
Uniform grid in the computational domain is mapped into physical domain.

2.2 Initial Grid

The **algebraic grid** is discretized uniformly as:

$$\begin{aligned}
X(i,j) &= \left(\frac{IL-i}{IL-1}\right) X(1,j) + \left(\frac{i-1}{IL-1}\right) X(IL,j), \\
Y(i,j) &= \left(\frac{JL-j}{JL-1}\right) Y(i,1) + \left(\frac{j-1}{JL-1}\right) Y(i,JL).
\end{aligned}$$

Here, for instance, when $i = 1$, $X(i,j) = X(1,j)$, and when $i = IL$, $X(i,j) = X(IL,j)$.

2.3 Numerical Discretizations

Use central difference approximations for elliptic proplem.

$$\begin{aligned}
x_\xi &= \frac{x_{i+1,j} - x_{i-1,j}}{2\triangle\xi} = \delta_\xi^0 x, \\
x_\eta &= \frac{x_{i,j+1} - x_{i,j-1}}{2\triangle\eta} = \delta_\eta^0 x, \\
x_{\xi\xi} &= \frac{x_{i+1,j} - 2x_{i,j} + x_{i-1,j}}{\triangle\xi^2} = \delta_{\xi\xi} x, \\
x_{\eta\eta} &= \frac{x_{i,j+1} - 2x_{i,j} + x_{i,j-1}}{\triangle\eta^2} = \delta_{\eta\eta} x, \\
x_{\xi\eta} &= \frac{\partial}{\partial\eta}(x_\xi) = \frac{\frac{x_{i+1,j+1} - x_{i-1,j+1}}{2\triangle\xi} - \frac{x_{i+1,j-1} - x_{i-1,j-1}}{2\triangle\xi}}{2\triangle\eta} \\
&= \frac{x_{i+1,j+1} - x_{i-1,j+1} - x_{i+1,j-1} + x_{i-1,j-1}}{4\triangle\xi\triangle\eta} = \delta_{\xi\eta} x.
\end{aligned}$$

We can evaluate α, β, γ, J numerically.
Nonlinear difference equations are solved using the iterative method.

2.4 Discretized Equations

Assume $\triangle\xi = 1,\ \triangle\eta = 1$.
We obtain the discretizations:

$$\begin{aligned} \alpha_{ij}\cdot\big(x_{i+1,j}-2x_{ij}+x_{i-1,j}\big) \ &- \ \frac{1}{2}\beta_{ij}\cdot\big(x_{i+1,j+1}-x_{i-1,j+1}-x_{i+1,j-1}+x_{i-1,j-1}\big) \\ &+ \ \gamma_{ij}\cdot\big(x_{i,j+1}-2x_{ij}+x_{i,j-1}\big) \ = \ 0. \qquad \circledast \end{aligned}$$

$$\begin{aligned} \alpha_{ij}\cdot\big(y_{i+1,j}-2y_{ij}+y_{i-1,j}\big) \ &- \ \frac{1}{2}\beta_{ij}\cdot\big(y_{i+1,j+1}-y_{i-1,j+1}-y_{i+1,j-1}+y_{i-1,j-1}\big) \\ &+ \ \gamma_{ij}\cdot\big(y_{i,j+1}-2y_{ij}+y_{i,j-1}\big) \ = \ 0. \qquad \circledcirc \end{aligned}$$

Use Jacobi or Gauss-Seidel iterations to solve the equations ($\circledast$) and ($\circledcirc$) above.

2.5 Jacobi or Gauss-Seidel Iterations

① **Jacobi iteration.** For Jacobi iteration, need to keep old values in the memory.

$$\begin{aligned} -2(\alpha^n_{ij}+\gamma^n_{ij})x^{n+1}_{ij} \ &= \ - \ \alpha^n_{ij}\cdot\big(x^n_{i+1,j}+x^n_{i-1,j}\big) \\ &\quad + \ \frac{1}{2}\beta^n_{ij}\cdot\big(x^n_{i+1,j+1}-x^n_{i-1,j+1}-x^n_{i+1,j-1}+x^n_{i-1,j-1}\big) \\ &\quad - \ \gamma^n_{ij}\cdot\big(x^n_{i,j+1}+x^n_{i,j-1}\big). \end{aligned}$$

$$\begin{aligned} -2(\alpha^n_{ij}+\gamma^n_{ij})y^{n+1}_{ij} \ &= \ - \ \alpha^n_{ij}\cdot\big(y^n_{i+1,j}+y^n_{i-1,j}\big) \\ &\quad + \ \frac{1}{2}\beta^n_{ij}\cdot\big(y^n_{i+1,j+1}-y^n_{i-1,j+1}-y^n_{i+1,j-1}+y^n_{i-1,j-1}\big) \\ &\quad - \ \gamma^n_{ij}\cdot\big(y^n_{i,j+1}+y^n_{i,j-1}\big). \end{aligned}$$

② **Gauss-Seidel iteration.** Use the latest available values in x and y in the iteration.
Define

$$\begin{cases} \triangle x^{n+1}_{ij} \ = \ x^{n+1}_{ij}-x^n_{ij} \\ \triangle y^{n+1}_{ij} \ = \ y^{n+1}_{ij}-y^n_{ij} \end{cases} \Rightarrow \begin{cases} x^{n+1}_{ij} \ = \ x^n_{ij}+\triangle x^{n+1}_{ij} \\ y^{n+1}_{ij} \ = \ y^n_{ij}+\triangle y^{n+1}_{ij} \end{cases} \qquad \circledast\circledast$$

$$\begin{aligned} \triangle x^{n+1}_{ij} \ &= \ -\frac{-\alpha^n\delta_{\xi\xi}x^n+2\beta\delta_{\xi\eta}x^n-\gamma\delta_{\eta\eta}x^n}{2(\alpha^n+\gamma^n)} \\ \triangle y^{n+1}_{ij} \ &= \ -\frac{-\alpha^n\delta_{\xi\xi}y^n+2\beta\delta_{\xi\eta}y^n-\gamma\delta_{\eta\eta}y^n}{2(\alpha^n+\gamma^n)} \end{aligned}$$

Solve for x^{n+1}_{ij} and y^{n+1}_{ij} using $\circledast\circledast$.
We set the convergence criterion $\varepsilon = 10^{-3}$. Iterate until

$$\begin{cases} \max|\triangle x^{n+1}_{ij}| \ < \ \varepsilon \\ \max|\triangle y^{n+1}_{ij}| \ < \ \varepsilon. \end{cases}$$

3 Governing Equations

The 2D Euler equations can be written in the following convervation-law form:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} \;=\; 0. \tag{5}$$

The vector of conserved quantities is given by

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix},$$

where ρ is the density, (u, v) is the velocity, and e is the total energy.
The flux vectors E and F are given by

$$E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{bmatrix}, \qquad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{bmatrix}.$$

In order to close the system, an equation of state needs to be specified:

$$e = \frac{p}{\gamma - 1} + \frac{\rho(u^2 + v^2)}{2},$$

or, equivalently,

$$p = (\gamma - 1)\Big(e - \frac{\rho(u^2 + v^2)}{2}\Big),$$

where p is pressure.
The gas is air, $\gamma = 1.4$, and gas constant is $R = 287.1\ Joules/kg - {}^\circ K$.
Auxiliary variables are calculated with the following formulas:

$$\begin{aligned} c &= \sqrt{\frac{\gamma p}{\rho}}, \\ M &= \frac{\sqrt{u^2 + v^2}}{c}, \\ C_v &= \frac{R}{\gamma - 1}, \\ S &= C_v \log\left(\frac{p}{\rho^\gamma}\right). \end{aligned}$$

4 Coordinate Transformation of the Euler Equations

We have found the following metric relations:

$$\begin{cases} \xi_x = \frac{y_\eta}{J}, \\ \xi_y = -\frac{x_\eta}{J}, \\ \eta_x = -\frac{y_\xi}{J}, \\ \eta_y = \frac{x_\xi}{J}, \end{cases} \tag{6}$$

where $J = x_\xi y_\eta - x_\eta y_\xi$.
The Chain Rule gives:

$$\begin{cases} \frac{\partial E}{\partial x} = \xi_x E_\xi + \eta_x E_\eta, \\ \frac{\partial F}{\partial y} = \xi_y F_\xi + \eta_y F_\eta. \end{cases} \tag{7}$$

Plugging this into (5), we obtain

$$\frac{\partial U}{\partial t} + \big(\xi_x E_\xi + \xi_y F_\xi\big) + \big(\eta_x E_\eta + \eta_y F_\eta\big) \ = \ 0. \tag{8}$$

To write equation (8) in conservation law form, we multiply the equation by J and use metric relations:

$$\begin{aligned} J\frac{\partial U}{\partial t} + \big(J\xi_x E_\xi + J\xi_y F_\xi\big) + \big(J\eta_x E_\eta + J\eta_y F_\eta\big) \ &= \ 0, \\ \frac{\partial (UJ)}{\partial t} + \big(y_\eta E_\xi - x_\eta F_\xi\big) + \big(-y_\xi E_\eta + x_\xi F_\eta\big) \ &= \ 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial (UJ)}{\partial t} \quad &+ \quad \big(y_\eta E - x_\eta F\big)_\xi - \big(y_{\eta\xi} E - x_{\eta\xi} F\big) \\ &+ \quad \big(-y_\xi E + x_\xi F\big)_\eta - \big(-y_{\xi\eta} E + x_{\xi\eta} F\big) \ = \ 0. \end{aligned}$$

Cancelling out the terms, we get

$$\frac{\partial (UJ)}{\partial t} + \big(y_\eta E - x_\eta F\big)_\xi + \big(-y_\xi E + x_\xi F\big)_\eta \ = \ 0.$$

Thus, the 2D Euler Equations may be written in **conservation law form**:

$$\boxed{\begin{cases} \frac{\partial U'}{\partial t} + \frac{\partial E'}{\partial \xi} + \frac{\partial F'}{\partial \eta} \ = \ 0, \\ U' \ = \ UJ, \\ E' \ = \ y_\eta E - x_\eta F \ = \ J(\xi_x E + \xi_y F), \\ F' \ = \ -y_\xi E + x_\xi F \ = \ J(\eta_x E + \eta_y F). \end{cases}}$$

5 Finite Volume Method

For the project we use the finite volume formulation for the discretization of the equations. We apply integral form of the equations to each individual grid cell. We have

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} \;=\; 0.$$

Denote

$$\vec{F} = E\vec{i} + F\vec{j}.$$

We can rewrite the Euler Equations as

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{F} \;=\; 0 \qquad (9)$$

Integrating (9) over the volume V_{ij}, we have

$$\int_{V_{ij}} \Big(\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}\Big)\, dV \;=\; 0.$$

The divergence theorem gives

$$\frac{\partial}{\partial t}\int_{V_{ij}} U\, dV + \oint_{S_{ij}} \vec{F} \cdot d\vec{S} \;=\; 0.$$

Denoting the average of U over the volume V_{ij} as $\overline{U} = \frac{\int U\, dV}{V}$, we get

$$\frac{\partial \overline{U}_{ij}}{\partial t} + \frac{1}{V_{ij}}\oint_{S_{ij}} \vec{F} \cdot d\vec{S} \;=\; 0,$$

$$\frac{\partial \overline{U}_{ij}}{\partial t} \;=\; -\frac{1}{V_{ij}}\oint_{S_{ij}} \vec{F} \cdot d\vec{S} \;=\; 0,$$

$$\frac{\partial \overline{U}_{ij}}{\partial t} \;=\; -\frac{1}{V_{ij}}\Big[(\vec{F}\cdot\vec{S})_{i+\frac{1}{2}} + (\vec{F}\cdot\vec{S})_{i-\frac{1}{2}} + (\vec{F}\cdot\vec{S})_{j+\frac{1}{2}} + (\vec{F}\cdot\vec{S})_{j-\frac{1}{2}}\Big], \qquad (10)$$

where

$$(\vec{F}\cdot\vec{S})_{i+\frac{1}{2}} \;=\; \vec{F}_{i+\frac{1}{2}} \cdot \vec{n}_{i+\frac{1}{2}} S_{i+\frac{1}{2}} \;=\; F'_{i+\frac{1}{2}} S_{i+\frac{1}{2}},$$

$$F' \;=\; \vec{F}\cdot\vec{n} \;=\; (E\vec{i} + F\vec{j})\cdot(n_x\vec{i} + n_y\vec{j}) \;=\; n_x E + n_y F$$

$$= \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (e+p)u \end{bmatrix} n_x + \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (e+p)v \end{bmatrix} n_y \;=\; \begin{bmatrix} \rho u' \\ \rho u u' + p n_x \\ \rho v u' + p n_y \\ (e+p)u' \end{bmatrix},$$

and $u' = u n_x + v n_y = \vec{v}\cdot\vec{n}$. Thus, equation (10) becomes

$$\frac{\partial \overline{U}_{ij}}{\partial t} \;=\; -\frac{1}{V_{ij}}\Big[F'_{i+\frac{1}{2}} S_{i+\frac{1}{2}} + F'_{i-\frac{1}{2}} S_{i-\frac{1}{2}} + F'_{j+\frac{1}{2}} S_{j+\frac{1}{2}} + F'_{j-\frac{1}{2}} S_{j-\frac{1}{2}}\Big],$$

where $F' = E(U)n_x + F(U)n_y$.

$$S \;=\; \sqrt{\triangle y^2 + \triangle x^2},$$

$$n_x \;=\; \frac{\triangle y}{S},$$

$$n_y \;=\; -\frac{\triangle x}{S},$$

$$\vec{S} \;=\; \vec{n} S \;=\; \triangle y \vec{i} - \triangle x \vec{j}.$$

Note that the counterclockwise direction is used when calculating $\triangle x$ and $\triangle y$.

$$
\begin{aligned}
\vec{S}_{i+\frac{1}{2}} &= (y_{i+1,j+1} - y_{i+1,j})\vec{i} - (x_{i+1,j+1} - x_{i+1,j})\vec{j}, \\
\vec{S}_{i-\frac{1}{2}} &= -\big[(y_{i,j+1} - y_{ij})\vec{i} - (x_{i,j+1} - x_{ij})\vec{j}\,\big], \\
\vec{S}_{j+\frac{1}{2}} &= -\big[(y_{i+1,j+1} - y_{i,j+1})\vec{i} - (x_{i+1,j+1} - x_{i,j+1})\vec{j}\,\big], \\
\vec{S}_{j-\frac{1}{2}} &= (y_{i+1,j} - y_{ij})\vec{i} - (x_{i+1,j} - x_{ij})\vec{j}.
\end{aligned}
$$

We can rewrite the finite volume formulation in the form which looks like finite differences. As a result, we obtain the formulation below.

5.1 Surface Variables

Denote

$$
\begin{aligned}
\triangle x_{i+\frac{1}{2}} &= x_{i+1,j+1} - x_{i+1,j}, \\
\triangle y_{i+\frac{1}{2}} &= y_{i+1,j+1} - y_{i+1,j}, \\
\triangle x_{i-\frac{1}{2}} &= x_{i,j+1} - x_{i,j}, \\
\triangle y_{i-\frac{1}{2}} &= y_{i,j+1} - y_{i,j}, \\
\triangle x_{j+\frac{1}{2}} &= x_{i+1,j+1} - x_{i,j+1}, \\
\triangle y_{j+\frac{1}{2}} &= y_{i+1,j+1} - y_{i,j+1}, \\
\triangle x_{j-\frac{1}{2}} &= x_{i+1,j} - x_{i,j}, \\
\triangle y_{j-\frac{1}{2}} &= y_{i+1,j} - y_{i,j}.
\end{aligned}
$$

The surface vectors are calculated with the following formulas:

$$
\begin{aligned}
S_{i+\frac{1}{2}} &= \sqrt{\triangle y^2_{i+\frac{1}{2}} + \triangle x^2_{i+\frac{1}{2}}}, \\
S_{i-\frac{1}{2}} &= \sqrt{\triangle y^2_{i-\frac{1}{2}} + \triangle x^2_{i-\frac{1}{2}}}, \\
S_{j+\frac{1}{2}} &= \sqrt{\triangle y^2_{j+\frac{1}{2}} + \triangle x^2_{j+\frac{1}{2}}}, \\
S_{j-\frac{1}{2}} &= \sqrt{\triangle y^2_{j-\frac{1}{2}} + \triangle x^2_{j-\frac{1}{2}}}.
\end{aligned}
$$

$$
\begin{aligned}
n_{x_{i+\frac{1}{2}}} &= \frac{\triangle y_{i+\frac{1}{2}}}{S_{i+\frac{1}{2}}}, & n_{y_{i+\frac{1}{2}}} &= -\frac{\triangle x_{i+\frac{1}{2}}}{S_{i+\frac{1}{2}}}, \\
n_{x_{i-\frac{1}{2}}} &= \frac{\triangle y_{i-\frac{1}{2}}}{S_{i-\frac{1}{2}}}, & n_{y_{i-\frac{1}{2}}} &= -\frac{\triangle x_{i-\frac{1}{2}}}{S_{i-\frac{1}{2}}}, \\
n_{x_{j+\frac{1}{2}}} &= -\frac{\triangle y_{j+\frac{1}{2}}}{S_{j+\frac{1}{2}}}, & n_{y_{j+\frac{1}{2}}} &= \frac{\triangle x_{j+\frac{1}{2}}}{S_{j+\frac{1}{2}}}, \\
n_{x_{j-\frac{1}{2}}} &= -\frac{\triangle y_{j-\frac{1}{2}}}{S_{j-\frac{1}{2}}}, & n_{y_{j-\frac{1}{2}}} &= \frac{\triangle x_{j-\frac{1}{2}}}{S_{j-\frac{1}{2}}}.
\end{aligned}
$$

5.2 Cell Volume

The volume of the cell is calculated as:

$$\begin{aligned}
V_{ij} &= V_1 + V_2, \\
V_1 &= \frac{1}{2}\, abs\left\{ \begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \right\} = \frac{1}{2}\Big|(x_3 - x_1)(y_4 - y_1) - (x_4 - x_1)(y_3 - y_1)\Big| \\
&= \frac{1}{2}\Big|(x_{i+1,j+1} - x_{ij})(y_{i,j+1} - y_{ij}) - (x_{i,j+1} - x_{ij})(y_{i+1,j+1} - y_{ij})\Big|, \\
V_2 &= \frac{1}{2}\, abs\left\{ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right\} = \frac{1}{2}\Big|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\Big| \\
&= \frac{1}{2}\Big|(x_{i+1,j} - x_{ij})(y_{i+1,j+1} - y_{ij}) - (x_{i+1,j+1} - x_{ij})(y_{i+1,j} - y_{ij})\Big|.
\end{aligned}$$

5.3 Flux Terms and the Finite Volume Formulation

The flux terms are given by

$$\begin{aligned}
E'_{i+\frac{1}{2}} &= E \times n_{x_{i+\frac{1}{2}}} + F \times n_{y_{i+\frac{1}{2}}}, \\
E'_{i-\frac{1}{2}} &= E \times n_{x_{i-\frac{1}{2}}} + F \times n_{y_{i-\frac{1}{2}}}, \\
F'_{j+\frac{1}{2}} &= E \times n_{x_{j+\frac{1}{2}}} + F \times n_{y_{j+\frac{1}{2}}}, \\
F'_{j-\frac{1}{2}} &= E \times n_{x_{j-\frac{1}{2}}} + F \times n_{y_{j-\frac{1}{2}}},
\end{aligned}$$

and the finite volume formulation becomes

$$\boxed{\frac{\partial U_{ij}}{\partial t} + \frac{1}{V_{ij}}\Big[E'_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - E'_{i-\frac{1}{2}} S_{i-\frac{1}{2}} + F'_{j+\frac{1}{2}} S_{j+\frac{1}{2}} - F'_{j-\frac{1}{2}} S_{j-\frac{1}{2}}\Big] = 0.}$$

6 Numerical Schemes

6.1 MacCormack Method

The finite volume formulation of the MacCormack method is:

$$U^*_{ij} = U^n_{ij} - \frac{dt}{V_{ij}}\Big[\big(E'^n_{i+1,j}S_{i+\frac{1}{2}} - E'^n_{ij}S_{i-\frac{1}{2}}\big) + \big(F'^n_{i,j+1}S_{j+\frac{1}{2}} - F'^n_{ij}S_{j-\frac{1}{2}}\big)\Big], \qquad \text{(predictor)}$$

$$U^{n+1}_{ij} = \frac{1}{2}\Big(U^n_{ij} + U^*_{ij} - \frac{dt}{V_{ij}}\big[E'^*_{ij}S_{i+\frac{1}{2}} - E'^*_{i-1,j}S_{i-\frac{1}{2}} + F'^*_{ij}S_{j+\frac{1}{2}} - F'^*_{i,j-1}S_{j-\frac{1}{2}}\big]\Big). \qquad \text{(corrector)}$$

The flux terms for the **predictor** are calculated as:

$$\begin{aligned}
E'^n_{i+1,j}\, S_{i+\frac{1}{2}} &= \Big[E_{i+1,j}\, n_{x_{i+\frac{1}{2}}} \; + \; F_{i+1,j}\, n_{y_{i+\frac{1}{2}}}\Big]\, S_{i+\frac{1}{2}},\\
E'^n_{ij}\, S_{i-\frac{1}{2}} &= \Big[E_{ij}\, n_{x_{i-\frac{1}{2}}} \; + \; F_{ij}\, n_{y_{i-\frac{1}{2}}}\Big]\, S_{i-\frac{1}{2}},\\
F'^n_{i,j+1}\, S_{j+\frac{1}{2}} &= \Big[E_{i,j+1}\, n_{x_{j+\frac{1}{2}}} \; + \; F_{i,j+1}\, n_{y_{j+\frac{1}{2}}}\Big]\, S_{j+\frac{1}{2}},\\
F'^n_{ij}\, S_{j-\frac{1}{2}} &= \Big[E_{ij}\, n_{x_{j-\frac{1}{2}}} \; + \; F_{ij}\, n_{y_{j-\frac{1}{2}}}\Big]\, S_{j-\frac{1}{2}},
\end{aligned}$$

where

$$E_{ij} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{bmatrix}_{ij}, \qquad E_{i\pm1,j} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{bmatrix}_{i\pm1,j},$$

$$F_{ij} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{bmatrix}_{ij}, \qquad F_{i\pm1,j} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{bmatrix}_{i\pm1,j}.$$

The flux terms for the **corrector** are calculated as:

$$\begin{aligned}
E'^*_{ij} S_{i+\frac{1}{2}} &= \Big[E^*_{ij}\, n_{x_{i+\frac{1}{2}}} \; + \; F^*_{ij}\, n_{y_{i+\frac{1}{2}}}\Big]\, S_{i+\frac{1}{2}},\\
E'^*_{i-1,j} S_{i-\frac{1}{2}} &= \Big[E^*_{i-1,j}\, n_{x_{i-\frac{1}{2}}} \; + \; F^*_{i-1,j}\, n_{y_{i-\frac{1}{2}}}\Big]\, S_{i-\frac{1}{2}},\\
F'^*_{ij} S_{j+\frac{1}{2}} &= \Big[E^*_{ij}\, n_{x_{j+\frac{1}{2}}} \; + \; F^*_{ij}\, n_{y_{j+\frac{1}{2}}}\Big]\, S_{j+\frac{1}{2}},\\
F'^*_{i,j-1} S_{j-\frac{1}{2}} &= \Big[E^*_{i,j-1}\, n_{x_{j-\frac{1}{2}}} \; + \; F^*_{i,j-1}\, n_{y_{j-\frac{1}{2}}}\Big]\, S_{j-\frac{1}{2}},
\end{aligned}$$

where

$$E^*_{ij} = E(U^*_{ij}) = \begin{bmatrix} \rho^* u^* \\ \rho^* u^{*2} + p^* \\ \rho^* u^* v^* \\ (e^* + p^*)u^* \end{bmatrix}_{ij}, \qquad F^*_{ij} = F(U^*_{ij}) = \begin{bmatrix} \rho^* v^* \\ \rho^* u^* v^* \\ \rho^* v^{*2} + p^* \\ (e^* + p^*)v^* \end{bmatrix}_{ij}.$$

6.2 MacCormack Method with Artificial Diffusion

The finite volume formulation of the MacCormack method with MacCormack-Baldwin diffusion is the following. The **predictor** step is:

$$U^*_{ij} \;=\; U^n_{ij} - \frac{dt}{V_{ij}}\Big[\big(E^{D\,n}_{i+\frac{1}{2}}\,S_{i+\frac{1}{2}} - E^{D\,n}_{i-\frac{1}{2}}\,S_{i-\frac{1}{2}}\big) + \big(F^{D\,n}_{j+\frac{1}{2}}\,S_{j+\frac{1}{2}} - F^{D\,n}_{j-\frac{1}{2}}\,S_{j-\frac{1}{2}}\big)\Big], \qquad \text{(predictor)}$$

where the flux terms are calculated as

$$\begin{aligned}
E^{D\,n}_{i+\frac{1}{2}} &= E'^n_{i+1,j} \;-\; \varepsilon\,(|u'|+c)^n_{i+\frac{1}{2},j}\,\frac{|p^n_{i+1,j}-2p^n_{ij}+p^n_{i-1,j}|}{p^n_{i+1,j}+2p^n_{ij}+p^n_{i-1,j}}\,(U^n_{i+1,j}-U^n_{ij}),\\
E^{D\,n}_{i-\frac{1}{2}} &= E'^n_{ij} \;-\; \varepsilon\,(|u'|+c)^n_{i-\frac{1}{2},j}\,\frac{|p^n_{ij}-2p^n_{i-1,j}+p^n_{i-2,j}|}{p^n_{i,j}+2p^n_{i-1,j}+p^n_{i-2,j}}\,(U^n_{ij}-U^n_{i-1,j}),\\
F^{D\,n}_{j+\frac{1}{2}} &= F'^n_{i,j+1} \;-\; \varepsilon\,(|u'|+c)^n_{i,j+\frac{1}{2}}\,\frac{|p^n_{i,j+1}-2p^n_{ij}+p^n_{i,j-1}|}{p^n_{i,j+1}+2p^n_{ij}+p^n_{i,j-1}}\,(U^n_{i,j+1}-U^n_{ij}),\\
F^{D\,n}_{j-\frac{1}{2}} &= F'^n_{ij} \;-\; \varepsilon\,(|u'|+c)^n_{i,j-\frac{1}{2}}\,\frac{|p^n_{ij}-2p^n_{i,j-1}+p^n_{i,j-2}|}{p^n_{ij}+2p^n_{i,j-1}+p^n_{i,j-2}}\,(U^n_{ij}-U^n_{i,j-1}),
\end{aligned}$$

for $i = 1, \ldots, IL-1, \quad j = 1, \ldots, JL-1,$ that is, for all cells in the interior. [4]
Here, we assume that

$$\begin{cases} p^n_{-1,j} \;=\; 2p^n_{0,j} - p^n_{1,j},\\ p^n_{IL+1,j} \;=\; 2p^n_{IL,j} - p^n_{IL-1,j}; \end{cases} \qquad \begin{cases} p^n_{i,-1} \;=\; 2p^n_{i,0} - p^n_{i,1},\\ p^n_{i,JL+1} \;=\; 2p^n_{i,JL} - p^n_{i,JL-1}. \end{cases}$$

E'^n and F'^n are MacCormack fluxes, the formulas for which were given in the previous section. Also,

$$\begin{aligned}
(|u'|+c)^n_{i+\frac{1}{2},j} &= |u'_{i+1,j}| + c_{i+1,j} \;=\; \big|u_{i+1,j}\,n_{x_{i+\frac{1}{2}}} + v_{i+1,j}\,n_{y_{i+\frac{1}{2}}}\big| + c_{i+1,j},\\
(|u'|+c)^n_{i-\frac{1}{2},j} &= |u'_{ij}| + c_{ij} \;=\; \big|u_{ij}\,n_{x_{i-\frac{1}{2}}} + v_{ij}\,n_{y_{i-\frac{1}{2}}}\big| + c_{ij}, \qquad \circledast\\
(|u'|+c)^n_{i,j+\frac{1}{2}} &= |u'_{i,j+1}| + c_{i,j+1} \;=\; \big|u_{i,j+1}\,n_{x_{j+\frac{1}{2}}} + v_{i,j+1}\,n_{y_{j+\frac{1}{2}}}\big| + c_{i,j+1},\\
(|u'|+c)^n_{i,j-\frac{1}{2}} &= |u'_{ij}| + c_{ij} \;=\; \big|u_{ij}\,n_{x_{j-\frac{1}{2}}} + v_{ij}\,n_{y_{j-\frac{1}{2}}}\big| + c_{ij}.
\end{aligned}$$

The **corrector** step is

$$U^{n+1}_{ij} \;=\; \frac{1}{2}\Big(U^n_{ij} + U^*_{ij} - \frac{dt}{V_{ij}}\big[E^{D\,*}_{i+\frac{1}{2}}\,S_{i+\frac{1}{2}} - E^{D\,*}_{i-\frac{1}{2}}\,S_{i-\frac{1}{2}} + F^{D\,*}_{j+\frac{1}{2}}\,S_{j+\frac{1}{2}} - F^{D\,*}_{j-\frac{1}{2}}\,S_{j-\frac{1}{2}}\big]\Big), \qquad \text{(corrector)}$$

where the flux terms are calculated as

$$\begin{aligned}
E^{D\,*}_{i+\frac{1}{2}} &= E'^*_{ij} \;-\; \varepsilon\,(|u'|+c)^*_{i+\frac{1}{2},j}\,\frac{|p^*_{i+1,j}-2p^*_{ij}+p^*_{i-1,j}|}{p^*_{i+1,j}+2p^*_{ij}+p^*_{i-1,j}}\,(U^*_{i+1,j}-U^*_{ij}),\\
E^{D\,*}_{i-\frac{1}{2}} &= E'^*_{i-1,j} \;-\; \varepsilon\,(|u'|+c)^*_{i-\frac{1}{2},j}\,\frac{|p^*_{ij}-2p^*_{i-1,j}+p^*_{i-2,j}|}{p^*_{ij}+2p^*_{i-1,j}+p^*_{i-2,j}}\,(U^*_{ij}-U^*_{i-1,j}),\\
F^{D\,*}_{j+\frac{1}{2}} &= F'^*_{ij} \;-\; \varepsilon\,(|u'|+c)^*_{i,j+\frac{1}{2}}\,\frac{|p^*_{i,j+1}-2p^*_{ij}+p^*_{i,j-1}|}{p^*_{i,j+1}+2p^*_{ij}+p^*_{i,j-1}}\,(U^*_{i,j+1}-U^*_{ij}),\\
F^{D\,*}_{j-\frac{1}{2}} &= F'^*_{i,j-1} \;-\; \varepsilon\,(|u'|+c)^*_{i,j-\frac{1}{2}}\,\frac{|p^*_{ij}-2p^*_{i,j-1}+p^*_{i,j-2}|}{p^*_{ij}+2p^*_{i,j-1}+p^*_{i,j-2}}\,(U^*_{ij}-U^*_{i,j-1}),
\end{aligned}$$

[4] Indexing is based on the C++ implementation.

for $\quad i = 1, \dots, IL-1, \quad j = 1, \dots, JL-1, \quad$ that is, for all cells in the interior.
Here, we assume that

$$\begin{cases} p^*_{-1,j} &= 2p^*_{0,j} - p^*_{1,j}, \\ p^*_{IL+1,j} &= 2p^*_{IL,j} - p^*_{IL-1,j}; \end{cases} \qquad \begin{cases} p^*_{i,-1} &= 2p^*_{i,0} - p^*_{i,1}, \\ p^*_{i,JL+1} &= 2p^*_{i,JL} - p^*_{i,JL-1}. \end{cases}$$

E'^* and F'^* are MacCormack fluxes, the formulas for which were given in the previous section. [5] Also, similar formulas as in ⊛ are used for the corrector.
In our experiments, we took $\varepsilon = 0.6$.

[5] Indexing is based on the C++ implementation.

6.3 Lax-Friedrichs Method

Lax-Friedrichs method can be formulated in the finite volume formulation as:

$$U_{ij}^{n+1} = U_{i,j}^{n} - \frac{dt}{V_{ij}}\Big[\big(E'^{n}_{i+\frac{1}{2},j}S_{i+\frac{1}{2}} - E'^{n}_{i-\frac{1}{2},j}S_{i-\frac{1}{2}}\big) + \big(F'^{n}_{i,j+\frac{1}{2}}S_{j+\frac{1}{2}} - F'^{n}_{i,j-\frac{1}{2}}S_{j-\frac{1}{2}}\big)\Big],$$

where the flux terms are calculated as

$$\begin{aligned}
E'^{n}_{i+\frac{1}{2},j} &= \left(\frac{E'^{n}_{i+1,j} + E'^{n}_{ij}}{2}\right) - \frac{\alpha_{i+\frac{1}{2}}(U^{n}_{i+1,j} - U^{n}_{ij})}{2} &= \frac{1}{2}\big\{(E'^{n}_{i+1,j} - \alpha_{i+\frac{1}{2}}U^{n}_{i+1,j}) + (E'^{n}_{ij} + \alpha_{i+\frac{1}{2}}U^{n}_{ij})\big\},\\
E'^{n}_{i-\frac{1}{2},j} &= \left(\frac{E'^{n}_{ij} + E'^{n}_{i-1,j}}{2}\right) - \frac{\alpha_{i-\frac{1}{2}}(U^{n}_{ij} - U^{n}_{i-1,j})}{2} &= \frac{1}{2}\big\{(E'^{n}_{ij} - \alpha_{i-\frac{1}{2}}U^{n}_{ij}) + (E'^{n}_{i-1,j} + \alpha_{i-\frac{1}{2}}U^{n}_{i-1,j})\big\},\\
E'^{n}_{i,j+\frac{1}{2}} &= \left(\frac{E'^{n}_{i,j+1} + E'^{n}_{ij}}{2}\right) - \frac{\alpha_{i+\frac{1}{2}}(U^{n}_{i,j+1} - U^{n}_{ij})}{2} &= \frac{1}{2}\big\{(E'^{n}_{i,j+1} - \alpha_{j+\frac{1}{2}}U^{n}_{i,j+1}) + (E'^{n}_{ij} + \alpha_{j+\frac{1}{2}}U^{n}_{ij})\big\},\\
E'^{n}_{i,j-\frac{1}{2}} &= \left(\frac{E'^{n}_{ij} + E'^{n}_{i,j-1}}{2}\right) - \frac{\alpha_{i+\frac{1}{2}}(U^{n}_{ij} - U^{n}_{i,j-1})}{2} &= \frac{1}{2}\big\{(E'^{n}_{ij} - \alpha_{j-\frac{1}{2}}U^{n}_{ij}) + (E'^{n}_{i,j-1} + \alpha_{j-\frac{1}{2}}U^{n}_{i,j-1})\big\},
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{i+\frac{1}{2}} &= |u'_{ij}| + c_{ij} = \left|u_{ij}\, n_{x_{i+\frac{1}{2}}} + v_{ij}\, n_{y_{i+\frac{1}{2}}}\right| + c_{ij},\\
\alpha_{i-\frac{1}{2}} &= |u'_{ij}| + c_{ij} = \left|u_{ij}\, n_{x_{i-\frac{1}{2}}} + v_{ij}\, n_{y_{i-\frac{1}{2}}}\right| + c_{ij},\\
\alpha_{j+\frac{1}{2}} &= |u'_{ij}| + c_{ij} = \left|u_{ij}\, n_{x_{j+\frac{1}{2}}} + v_{ij}\, n_{y_{j+\frac{1}{2}}}\right| + c_{ij},\\
\alpha_{j-\frac{1}{2}} &= |u'_{ij}| + c_{ij} = \left|u_{ij}\, n_{x_{j-\frac{1}{2}}} + v_{ij}\, n_{y_{j-\frac{1}{2}}}\right| + c_{ij}.
\end{aligned}$$

We can rewrite the above fluxes more explicitly as

$$\begin{aligned}
E'^{n}_{i+\frac{1}{2},j}S_{i+\frac{1}{2}} &= \left[\frac{1}{2}\Big(\big(E_{i+1,j} + E_{ij}\big)\, n_{x_{i+\frac{1}{2}}} + \big(F_{i+1,j} + F_{ij}\big)\, n_{y_{i+\frac{1}{2}}}\Big) - \frac{1}{2}\alpha_{i+\frac{1}{2}}(U^{n}_{i+1,j} - U^{n}_{ij})\right] S_{i+\frac{1}{2}},\\
E'^{n}_{i-\frac{1}{2},j}S_{i-\frac{1}{2}} &= \left[\frac{1}{2}\Big(\big(E_{ij} + E_{i-1,j}\big)\, n_{x_{i-\frac{1}{2}}} + \big(F_{ij} + F_{i-1,j}\big)\, n_{y_{i-\frac{1}{2}}}\Big) - \frac{1}{2}\alpha_{i-\frac{1}{2}}(U^{n}_{ij} - U^{n}_{i-1,j})\right] S_{i-\frac{1}{2}},\\
E'^{n}_{i,j+\frac{1}{2}}S_{j+\frac{1}{2}} &= \left[\frac{1}{2}\Big(\big(E_{i,j+1} + E_{ij}\big)\, n_{x_{j+\frac{1}{2}}} + \big(F_{i,j+1} + F_{ij}\big)\, n_{y_{j+\frac{1}{2}}}\Big) - \frac{1}{2}\alpha_{j+\frac{1}{2}}(U^{n}_{i,j+1} - U^{n}_{ij})\right] S_{j+\frac{1}{2}},\\
E'^{n}_{i,j-\frac{1}{2}}S_{j-\frac{1}{2}} &= \left[\frac{1}{2}\Big(\big(E_{ij} + E_{i,j-1}\big)\, n_{x_{j-\frac{1}{2}}} + \big(F_{ij} + F_{i,j-1}\big)\, n_{y_{j-\frac{1}{2}}}\Big) - \frac{1}{2}\alpha_{j-\frac{1}{2}}(U^{n}_{ij} - U^{n}_{i,j-1})\right] S_{j-\frac{1}{2}},
\end{aligned}$$

where

$$E_{ij} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (e+p)u \end{bmatrix}_{ij}, \qquad E_{i\pm1,j} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (e+p)u \end{bmatrix}_{i\pm1,j},$$

$$F_{ij} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (e+p)v \end{bmatrix}_{ij}, \qquad F_{i\pm1,j} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (e+p)v \end{bmatrix}_{i\pm1,j}.$$

7 Numerical Experiments

7.1 Methods and Grids Considered

1) **Lax-Friedrich's**

40x20
80x40
160x80
320x160
640x320

2) **MacCormack** with no diffusion

40x20
80x40
160x80

3) **MacCormack** with MacCormack-Baldwin artificial dissipation

40x20
80x40
160x80
320x160

7.2 Initial Conditions

We have chosen uniform initial conditions for all the experiments.

7.3 Boundary Conditions

The following boundary conditions were used for the internal flow in a 2D channel. The values at the ghost cells are updated once the calculations has been done in the interior of the domain.

Upper wall:

- In front of shock: $i < IS$

$$\begin{cases} \rho_{i,JL} = \rho_{i,JL-1}, \\ p_{i,JL} = p_{i,JL-1}, \\ u_{i,JL} = u_{i,JL-1}, \\ v_{i,JL} = -v_{i,JL-1}. \end{cases}$$

- Behind the shock: [6] $i \geq IS$

$$\begin{cases} \rho_{i,JL} = \rho_{i,JL-1}, \\ p_{i,JL} = p_{i,JL-1}, \\ u_{i,JL} = \cos 2\theta \, u_{i,JL-1} - \sin 2\theta \, v_{i,JL-1}, \\ v_{i,JL} = -\sin 2\theta \, u_{i,JL-1} - \cos 2\theta \, v_{i,JL-1}. \end{cases}$$

[6] The result is derived in Professor X. Zhong's lecture notes.

Lower wall:

$$\begin{cases} \rho_{i,0} = \rho_{i,1}, \\ p_{i,0} = p_{i,1}, \\ u_{i,0} = u_{i,1}, \\ v_{i,0} = -v_{i,1}. \end{cases}$$

Left entry conditions:

$$\begin{cases} \rho_{0,j} = \text{constant (fixed)}, \\ p_{0,j} = \text{constant (fixed)}, \\ u_{0,j} = \text{constant (fixed)}, \\ v_{0,j} = \text{constant (fixed)}. \end{cases}$$

Right exit conditions:

$$\begin{cases} \rho_{IL,j} = \rho_{IL-1,j}, \\ p_{IL,j} = p_{IL-1,j}, \\ u_{IL,j} = u_{IL-1,j}, \\ v_{IL,j} = v_{IL-1,j}. \end{cases}$$

7.4 Courant Number

The timestep dt is calculated adaptively.

$$\begin{aligned} dt_{ij} &= \left. \frac{\nu}{\frac{|u|}{dx} + \frac{|v|}{dy} + c\sqrt{\frac{1}{dx^2} + \frac{1}{dy^2}}} \right|_{ij}, \\ dt &= \min(dt_{ij}). \end{aligned}$$

In our experiments, $\nu = 0.7$.

7.5 Stopping Condition

The calculations are done while the following condition is true:

$$\frac{\sum_j \left|p^{n+1}_{IL-1,j} - p^{n}_{IL-1,j}\right|}{\sum_j \left|p^{n+1}_{IL-1,j}\right|} \;<\; 10^{-3}.$$

Note that we consider values of pressure p only on the last (rightmost) x-level in the interior of the computational domain.

7.6 Analysis and Observations

7.6.1 Lax-Friedrichs

Lax-Friedrich's method, a first order method, is always stable. However, it shows excessive dissipation when used on coarse grids. As a result, discontinuities are overly smeared. As grid gets refined, Lax-Friedrichs scheme resolves shocks better. However, in order to get a result that gives reasonably resolved shocks, the grid needs to be very fine, which is expensive to do. For example, to get the solution on a 640x320 grid, it took over 36 hours of computational time.

7.6.2 MacCormack

MacCormack's method is second order accurate for smooth data, and therefore, its approximations do not show the dissipative behavior that Lax-Friedrichs approximations do. However, since MacCormack is a second order method, it is dispersive and results in spurious oscillations. When used with no dissipation, MacCormack method is unstable. As displayed in figures, it gives very dispersive approximations for the first few iterations and later blows up. In order to overcome this problem, artificial viscosity has to be added to make MacCormack method stable. We added the MackCormack-Baldwin diffusion term (with $\varepsilon = 0.6$) for our numerical calculations. As a result, oscillations are smaller. MacCormack method with diffusion added gives more correct results in L_1-norm than Lax-Friedrichs method does. Also, even though MacCormack method takes more time to execute a single iteration, it gives convergence in fewer iterations. Since MacCormack method is second order accurate, we observed that coarser grids can be used to obtain an approximation which is more accurate than that of the Lax-Friedrichs approximation on finer grids. As the grid is refined, both Lax-Friedrichs and MacCormack method give *less* accurate approximations than their order would suggest. We would expect this, since the problem we considered has discontinuities (shocks), and both methods are at most first order accurate near discontinuities. Overall, MacCormack's approximation converges more rapidly to the exact solutions. Also, when we restrict our observations to the smooth regions, both Lax-Friedrichs and MacCormack methods give first and second order accurate solutions, respectively.

8 Appendix

8.1 Finite Difference Methods (Structured Grids)

We have used the finite volume formulation for the project. There is also a finite difference approach. The finite difference discretizations for MacCormack are given here for the rectangular grids and structured non-rectangular grids.

8.1.1 Cartesian Grid

MacCormack Method

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0.$$

$$\begin{aligned} U_{ij}^{*} &= U_{ij}^{n} - dt\left(\frac{E_{i+1,j}^{n} - E_{ij}^{n}}{dx} + \frac{F_{i,j+1}^{n} - F_{ij}^{n}}{dy}\right), && \text{(predictor)} \\ U_{ij}^{n+1} &= \frac{1}{2}\left(U_{ij}^{n} + U_{ij}^{*} - dt\left(\frac{E_{ij}^{*} - E_{i-1,j}^{*}}{dx} + \frac{F_{i,j}^{*} - F_{i,j-1}^{*}}{dy}\right)\right). && \text{(corrector)} \end{aligned}$$

8.1.2 Non-Cartesian Grid

MacCormack Method

$$\begin{aligned} U_{ij}^{*} &= U_{ij}^{n} - \frac{dt}{J_{ij}}\left(\frac{E_{i+1,j}^{\prime n} - E_{ij}^{\prime n}}{dx} + \frac{F_{i,j+1}^{\prime n} - F_{ij}^{\prime n}}{dy}\right), && \text{(predictor)} \\ U_{ij}^{n+1} &= \frac{1}{2}\left(U_{ij}^{n} + U_{ij}^{*} - \frac{dt}{J_{ij}}\left(\frac{E_{ij}^{\prime *} - E_{i-1,j}^{\prime *}}{dx} + \frac{F_{i,j}^{\prime *} - F_{i,j-1}^{\prime *}}{dy}\right)\right). && \text{(corrector)} \end{aligned}$$

where

$$\begin{aligned} E' &= y_\eta E - x_\eta F, \\ F' &= -y_\xi E + x_\xi F, \\ J &= x_\xi y_\eta - x_\eta y_\xi. \end{aligned}$$