# Monochromatic paths and the cohomology of ordinals

UCLA Logic seminar

Xiang Li

# Basic setup

**Classical Ramsey theory**

Given an edge coloring of a complete graph, must there be a sufficiently large monochromatic clique?

We replace cliques by sets connected by **monochromatic paths**.

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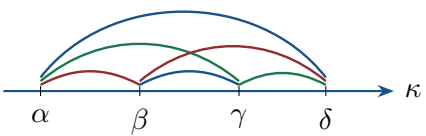
Given an edge coloring of a complete graph, must there be a sufficiently large monochromatic clique?

We replace cliques by sets connected by **monochromatic paths**.

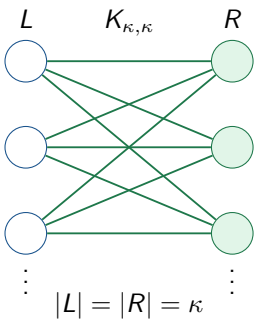
- $\kappa$ is a **regular uncountable cardinal**.
- Vertices are ordinals below $\kappa$.
- A coloring is $c : [\kappa]^2 \to \omega$.
- A subset of $\kappa$ is **small** if it has size less than $\kappa$.



# Highly connected Ramsey relations

## Highly connected graphs

A graph on $\kappa$ vertices is highly connected if it remains connected after any small deletion.



# Highly connected Ramsey relations

## Highly connected graphs

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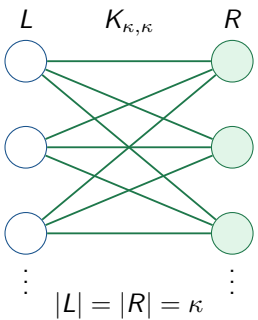
## Bounded-path relations (Hrušák–Shelah–Zhang)

$$\kappa \longrightarrow_{\mathrm{hc},<5} (\kappa)^2_\omega.$$

Every coloring has a set $B$ of size $\kappa$ and a color $i$ such that, after every small deletion $Z$, any two remaining vertices have an $i$-colored path **inside** $B \setminus Z$ of $< 5$ **edges**.



# Stationarity gives many choices

## Clubs and stationary subsets of $\kappa$

- A **club** is a closed unbounded subset of $\kappa$.
- A set is **stationary** if it meets every club.
- Removing a nonstationary set preserves stationarity.

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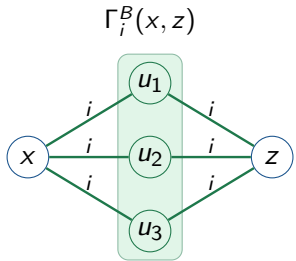
**Common neighbors in $B$ of color $i$**

$\Gamma_i^B(x,z) = \{u \in B : c(x,u) = c(z,u) = i\}$.



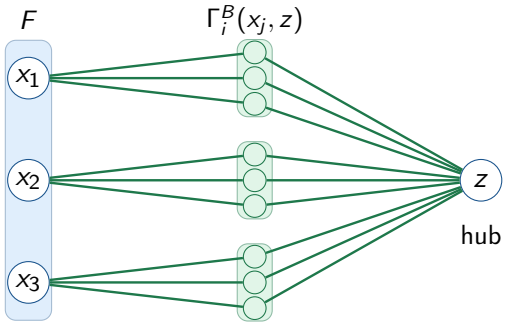

A stationary set of common neighbors.

# The stationary hub property

**Stationary hub property for $B \subseteq \kappa$ and color $i$**

For every small $F \subseteq B$,

$$\{z \in B : \text{for every } x \in F,\ \Gamma_i^B(x, z) \text{ is stationary}\} \text{ is stationary.} \qquad \text{(H)}$$



# The weakly compact collapse theorem

**Theorem (L)**

Suppose $\kappa$ is weakly compact and $\nu < \kappa$ is uncountable regular. After forcing with $\mathrm{Coll}(\nu, <\kappa)$, $\kappa = \nu^+$ and every $c : [\kappa]^2 \to \omega$ has property (H) on some $B, i$.

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**When $\nu = \omega_1$**

$$\omega_2 \longrightarrow_{\mathrm{hc},<5} (\omega_2)^2_\omega, \qquad \mathsf{PH}_1(\omega_2).$$

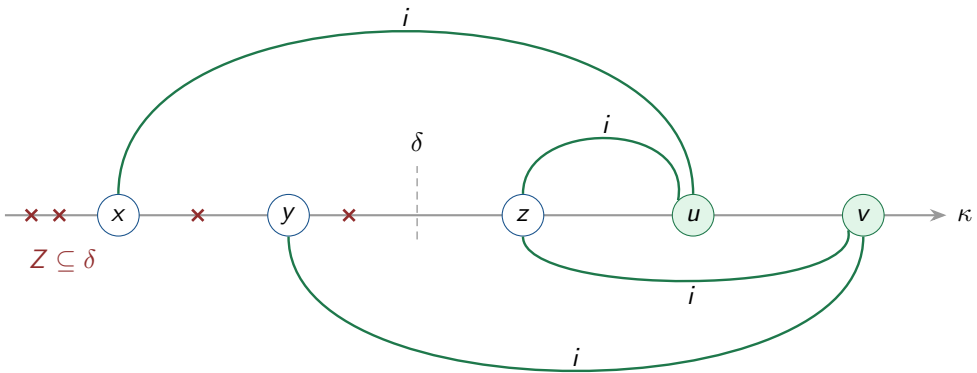
This answers questions of Hrušák–Shelah–Zhang and Bannister–Bergfalk–Moore–Todorcevic.

**When $\nu = \omega_2$**

Both conclusions hold at $\omega_3$. Later, we add a nontrivial two-coherent family while preserving both conclusions.

# Four edges survive deletions

Assume $B, i$ satisfy (H) for $c$. Let $Z \subseteq B$ be small and $x < y$ lie in $B \setminus Z$. Choose $\delta < \kappa$ with $Z \cup \{x, y\} \subseteq \delta$.



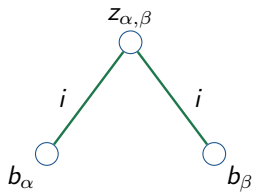

$x - u - z - v - y$ in $B \setminus Z$

# The first partition hypothesis

**$\mathrm{PH}_1(\kappa)$ (BBMT), in an equivalent form**

For every $d : \kappa^2 \to \omega$, there are a color $i$ and ordinals $b_\alpha, z_{\alpha,\beta} < \kappa$ such that

$$b_\alpha \geq \alpha, \qquad z_{\alpha,\beta} \geq b_\alpha, b_\beta,$$
$$d(b_\alpha, z_{\alpha,\beta}) = d(b_\beta, z_{\alpha,\beta}) = i \qquad (\alpha, \beta < \kappa).$$



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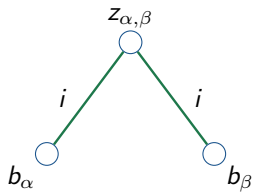



**From the stationary hub property to $\mathrm{PH}_1$**

Set $c(\{x, y\}) = d(x, y)$ for $x < y$ and get (H) on some $B, i$.

- Recursively choose $b_\alpha \geq \alpha$ in $B$ as a hub for $\{b_\xi : \xi < \alpha\}$, above all earlier choices.
- Choose $z_{\alpha,\beta} > b_\alpha, b_\beta$ in $\Gamma_i^B(b_\alpha, b_\beta)$ for $\alpha \neq \beta$. Set $z_{\alpha,\alpha} = z_{\alpha,\alpha+1}$.

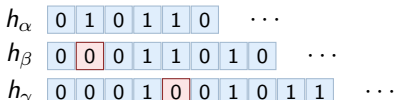
# Gluing coherent functions

**Local functions, compatible up to finite errors**

$$h_\alpha : \alpha \longrightarrow \{0,1\}, \qquad h_\beta \restriction \alpha =^* h_\alpha \quad (\alpha < \beta < \kappa).$$

Here $=^*$ means equality except at finitely

Exact agreement on overlaps would give a union. Finite errors can prevent this.



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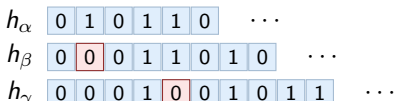


## Triviality

The family is trivial if some $h : \kappa \to \{0,1\}$ satisfies $h \restriction \alpha =^* h_\alpha$ for every $\alpha < \kappa$.

# A common upper row bounds the errors

Coherence gives the coloring $d : \kappa^2 \to \omega$ defined by

$$d(a, b) = \big|\{\xi < \min(a, b) : h_a(\xi) \neq h_b(\xi)\}\big|.$$

By $\mathrm{PH}_1$, choose $b_\alpha \geq \alpha$ and a color $k$ such that

$$z_{\alpha,\beta} \geq b_\alpha, b_\beta,$$
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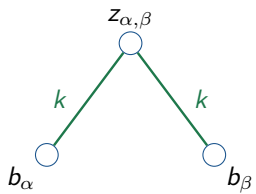

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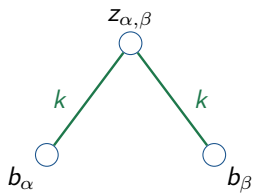


**Uniform bound**

For $\xi < \min(b_\alpha, b_\beta)$, if $h_{b_\alpha}(\xi) \neq h_{b_\beta}(\xi)$, at least one differs from $h_{z_{\alpha,\beta}}(\xi)$. Hence,

$$d(b_\alpha, b_\beta) \leq d(b_\alpha, z_{\alpha,\beta}) + d(b_\beta, z_{\alpha,\beta}) = 2k.$$

## Compactness glues the selected rows

Let $N = 2k$. For every $\alpha < \kappa$ and $E \in [b_\alpha]^{N+1}$, require

$$\bigvee_{\xi \in E} \left(h(\xi) = h_{b_\alpha}(\xi)\right)$$

This permits at most $N$ errors per selected row.

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**Finite satisfiability**

A finite collection involves rows $b_{\alpha_1}, \ldots, b_{\alpha_m}$. Choose $b_\gamma$ above them all. By the uniform bound, $h_{b_\gamma}$ satisfies every requirement.

Propositional compactness gives $h : \kappa \to \{0, 1\}$ satisfying these requirements.

Since $b_\alpha \geq \alpha$, for every $\alpha < \kappa$,

$$h \restriction \alpha =^* h_{b_\alpha} \restriction \alpha =^* h_\alpha.$$

## First cohomology measures the failure to glue

For an abelian group $A$, coherent families and trivial families form groups under pointwise addition.

**Theorem (Bergfalk–Lambie-Hanson)**

$$\check{H}^1(\kappa, A_d) \cong \frac{\{\text{one-coherent } A\text{-valued families}\}}{\{\text{trivial families}\}}.$$

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**The degree-one implication**

$$\mathrm{PH}_1(\kappa) \implies \check{H}^1(\kappa, A_d) = 0 \qquad \text{for every abelian group } A.$$

**General theorem (L)**

$$\mathrm{PH}_n(\kappa) \implies \check{H}^n(\kappa, A_d) = 0 \qquad (1 \leq n < \omega).$$

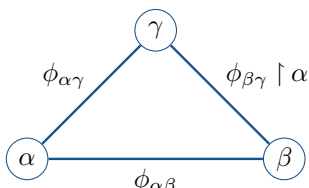
The ordinal has its order topology. $A_d$ is the sheaf of locally constant $A$-valued functions.

# Functions on edges

Let $\phi_{\alpha\beta} : \alpha \to A$ for $\alpha < \beta < \kappa$.

**Two-coherence around every triangle**

$$\phi_{\alpha\beta} + \phi_{\beta\gamma} \restriction \alpha =^* \phi_{\alpha\gamma} \qquad (\alpha < \beta < \gamma).$$



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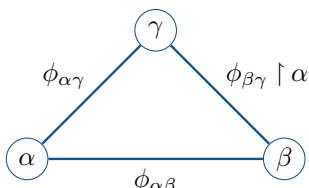


**Triviality.** There are vertex functions $\psi_\alpha : \alpha \to A$ with

$$\phi_{\alpha\beta} =^* \psi_\beta \restriction \alpha - \psi_\alpha.$$

A nontrivial family represents a nonzero class in $\check{H}^2(\kappa, A_d)$.

# The separation theorem at $\omega_3$

**Theorem (L)**

Starting from one weakly compact cardinal $\kappa$, we can force a model in which every $c : [\omega_3]^2 \to \omega$ admits $B \subseteq \omega_3$ and $i < \omega$ satisfying (H), and for every nontrivial abelian group $A$,

$$\check{H}^1(\omega_3, A_d) = 0, \qquad \check{H}^2(\omega_3, A_d) \neq 0.$$

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Since $\mathrm{PH}_2$ would imply $\check{H}^2 = 0$, we obtain

$$\boxed{\mathrm{PH}_1(\omega_3) + \neg\, \mathrm{PH}_2(\omega_3)}$$

First force with $\mathrm{Coll}(\omega_2, <\kappa)$, then add a two-coherent family so that every coloring still admits $B, i$ satisfying (H).

## An extra vertex suggests common neighbors

Let $G \subseteq \mathrm{Coll}(\nu, <\kappa)$ be generic and set $\mathbb{V} = V[G]$.
Fix $c : [\kappa]^2 \to \omega$ in $\mathbb{V}$. We need $B \subseteq \kappa$ and $i < \omega$ satisfying (H).

### The elementary embedding

In $\mathbb{V}$, there are a model $\mathbb{M} \prec H(\kappa^+)^{\mathbb{V}}$ containing $c$ and a countably closed forcing $\mathbb{Q}$ adding an elementary embedding

$$j : \mathbb{M} \longrightarrow \mathbb{N}, \qquad \mathrm{crit}(j) = \kappa.$$

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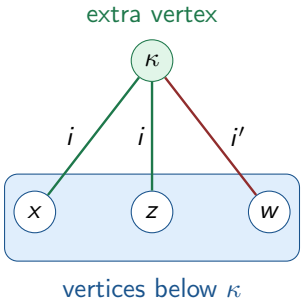


### Colors at $\kappa$

$$f(\alpha) = j(c)(\alpha, \kappa) \qquad (\alpha < \kappa).$$

If $f(x) = f(z) = i$, then $x - \kappa - z$ has color $i$. We need common neighbors below $\kappa$ to replace this extra vertex.

## Replacing the extra vertex

To build $B$, we first use $Y_i = f^{-1}\{i\}$ as **temporary vertex sets**.

Fix distinct $x, z \in Y_i$. Then, $\Gamma_i^\kappa(x, z) \in \mathbb{M}$.

### Stationarity in $\mathbb{V}$

- Since $j(c)(x, \kappa) = j(c)(z, \kappa) = i$, $\kappa \in j(\Gamma_i^\kappa(x, z))$.

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- Since $j(c)(x,\kappa) = j(c)(z,\kappa) = i$, $\kappa \in j(\Gamma_i^\kappa(x,z))$.
- If $\Gamma_i^\kappa(x,z)$ were nonstationary, some club $C \in \mathbb{M}$ would be disjoint from it, contradicting $\kappa \in j(C) \cap j(\Gamma_i^\kappa(x,z))$.

**From external to internal neighbors**

These common neighbors may lie outside $Y_i$.

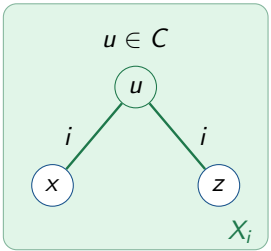
We next select sets $Y_i$ with enough common color-$i$ neighbors **inside** $Y_i$.

# Selecting sets with internal neighbors

**Extraction lemma (L)**

In the extension by $\mathbb{Q}$, choose $X_i \in \{Y_i, \varnothing\}$ such that

- $A \subseteq \bigcup_{i<\omega} X_i$ for some $A$ stationary in $\mathbb{V}$.
- For every club $C$ in $\mathbb{V}$, any two vertices in $X_i$ have a common color-$i$ neighbor in $X_i \cap C$.



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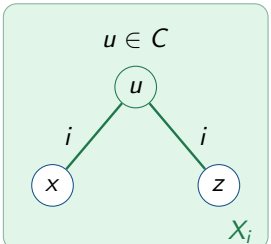


**Constructing $B$ in $\mathbb{V}$**

The $X_i$ need not be in $\mathbb{V}$. In $\mathbb{V}$, define

$$U_i(q) = \{\alpha < \kappa : \exists r \leq q \ (r \Vdash \alpha \in \dot{X}_i)\}.$$

We will take $B = U_i(p)$ for suitable $p, i$. Common neighbors forced into $\dot{X}_i$ below $p$ are then in $B$.

# The overlap needed for (H)

**Overlap lemma (L)**

In this setting, there are $p \in \mathbb{Q}$ and $i < \omega$ such that

$$\bigcap_{q \in D} U_i(q) \text{ is stationary} \qquad (\varnothing \neq D \in [\mathbb{Q} \downarrow p]^{<\kappa}).$$

**Proof of (H) for $B = U_i(p)$ and $i$.** For $F \in [B]^{<\kappa}$, choose

$$q_x \leq p, \qquad q_x \Vdash x \in \dot{X}_i \qquad (x \in F).$$

Then, $H = B \cap \bigcap_{x \in F} U_i(q_x)$ is stationary.

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Then, $H = B \cap \bigcap_{x \in F} U_i(q_x)$ is stationary.

Fix $z \in H \setminus F$, $x \in F$, and a club $C \subseteq \kappa$. The internal-neighbor property gives

$$\exists r \leq q_x \; \exists u \in C : \quad r \Vdash \big(\{x, z, u\} \subseteq \dot{X}_i \;\wedge\; c(x,u) = c(z,u) = i\big).$$

Hence,

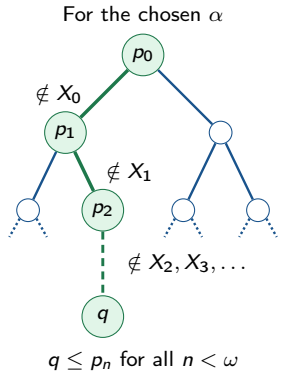
$$r \leq p \implies u \in U_i(p) = B \implies u \in \Gamma_i^B(x,z) \cap C.$$

Thus, every $z \in H \setminus F$ is a hub for $F$, so $B, i$ satisfy (H).

# A baby version of the overlap lemma

## Binary splitting

Suppose the two-condition overlap fails. At each node $p$ on level $n$, choose $q_0, q_1 \leq p$ with $U_n(q_0) \cap U_n(q_1)$ nonstationary.



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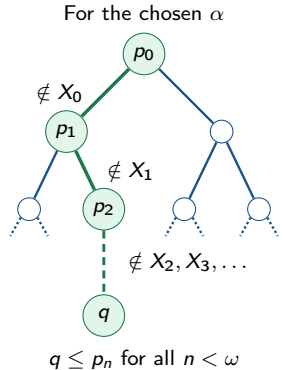
Suppose the two-condition overlap fails. At each node $p$ on level $n$, choose $q_0, q_1 \leq p$ with $U_n(q_0) \cap U_n(q_1)$ nonstationary.

- Build the whole binary tree. It is countable, so the union of its exceptional sets is nonstationary.
- Choose $\alpha \in A$ outside that union.
- At step $n$, follow a child forcing $\alpha \notin \dot{X}_n$.
- Countable closure gives a lower bound $q$, forcing

$$\alpha \notin \bigcup_{n<\omega} \dot{X}_n.$$

This contradicts the cover of $A$.




The same proof with $< \kappa$ children per node gives the full overlap for (H).

## Adding $\Phi$ while every coloring admits $B, i$ satisfying (H)

Let $R$ add a nontrivial two-coherent family $\Phi = \bigcup r$, where $r \subseteq R$ is generic.

$$V \xrightarrow{\mathrm{Coll}(\omega_2, <\kappa)} \mathbb{V} \xrightarrow{R} U = \mathbb{V}[r].$$

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**Lemma (L)**

In $U$, with $\kappa = \omega_3$, every coloring $c : [\kappa]^2 \to \omega$ belongs to a model $W \prec H(\kappa^+)^U$ such that a countably closed forcing adds an elementary embedding

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**Lemma (L)**

In $U$, with $\kappa = \omega_3$, every coloring $c : [\kappa]^2 \to \omega$ belongs to a model $W \prec H(\kappa^+)^U$ such that a countably closed forcing adds an elementary embedding

$$j : W \longrightarrow W', \qquad \mathrm{crit}(j) = \kappa.$$

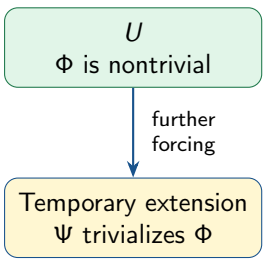
In $U$, every $c : [\omega_3]^2 \to \omega$ admits $B, i$ satisfying (H).

**Nonvanishing in $U$**

Use $R$ with coefficients in

$$A_* = \mathbb{Z} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}.$$

Then, $\check{H}^2(\omega_3, A_d) \neq 0$ for every $A \neq 0$.



## Why the lift needs a trivializer

In a further extension of $U$, we lift

$$j : \mathbb{M} \longrightarrow \mathbb{N} \quad \text{to} \quad j^* : \mathbb{M}[r] \longrightarrow \mathbb{N}[r^*].$$

**The $\kappa$-th row of a lift**

For a lift $j^*$, let $\phi_{\alpha\kappa} = (j^*(\Phi))_{\alpha\kappa}$. For $\alpha < \beta < \kappa$,

$$\phi_{\alpha\beta} + \phi_{\beta\kappa} \restriction \alpha =^* \phi_{\alpha\kappa}.$$

Thus, $\psi_\alpha = -\phi_{\alpha\kappa}$ satisfies

$$\phi_{\alpha\beta} =^* \psi_\beta \restriction \alpha - \psi_\alpha.$$

Since $\Phi$ is nontrivial in $U$, a lift requires a trivializer outside $U$.

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**A trivializer supplies the row**

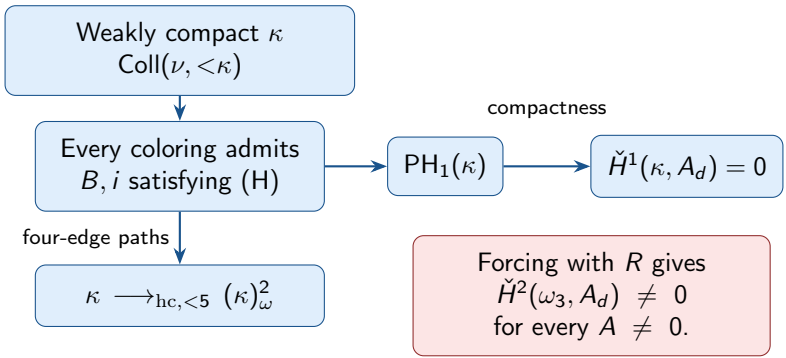
A further forcing over $U$ adds a trivializer $\Psi$ of $\Phi$. The construction also gives $\Phi, \Psi \in \mathbb{N}$. Define $m$ by

$$m \restriction [\kappa]^2 = \Phi, \qquad m_{\alpha\kappa} = -\psi_\alpha.$$

Then, $m \in \mathbb{N}$, and its two-coherence gives $m \in j(R)$. Moreover, $m \leq j(s) = s$ for every $s \in r$.

In this further extension, force with $j(R) \downarrow m = \{q \in j(R) : q \leq m\}$. The upward closure $r^*$ of the generic filter in $j(R)$ is $\mathbb{N}$-generic and contains $j``r$.

# Two consequences of the same combinatorics



# Can four edges be reduced to two?

**Open upper-bound question**

Starting from a weakly compact cardinal, can we force

$$\omega_2 \longrightarrow_{\mathrm{hc},<3} (\omega_2)^2_\omega \ ?$$

- Two edges must suffice after every small deletion.

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- Two edges must suffice after every small deletion.
- A stronger Ramsey hypothesis gives such a model (L), answering HSZ's two-edge consistency question (8.2).
- $\mathrm{PH}_1$ allows common upper neighbors outside $\{b_\alpha : \alpha < \kappa\}$. The two-edge relation requires the common neighbor to lie **inside the surviving graph**.

Thank you!