Weak Vopěnka's principle and cotorsion-pair completeness

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Abstract

Weak Vopěnka's principle (WVP) implies that every cotorsion pair over every ring is complete (CPC), and CPC implies generic weak Vopěnka's principle (gWVP) for every set-sized relational language. If there is an ω_1-strongly compact cardinal or if some set-forcing extension satisfies the P-ideal dichotomy or $\mathrm{ISP}(\mu)$ for some regular $\mu \geq \aleph_2$, then CPC implies WVP. Moreover, WVP holds iff for every continuous inverse tower $(P_\alpha, p_{\alpha\beta})_{\alpha\leq\beta\in\mathrm{Ord}}$ of abelian groups, some $p_{\alpha,\alpha+1}$ has a section. Let I be the class of completely ineffable cardinals. Under gWVP, for every ordinal θ there is a $\kappa \in I$ above θ with $I \cap \kappa$ completely ineffable in κ. Every set forcing preserves CPC.

Introduction

We assume global choice. Let WVP denote weak Vopěnka's principle and gWVP denote generic weak Vopěnka's principle for set-sized relational languages. Thus, ¬gWVP holds iff for some set-sized language $\mathcal{L}$ of finitary relations there are $\mathcal{L}$-structures $(H_\alpha)_{\alpha\in\mathrm{Ord}}$ and homomorphisms $p_{\alpha\beta} : H_\beta \to H_\alpha$ in V for all $\alpha \leq \beta$ such that in every set-forcing extension $V[G]$,

$$\mathrm{Hom}_{\mathcal{L}}^{V[G]}(H_\beta, H_\alpha) = \begin{cases} \{p_{\alpha\beta}\} & \alpha \leq \beta, \\ \varnothing & \beta < \alpha. \end{cases}$$

Let CPC mean that for every ring R, every cotorsion pair in the category Mod-R of right R-modules is complete. An extension of A by B is a short exact sequence $0 \to B \xrightarrow{i} E \xrightarrow{q} A \to 0$. In Mod-$R$, exactness means $\ker i = 0$, $i[B] = \ker q$ and $q[E] = A$. The sequence splits iff $qs = \mathrm{id}_A$ for some $s : A \to E$ (i.e., q has a section s). In an abelian category, $A \perp B$ means that every extension of A by B splits, equivalently $\mathrm{Ext}^1(A, B) = 0$. A cotorsion pair consists of two classes $\mathcal{A}, \mathcal{B}$ with

$$\mathcal{A} = {}^{\perp}\mathcal{B} = \{A : (\forall B \in \mathcal{B})\ A \perp B\},\ \mathcal{B} = \mathcal{A}^{\perp} = \{B : (\forall A \in \mathcal{A})\ A \perp B\}.$$

In Mod-R, $(\mathcal{A}, \mathcal{B})$ is complete iff every M has a special $\mathcal{A}$-precover, that is, a map q in a short exact sequence $0 \to B \to A \xrightarrow{q} M \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ [17, Lemma 1.2].

Salce defined cotorsion pairs of abelian groups in 1979 [14]. Salce's problem is whether every cotorsion pair over $\mathbb{Z}$ is complete [3]. By the Eklof–Trlifaj theorem, $(\mathcal{A}, \mathcal{B})$ is complete if $\mathcal{B} = \mathcal{S}^{\perp}$ for some set $\mathcal{S}$ of modules [5, Theorem 10]. Eklof and Shelah give a model of ZFC + GCH in which some abelian group has no special ${}^{\perp}\{\mathbb{Z}\}$-precover [4, Theorem 0.4]. Relative to the consistency of Vopěnka's principle, Cox obtains a model in which every cotorsion pair over every right hereditary ring is complete. Cox also asks whether ZFC gives an example of a ring with an incomplete cotorsion pair [3].

1 WVP and splitting in inverse towers

Let Ord be the class of ordinals. Let $T = (P_\alpha, p_{\alpha\beta})_{\alpha\le\beta\in\mathrm{Ord}}$ be an inverse tower in Mod-R, which means $p_{\alpha\beta} : P_\beta \to P_\alpha$, $p_{\alpha\alpha} = \mathrm{id}_{P_\alpha}$, and $p_{\alpha\gamma} = p_{\alpha\beta} \circ p_{\beta\gamma}$ for all $\alpha \le \beta \le \gamma$. At each nonzero limit λ, define

$$\varprojlim_{\xi<\lambda} P_\xi = \left\{ x \in \prod_{\xi<\lambda} P_\xi : p_{\xi\zeta}(x(\zeta)) = x(\xi) \text{ for all } \xi \le \zeta < \lambda \right\}.$$

The tower is continuous iff $z \mapsto (p_{\xi\lambda}(z))_{\xi<\lambda}$ is an isomorphism $P_\lambda \to \varprojlim_{\xi<\lambda} P_\xi$ at every nonzero limit λ.

Let **Graph** be the category of directed graphs. A directed graph is a set with a binary relation, and a graph homomorphism is a function preserving the relation. In the category **Ord** of ordinals there is one arrow $\alpha \to \beta$ when $\alpha \le \beta$, and none otherwise. Then, ¬WVP iff there is a full embedding $\mathbf{Ord}^{\mathrm{op}} \to \mathbf{Graph}$, equivalently a sequence of graphs $(H_\alpha)_{\alpha\in\mathrm{Ord}}$ with

$$|\operatorname{Hom}(H_\beta, H_\alpha)| = \begin{cases} 1 & \alpha \le \beta, \\ 0 & \beta < \alpha. \end{cases}$$

Theorem 1.1. *The following are equivalent.*

1. WVP.
2. *For every ring R and every continuous inverse tower $(P_\alpha, p_{\alpha\beta})_{\alpha\le\beta\in\mathrm{Ord}}$ in* Mod-R*, some $p_{\alpha,\alpha+1}$ has a section $s : P_\alpha \to P_{\alpha+1}$.*
3. *Statement (2) with $R = \mathbb{Z}$.*

Proof. (1) ⇒ (2). Assume WVP. Let T be a continuous tower in Mod-R.

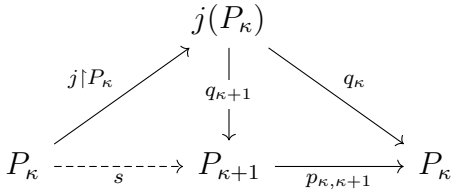
Let $\mathcal{L}$ be the language of set theory with two constant symbols c_R, c_T. For each $\mu \in \mathrm{Ord}$, define an $\mathcal{L}$-structure $\mathcal{M}_\mu = (V_{\theta_\mu}, \in, R, T_\mu)$ where $T_\mu = (\langle P_\xi : \xi \le \mu\rangle, \langle p_{\xi\zeta} : \xi \le \zeta \le \mu\rangle)$ and $\theta_\mu = \mathrm{rank}(\{R, T_\mu\}) + \omega$.

Let $C = \langle \mathcal{M}_\mu : \mu \in \mathrm{Ord}\rangle$. By WVP, choose a C-strong cardinal κ [19, Theorem 1.6]. Choose a successor ordinal λ with $(\kappa+1, \mathcal{M}_{\kappa+1}) \in V_\lambda$. By [19, Definition 1.3 and the following observation], there is an elementary embedding $j : V \to N \supseteq V_\lambda$ with $\mathrm{crit}(j) = \kappa$, $\tau = j(\kappa) > \lambda$ and $j(C) \cap V_\lambda = C \cap V_\lambda$.

Then, $j(C)(\kappa+1) = \mathcal{M}_{\kappa+1}$. By elementarity, $j(C)(\tau) = j(\mathcal{M}_\kappa)$. Since $\kappa+1 < \tau$, $j(R) = R$ and $j(T_\kappa) \restriction (\kappa+2) = T_{\kappa+1}$ by elementarity again. For each $\xi \le \kappa+1$, let $q_\xi : j(P_\kappa) \to P_\xi$ be the map from stage τ to ξ in $j(T_\kappa)$.

For each $\xi < \kappa$, let $\gamma = \mathrm{rank}(\{R, P_\xi\})$. Since $j(\xi) = \xi$, $j(P_\xi) = P_\xi$. Thus, $j(\gamma) = \gamma < \lambda < j(\kappa)$, so $\gamma < \kappa$. Thus, $j(r) = r$ for all $r \in R$ and $j(x) = x$ for all $x \in P_\xi$.

Let $s = q_{\kappa+1} \circ (j \restriction P_\kappa)$. The following diagram commutes because $j(T_\kappa) \restriction (\kappa+2) = T_{\kappa+1}$.



For each $z \in P_\kappa$ and each $\xi < \kappa$, $q_\xi = j(p_{\xi\kappa})$ since $j(\xi) = \xi$. Thus,

$$p_{\xi\kappa}q_\kappa(j(z)) = q_\xi(j(z)) = j(p_{\xi\kappa}(z)) = p_{\xi\kappa}(z).$$

By continuity at κ, $q_\kappa(j(z)) = z$. The map s is R-linear since $q_{\kappa+1}$ is R-linear, $j(u+v) = j(u) + j(v)$ and $j(ur) = j(u)j(r) = j(u)r$ for all $u, v \in P_\kappa$ and $r \in R$. Hence, s is a section.

(2) $\Rightarrow$ (3). Let $R = \mathbb{Z}$.

(3) $\Rightarrow$ (1). Suppose $\neg$WVP. Let $G : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Graph}$ be a full embedding. By [13, Theorem 3.14], there is a functor $A : \mathbf{Graph} \to \mathbf{Ab}$ and natural isomorphisms

$$\gamma_{X,Y} : \mathbb{Z}[\mathrm{Hom}_{\mathbf{Graph}}(X, Y)] \xrightarrow{\cong} \mathrm{Hom}(A(X), A(Y)).$$

For $u : X \to Y$, naturality gives the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[\mathrm{Hom}_{\mathbf{Graph}}(X,X)] & \xrightarrow{\gamma_{X,X}} & \mathrm{Hom}(A(X),A(X)) \\ {\scriptstyle \mathbb{Z}[u\circ -]}\downarrow & & \downarrow{\scriptstyle A(u)\circ -} \\ \mathbb{Z}[\mathrm{Hom}_{\mathbf{Graph}}(X,Y)] & \xrightarrow[\gamma_{X,Y}]{} & \mathrm{Hom}(A(X),A(Y)) \end{array}$$

By injectivity of $\gamma_{X,Y}$, $0 \neq \gamma_{X,Y}(u) = A(u) \circ \gamma_{X,X}(\mathrm{id}_X)$. Thus, $A(u) \neq 0$.

Let $A_\alpha = A(G(\alpha))$. For each $\alpha \leq \beta$, let $a_{\alpha\beta} : A_\beta \to A_\alpha$ be the image of $\beta \to \alpha$ under $A \circ G$. Then, $a_{\alpha\beta} \neq 0$. Since G is full and faithful, $\mathrm{Hom}(A_\alpha, A_{\alpha+1}) = 0$.

For each ordinal α, let

$$P_\alpha = \left\{ (x,t) \in A_0 \times \prod_{\xi<\alpha} A_{\xi+1} : a_{0,\xi+1}(t(\xi)) = x \text{ for all } \xi < \alpha \right\}.$$

Identify P_0 with A_0. Define $p_{\alpha\beta}(x,t) = (x, t \restriction \alpha)$ for all $\alpha \leq \beta$. At each nonzero limit λ, the inverse of $P_\lambda \to \varprojlim_{\alpha<\lambda} P_\alpha$ is

$$\Big((x_\alpha, t_\alpha)\Big)_{\alpha<\lambda} \longmapsto \left(x_0, \bigcup_{\alpha<\lambda} t_\alpha\right),$$

since $x_\beta = x_\alpha$ and $t_\beta \restriction \alpha = t_\alpha$ for all $\alpha \leq \beta < \lambda$.

By (3), choose a section s of $p_{\alpha,\alpha+1}$ for some α. Define $f : A_\alpha \to P_\alpha$ by

$$f(y) = (a_{0\alpha}(y), (a_{\xi+1,\alpha}(y))_{\xi<\alpha}).$$

Indeed, $a_{0,\xi+1}a_{\xi+1,\alpha} = a_{0\alpha}$ for every $\xi < \alpha$. Let $\pi : P_{\alpha+1} \to A_{\alpha+1}$ be defined by $\pi(x,t) = t(\alpha)$. Since $p_{\alpha,\alpha+1}s = \mathrm{id}$ and $a_{0,\alpha+1}\pi(x,t) = x$, we get

$$\begin{array}{ccccccc} A_\alpha & \xrightarrow{f} & P_\alpha & \xrightarrow{s} & P_{\alpha+1} & \xrightarrow{\pi} & A_{\alpha+1} \\ {\scriptstyle a_{0\alpha}}\downarrow & & {\scriptstyle p_{0\alpha}}\downarrow & & \downarrow{\scriptstyle p_{0,\alpha+1}} & & \downarrow{\scriptstyle a_{0,\alpha+1}} \\ A_0 & = & A_0 & = & A_0 & = & A_0 \end{array}$$

But $\pi s f \in \mathrm{Hom}(A_\alpha, A_{\alpha+1}) = 0$, so $a_{0\alpha} = a_{0,\alpha+1}\pi s f = 0$, a contradiction to $a_{0\alpha} \neq 0$. $\square$

Corollary 1.2. WVP *implies* CPC.

Proof. Assume WVP. Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in Mod-R and let $P_0 = M$ be any right R-module. At each stage α let $p_{\alpha\alpha} = \mathrm{id}_{P_\alpha}$ and stop if $P_\alpha \in \mathcal{A}$. Otherwise $\mathrm{Ext}^1_R(P_\alpha, B_\alpha) \neq 0$ for some $B_\alpha \in \mathcal{B}$. Choose a nonsplit short exact sequence

$$0 \longrightarrow B_\alpha \longrightarrow P_{\alpha+1} \xrightarrow{p_{\alpha,\alpha+1}} P_\alpha \longrightarrow 0.$$

Define $p_{\xi,\alpha+1} = p_{\xi\alpha} \circ p_{\alpha,\alpha+1}$ for all $\xi < \alpha$. At each nonzero limit λ, let $P_\lambda = \varprojlim_{\xi<\lambda} P_\xi$ and $p_{\xi\lambda}(x) = x(\xi)$ for each $\xi < \lambda$.

For all stages $\alpha < \beta$ of the recursion, $p_{\alpha\beta}$ is onto. Fix $x_\alpha \in P_\alpha$. Let $x_\xi = p_{\xi\alpha}(x_\alpha)$ for all $\xi < \alpha$. Recursively choose

$$\begin{aligned} x_{\xi+1} &\in p^{-1}_{\xi,\xi+1}(\{x_\xi\})\,(\alpha \leq \xi < \beta),\\ x_\lambda &= (x_\xi)_{\xi<\lambda}\,(\alpha < \lambda \leq \beta,\ \lambda \text{ a limit}). \end{aligned}$$

At a successor $\gamma+1$, $p_{\xi,\gamma+1}(x_{\gamma+1}) = p_{\xi\gamma}(x_\gamma) = x_\xi$ for every $\xi \leq \gamma$. At a limit λ, the preceding coordinates are compatible, so $x_\lambda \in P_\lambda$ and $p_{\xi\lambda}(x_\lambda) = x_\xi$. Thus, $p_{\alpha\beta}(x_\beta) = x_\alpha$.

The recursion stops at some η with $P_\eta \in \mathcal{A}$. Otherwise Theorem 1.1 gives a section of a nonsplit $p_{\alpha,\alpha+1}$. Let $K_\alpha = \ker p_{0\alpha}$ for each $\alpha \leq \eta$. Then, $K_0 = 0$. Since $p_{0\beta} = p_{0\alpha}p_{\alpha\beta}$ and $p_{\alpha\beta}$ is onto, $p_{\alpha\beta}[K_\beta] = K_\alpha$. For each $\alpha < \eta$, $\ker p_{\alpha,\alpha+1} \subseteq K_{\alpha+1}$, so restriction gives

$$0 \longrightarrow B_\alpha \longrightarrow K_{\alpha+1} \xrightarrow{p_{\alpha,\alpha+1}} K_\alpha \longrightarrow 0.$$

At each limit $0 < \lambda \leq \eta$, $p_{0\xi}(x(\xi)) = x(0)$ for all $x \in P_\lambda$ and $\xi < \lambda$. Hence,

$$K_\lambda = \{x \in P_\lambda : x(0) = 0\} = \{x \in P_\lambda : (\forall \xi < \lambda)\ x(\xi) \in K_\xi\} = \varprojlim_{\xi<\lambda} K_\xi.$$

Since $B_\alpha \in \mathcal{B}$ for all $\alpha < \eta$, [17, Lemma 2.3] gives $K_\eta \in \mathcal{A}^\perp = \mathcal{B}$. Hence, $p_{0\eta}$ is a special $\mathcal{A}$-precover in

$$0 \longrightarrow K_\eta \longrightarrow P_\eta \xrightarrow{p_{0\eta}} M \longrightarrow 0. \qquad \square$$

2 CPC, WVP and precovers over one ring

A map $q : A \to X$ is an $\mathcal{A}$-precover if $A \in \mathcal{A}$ and $\{qg : g \in \mathrm{Hom}(P, A)\} = \mathrm{Hom}(P, X)$ for every $P \in \mathcal{A}$.

Let $0 \to B \to A \xrightarrow{q} X \to 0$ be exact. For each $f : P \to X$, let $P \times_X A = \{(y, a) \in P \times A : f(y) = q(a)\}$. The rows are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & P \times_X A & \xrightarrow{\pi_P} & P & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & B & \longrightarrow & A & \xrightarrow[q]{} & X & \longrightarrow & 0. \end{array}$$

The sections of π_P are the maps (id_P, g) with $qg = f$. If $P \perp B$, the first row splits. Thus, every special $\mathcal{A}$-precover is an $\mathcal{A}$-precover.

Let $k = \mathbb{F}_2$. For each infinite cardinal κ, let $\mathcal{G}_\kappa$ be the category with one representative of each isomorphism class of graphs of size $\leq \kappa$ and with injective graph homomorphisms as arrows. Let R_κ be a ring with k-basis $\{1\} \sqcup \mathrm{Mor}(\mathcal{G}_\kappa)$, with $1r = r1 = r$. For all arrows a, b, define $ba = b \circ a$ if the codomain of a is the domain of b, and $ba = 0$ otherwise. Define $\underline{k} = \bigoplus_{C \in \mathcal{G}_\kappa} k\,\delta_C \in \text{Mod-}R_\kappa$ where δ_C's are formal k-basis vectors. Each arrow $a : C' \to C$ acts by $\delta_C a = \delta_{C'}$ and by zero on the other basis vectors. Let $H_0 = (\kappa, \kappa \times \kappa)$ be the representative of its isomorphism class. Every $C \in \mathcal{G}_\kappa$ admits an injective homomorphism to H_0, so $\underline{k} = \delta_{H_0} R_\kappa$.

For each poset D, let $D^{[2]} = \{(x, y) \in D^2 : x < y\}$. A coloring $c : D^{[2]} \to \omega$ is *subadditive* if for all $x < y < z$ in D,

$$c(x, z) \leq \max\{c(x, y), c(y, z)\},$$
$$c(x, y) \leq \max\{c(x, z), c(y, z)\}.$$

For each $x \in D$, we denote $c(x, x) = 0$. The coloring is *strongly unbounded* if $c[A^{[2]}]$ is unbounded in ω for every cofinal $A \subseteq D$ [12, Definition 4.1].

Theorem 2.1. *Let $\kappa \geq \aleph_1$ be a cardinal. Assume that for every set X of size $> \kappa$ there is no strongly unbounded subadditive coloring $([X]^{\leq \kappa})^{[2]} \to \omega$. The following are equivalent.*

1. WVP.
2. CPC.
3. *For every cotorsion pair $(\mathcal{A}, \mathcal{B})$ over R_κ, $\underline{k}$ has an $\mathcal{A}$-precover.*

For example, the narrow system property on $[X]^{\leq \kappa}$ implies the coloring hypothesis at X [12, Corollary 4.3]. The coloring hypothesis holds if κ^+ is ω_1-strongly compact; under the P-ideal dichotomy or $\mathrm{ISP}(\aleph_2)$, one can take $\kappa = \aleph_1$ (Proposition B.1).

In Corollary A.1, the same equivalence holds with (S, N) in place of $(R_\kappa, \underline{k})$, where S is right coherent and $0 \to S \to S \to N \to 0$ is exact.

Proof. (1) $\Rightarrow$ (2) is Corollary 1.2. (2) $\Rightarrow$ (3) since every special $\mathcal{A}$-precover is an $\mathcal{A}$-precover.

(3) $\Rightarrow$ (1). Suppose $\neg$WVP. Let $H : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Graph}$ be a full embedding. Then, $H(\xi)$ are pairwise nonisomorphic. By pigeonhole, recursively define a subsequence $(H_\alpha)_{\alpha > 0}$ such that $\kappa < |H_\alpha| < |H_\beta|$ for all $0 < \alpha < \beta$. To get a contradiction, we aim for a precover $P \to \underline{k}$ such that $\forall \beta > 0, |H_\beta| \leq |P|$.

Let $\mathrm{Inj}(C, G)$ be the set of injective graph homomorphisms $C \to G$. For each $\alpha > 0$ and each graph C, define

$$\mathcal{F}_\alpha(C) = \mathrm{Hom}(C, H_\alpha) \sqcup \bigsqcup_{\xi < \alpha} \mathrm{Inj}(C, H_\xi).$$

By fullness, $p_{\alpha\beta} : H_\beta \to H_\alpha$ is unique if $0 < \alpha \leq \beta$, and no arrow exists if $0 < \beta < \alpha$. Since $|H_\xi| < |H_\beta|$ for all $\xi < \beta$,

$$\mathcal{F}_\alpha(H_\beta) = \begin{cases} \{p_{\alpha\beta}\} & 0 < \alpha \leq \beta, \\ \{\mathrm{id}_{H_\beta}\} \text{ in summand } \beta & 0 < \beta < \alpha. \end{cases}$$

For all $\beta \geq 0$ and $\alpha > 0$, define

$$M_\beta = \bigoplus_{\substack{C \in \mathcal{G}_\kappa \\ f \in \mathrm{Inj}(C, H_\beta)}} k\delta_f, \quad E_\alpha = \bigoplus_{\substack{C \in \mathcal{G}_\kappa \\ f \in \mathcal{F}_\alpha(C)}} k\delta_f.$$

Each arrow a acts by $\delta_f a = \delta_{fa}$ if fa is defined, and by zero otherwise. Precomposition preserves the summand index in $\mathcal{F}_\alpha$. Define $u_\beta : M_\beta \to \underline{k}$ and $\varepsilon_\alpha : E_\alpha \to \underline{k}$ by $\delta_f \mapsto \delta_{\mathrm{dom}\, f}$, and let $K_\alpha = \ker \varepsilon_\alpha$. Since $\mathrm{Inj}(C, H_0) \neq \varnothing$ for every C, u_0 and each ε_α are onto. For each $\beta > 0$ let $\iota_\beta : M_\beta \hookrightarrow E_\beta$ be inclusion into the first summand, so $\varepsilon_\beta \iota_\beta = u_\beta$.

Let $\alpha, \beta > 0$, $\mathcal{F}_\alpha(H_\beta) = \{g\}$ and $D = [\mathrm{dom}\, H_\beta]^{\leq \kappa}$. For each $C \in D$ choose an identification of the induced graph with its representative and let $i_C : C \hookrightarrow H_\beta$ be inclusion. For each right R_κ-module X, let $X(C) = X\mathrm{id}_C$ and $x|_C = xa$ for all $x \in X(C')$ and inclusions $a : C \hookrightarrow C'$. Since $\delta_{i_{C'}}|_C = \delta_{i_C}$, evaluation $\varphi \mapsto (\varphi(\delta_{i_C}))_{C \in D}$ gives

$$\mathrm{Hom}_{R_\kappa}(M_\beta, X) \;\cong\; \left\{ (x_C) \in \prod_{C \in D} X(C) : x_{C'}|_C = x_C \text{ for all } C \subseteq C' \text{ in } D \right\}.$$

Conversely, define $\varphi(\delta_{i_C a}) = x_C a$. Every f factors through $C = \mathrm{im}\, f$, and enlarging C preserves $x_C a$ by compatibility. For each arrow u, $\varphi(\delta_{i_C a} u) = x_C a u = \varphi(\delta_{i_C a}) u$, with both sides zero if a and u do not compose.

For each $C \in D$, the space $K_\alpha(C)$ of vectors of coefficient sum zero has basis

$$B_C = \{\delta_h - \delta_{g|_C} : h \in \mathcal{F}_\alpha(C),\ h \neq g|_C\}.$$

For all $C \subseteq C'$, $(\delta_h - \delta_{g|_{C'}})|_C = \delta_{h|_C} - \delta_{g|_C} \in B_C \cup \{0\}$.

Let $(x_C)_{C \in D}$ be compatible in the $K_\alpha(C)$ and let $S_C = \mathrm{supp}\, x_C$. Since $x_C = x_{C'}|_C$, $S_C \subseteq S_{C'}|_C = \{u|_C : u \in S_{C'}\}$ for all $C \subseteq C'$. For each attained size n, choose C_n with $|S_{C_n}| = n$. Then, $A = \bigcup_n C_n \in D$ and $|S_A|$ is maximal since $S_{C_n} \subseteq S_A|_{C_n}$ for each n. For all $A \subseteq C \subseteq C'$, restriction $S_{C'} \to S_C$ is bijective by maximality. If $S_A \neq \varnothing$, pick $\delta_{h_A} - \delta_{g|_A} \in S_A$. The unique extensions $\delta_{h_C} - \delta_{g|_C} \in S_C$ are compatible. The h_C are in one summand, and every finite tuple is in some $C \supseteq A$. Thus, $\bigcup_{C \supseteq A} h_C \in \mathcal{F}_\alpha(H_\beta) \setminus \{g\} = \varnothing$. Hence, every $x_C = 0$ and

$$\mathrm{Hom}_{R_\kappa}(M_\beta, K_\alpha) \cong \varprojlim_{C \in D} K_\alpha(C) = 0. \tag{1}$$

Claim. *For all $\alpha, \beta > 0$, every exact sequence $0 \to K_\alpha \to L \xrightarrow{q} M_\beta \to 0$ has a unique section of q.*

Proof. For each $C \in D$, choose $z \in L$ above δ_{i_C} and let $y_C = z\mathrm{id}_C$. A section is a compatible family of lifts in $T_C = y_C + K_\alpha(C)$. Two sections differ by a map $M_\beta \to K_\alpha$, hence agree by (1). We obtain finite nonempty $P_C \subseteq T_C$ preserved by restriction.

For each $A \in D$, let $D_A = \{C \in D : A \subseteq C\}$. For all $C \in D_A$, $t \in T_C$ and $b \in B_A$, let $[t]_b = \{s \in T_C : b \notin (\mathrm{supp}(s - t))|_A\}$. Restriction from C' to C in D_A induces $[t]_b \mapsto [t|_C]_b$ since $(u|_C)|_A = u|_A$. A compatible family on a cofinal subset extends by $z_C = z_E|_C$ from any larger member E. Two choices agree by a common upper bound. If (s_C) and (t_C) represent

compatible b-classes, the vectors $\sum\{u \in \mathrm{supp}(s_C - t_C) : u|_A = b\}$ are compatible by the same restriction formula. By cofinality of D_A and (1), the vectors vanish, so $[s_C]_b = [t_C]_b$.

All but finitely many $b \in B_A$ admit a compatible family with value $[y_A]_b$ at A. Otherwise choose distinct exceptions b_n ($n < \omega$). For all $C \subseteq C'$ in D_A, let

$$F(C, C') = \{n < \omega : [y_{C'}|_C]_{b_n} \neq [y_C]_{b_n}\}.$$

The set is finite since $y_{C'}|_C - y_C$ has finite support. Let $F(C, C') = \varnothing$ if $A \nsubseteq C$. For all $C \subseteq C' \subseteq C''$, $F(C, C') \triangle F(C, C'') \subseteq F(C', C'')$ since equal classes at C' restrict equally to C. Thus, $c(C, C') = \max(\{0\} \cup F(C, C'))$ is subadditive on D. By hypothesis, pick a cofinal $\mathcal{H} \subseteq D$ with $c[\mathcal{H}^{[2]}] \subseteq m+1$ for some $m < \omega$. Pick $C \in \mathcal{H}$ above A and $n > \max\{m, c(A, C)\}$. The classes $[y_E]_{b_n}$ are compatible on the cofinal set $\mathcal{H} \cap D_A$ and extend to D_A with value $[y_C|_A]_{b_n} = [y_A]_{b_n}$ at A, a contradiction.

Let P_A be the set of $t \in T_A$ with $[t]_b = z_A$ for every $b \in B_A$ and every compatible family $(z_C)_{C \in D_A}$ of b-classes. Each condition prescribes one coefficient of $t - y_A$. By uniqueness, the conditions agree, and all but finitely many coefficients are prescribed to be zero. Thus, P_A is finite and nonempty since $k = \mathbb{F}_2$.

Let $A \subseteq C$, $t \in P_C$ and let $(z_E)_{E \in D_A}$ be a compatible family of b-classes. Pick $s \in z_C$. For each $a \in B_C$ with $a|_A = b$, the classes z_E ($E \supseteq C$) are contained in compatible a-classes since $(u|_C)|_A = u|_A$. Hence, $[t]_a = [s]_a$ whenever $a|_A = b$. Thus, $b \notin (\mathrm{supp}(t - s))|_A$ and $[t|_A]_b = z_A$. The restrictions $P_C \to P_A$ form an inverse system of finite nonempty sets. By [15, Tag 086J], $\varprojlim_{A \in D} P_A \neq \varnothing$, so q has a section. □

Let $\mathcal{A} = {}^{\perp}\{K_\alpha : \alpha > 0\}$. Since each $K_\alpha \in \mathcal{A}^\perp$, $\mathcal{A} \subseteq {}^{\perp}(\mathcal{A}^\perp) \subseteq \mathcal{A}$. By (3) with $(\mathcal{A}, \mathcal{A}^\perp)$, pick an $\mathcal{A}$-precover $p : P \to \underline{k}$. For each $\beta > 0$, $M_\beta \in \mathcal{A}$ by the claim, so u_β factors through p. By $P \perp K_\beta$ and the pullback above, p lifts through ε_β.

$$\begin{array}{ccccc} M_\beta & \xrightarrow{a_\beta} & P & \xrightarrow{b_\beta} & E_\beta \\ & \searrow^{u_\beta} & \downarrow p & \swarrow_{\varepsilon_\beta} & \\ & & \underline{k}. & & \end{array}$$

Since $\varepsilon_\beta(b_\beta a_\beta - \iota_\beta) = 0$, $b_\beta a_\beta = \iota_\beta$ by (1). Thus, $|H_\beta| \leq |P|$ by the injection $x \mapsto a_\beta(\delta_{i_{\{x\}}})$ for every $\beta > 0$, a contradiction. □

3 gWVP and completely ineffable cardinals under CPC

Theorem 3.1. CPC *implies* gWVP *for every set-sized relational language.*

Proof. Assume CPC+¬gWVP. Let $\mathcal{L}, (H_\alpha), (p_{\alpha\beta})$ witness ¬gWVP. We obtain one precover $P \to \underline{k}$ with $|H_\beta| \leq |P|$ for every β. Since the H_α are pairwise nonisomorphic and the $\mathcal{L}$-structures on subsets of each cardinal form a set, pick a subsequence with $\aleph_1 < |H_\alpha| < |H_\beta|$ for all $\alpha < \beta$. For all $\alpha < \beta$, $p_{\alpha\beta}$ identifies two points of H_β in V, hence in every $V[G]$.

Choose one representative of each isomorphism class of $\mathcal{L}$-structures of size $\leq \aleph_1$ admitting an injective homomorphism to some H_β. Let $\mathcal{C}$ be the category of the representatives and injective homomorphisms. In V, define $R, \underline{k}, M_\beta, E_\alpha, u_\beta, \varepsilon_\alpha, \iota_\beta$ as in Section 2, with

$k = \mathbb{F}_2$, $\mathcal{C}$ in place of $\mathcal{G}_\kappa$ and all ordinal indices. Let $K_\alpha = \ker \varepsilon_\alpha$. In each $V[G]$, let $\mathcal{F}_\alpha(C) = \mathrm{Hom}_{\mathcal{L}}(C, H_\alpha) \sqcup \bigsqcup_{\xi<\alpha} \mathrm{Inj}(C, H_\xi)$. By ¬gWVP and noninjectivity, $\mathcal{F}_\alpha(H_\beta) = \{g_{\alpha\beta}\}$, where $g_{\alpha\beta} = p_{\alpha\beta}$ if $\alpha \le \beta$ and $g_{\alpha\beta} = \mathrm{id}_{H_\beta}$ in summand β otherwise. For each $C \in \mathcal{C}$, pick an injective $f : C \to H_\beta$. Then, $\varepsilon_\alpha(\delta_{g_{\alpha\beta} f}) = \delta_C$ for every α, so each ε_α is onto.

Claim. *$M_\beta \perp K_\alpha$ for all α, β.*

Proof. In V, let $0 \to K_\alpha \to L \xrightarrow{q} M_\beta \to 0$ be exact and $g = g_{\alpha\beta}$. For each $C \in D = [\mathrm{dom}\, H_\beta]^{\le \aleph_1}$, use $i_C : C \hookrightarrow H_\beta$ and the basis B_C from Section 2. A section is a family

$$t_C \in T_C = \{t \in L(C) : q(t) = \delta_{i_C}\},\ t_{C'}|_C = t_C \text{ for all } C \subseteq C' \text{ in } D.$$

At turn $n < \omega$, player I plays $C_n \in D$ with $C_m \subseteq C_n$ for all $m < n$. Player II plays $t_n \in T_{C_n}$ with $t_n|_{C_m} = t_m$ for all $m < n$ and wins iff the play is infinite.

Let σ be a strategy of player I. For some sufficiently large regular χ, choose increasing countable $N_i \prec H(\chi)$ $(i < \omega_1)$ with $\sigma, D, (L(C), T_C, B_C)_{C \in D}$ and the restrictions in N_0, and $A_i = \bigcup(D \cap N_i) \in N_{i+1}$. Let $N = \bigcup_i N_i$. The A_i are cofinal in $D \cap N$, and $A = \bigcup_i A_i \in D$ since $|A| \le \aleph_1$.

Pick $z \in L$ with $q(z) = \delta_{i_A}$ and let $y = z\mathrm{id}_A \in T_A$. For each $i < \omega_1$, pick $u \in T_{A_i} \cap N$ by elementarity. The lifts in $T_{A_i} \cap N$ differ from u by exactly the finite sums from $B_{A_i} \cap N$. Thus, adding to u all terms of $y|_{A_i} - u$ in N gives the unique x_i minimizing $n_i = |\,\mathrm{supp}(y|_{A_i} - x_i)|$. For all $i \le j$, $x_j|_{A_i} \in T_{A_i} \cap N$ and restriction sends basis elements to basis elements or zero, so

$$n_i \le |\,\mathrm{supp}((y|_{A_j} - x_j)|_{A_i})| \le n_j.$$

Since ω_1 is regular, the n_i are constant on a tail, where $x_j|_{A_i} = x_i$ by uniqueness. For each $C \in D \cap N$, reply with $x_i|_C \in N$ for some i on the tail with $C \subseteq A_i$. Since $\sigma \in N$ and N is closed under finite sequences, the play is infinite.

By open determinacy [6], player II has a winning strategy $\tau \in V$. Let $t_C = \tau(\langle C \rangle)$. Suppose $t_{C'}|_C \ne t_C$ for some $C \subseteq C'$. In a $\mathrm{Coll}(\omega, |H_\beta|)$-generic extension, let $\mathrm{dom}\, H_\beta = \{a_n : n < \omega\}$ and $C_n = C' \cup \{a_i : i < n\}$. Against τ play $(C, C_0, C_1, \ldots)$ and $(C_0, C_1, \ldots)$. All finite histories are in V, so the differences v_n of the replies at C_n satisfy

$$v_0|_C = t_C - t_{C'}|_C \ne 0, \quad v_n \in K_\alpha(C_n), \quad v_{n+1}|_{C_n} = v_n.$$

Each term in v_n extends to a term in v_{n+1}. Choose compatible $\delta_{h_n} - \delta_{g|_{C_n}} \in \mathrm{supp}\, v_n$. The C_n cover H_β, so $\bigcup_n h_n \in \mathcal{F}_\alpha(H_\beta) \setminus \{g\} = \varnothing$ as in (1). Thus, (t_C) is a section in V. □

Let $\mathcal{A} = {}^{\perp}\{K_\alpha : \alpha \in \mathrm{Ord}\}$ and pick an $\mathcal{A}$-precover $p : P \to \underline{k}$ by CPC. For each β, $M_\beta \in \mathcal{A}$, $P \perp K_\beta$ and $\mathrm{Hom}_R(M_\beta, K_\beta) = 0$ as in (1). By the diagram in Theorem 2.1,

$$M_\beta \xrightarrow{a_\beta} P \xrightarrow{b_\beta} E_\beta,\ b_\beta a_\beta = \iota_\beta.$$

Hence, $|H_\beta| \le |M_\beta| \le |P|$ for every β, a contradiction. □

Let λ be regular and uncountable. A set $S \subseteq \lambda$ is *completely ineffable in* λ if S is in an upward closed family $\mathcal{S}$ of stationary subsets of λ with $T \setminus \{0\} \in \mathcal{S}$ for each $T \in \mathcal{S}$ and with the following property. For every $T \in \mathcal{S}$ and every T-list $(A_\xi)_{\xi \in T}$ with $A_\xi \subseteq \xi$, there are $B \subseteq \lambda$ and $H \in \mathcal{S}$ with $H \subseteq \{\xi \in T : A_\xi = B \cap \xi\}$ [9, Definition 11.1 and Lemma 11.3], [2, p. 852]. Let I be the class of cardinals λ completely ineffable in λ.

Corollary 3.2. *Assume* gWVP*, which follows from* CPC *by Theorem 3.1. For every ordinal θ and every I-list $(U_\xi)_{\xi\in I}$ with $U_\xi \subseteq \xi$, there is a $\kappa \in I$ above θ such that*

$$\{\xi \in I \cap \kappa : U_\xi = U_\kappa \cap \xi\}$$

is completely ineffable in κ.

Proof. We may assume $\theta \geq \omega$. Extend the I-list by $U_\xi = \varnothing$ for all $\xi \notin I$. We obtain κ in $V[G]$ and the family $\mathcal{S}$ in V.

For each $\alpha > \theta$, let H_α have sorts $\mathcal{P}(\alpha+1)$ and $\prod_{\xi\leq\alpha}\mathcal{P}(\xi)$. Define

$$e(\vec{A}, B) = \{\xi \leq \alpha : A_\xi = B \cap \xi\} \quad (B \subseteq \alpha + 1).$$

Name the sorts, $I\cap(\alpha+1)$, $\vec{U}_\alpha = (U_\xi)_{\xi\leq\alpha}$, $\vec{0}_\alpha = (\varnothing)_{\xi\leq\alpha}$ and $\{\xi\}$ $(\xi \leq \theta)$ by unary predicates. With the graphs of e and the Boolean operations, $|\mathcal{L}| \leq |\theta|$.

Let $p_{\alpha\beta} : H_\beta \to H_\alpha$ restrict both sorts to $\alpha+1$. By gWVP and the truth lemma, some $p \in \mathbb{P}$ forces a homomorphism $\dot{h} : H_\beta \to H_\alpha$ with $\beta \geq \alpha$ and $\dot{h} \neq p_{\alpha\beta}$. Indeed, if $u : H_\alpha \to H_\beta$ and $\alpha < \beta$, then $up_{\alpha\beta} \neq \mathrm{id}$ since $p_{\alpha\beta}(\{\beta\}) = p_{\alpha\beta}(\varnothing)$.

In $V[G]$ with $p \in G$, let $h = \dot{h}^G$. For all $\xi \leq \alpha$ with $\xi \in h(\{\xi\})$ and all $B \in \mathcal{P}(\beta+1)^V$,

$$\xi \in h(B) \iff \xi \in h(B \cap \{\xi\}) \iff \xi \in B.$$

If $\xi \in h(\{\xi\})$ for all $\xi \leq \alpha$, then $h = p_{\alpha\beta}$ on subsets. On lists, $\xi \in h(e(\vec{A}, A_\xi))$ gives $h(\vec{A})_\xi = h(A_\xi) \cap \xi = A_\xi$, a contradiction.

Let $\kappa = \min\{\xi \leq \alpha : \xi \notin h(\{\xi\})\} > \theta$ since h preserves the named singletons. Strengthen p in G to decide κ. In V, let

$$\mathcal{S} = \{S \subseteq \kappa : (\exists q \leq p)\ q \Vdash \kappa \in \dot{h}(S)\}.$$

In every extension through p, $h(B) \cap \kappa = B \cap \kappa$ by minimality. Since h preserves e,

$$\kappa \in h(e(\vec{A}, B)) \iff h(\vec{A})_\kappa = B \cap \kappa \quad (\vec{A}, B \in H_\beta).$$

Since $e(\vec{0}_\beta, \{\eta\}) = \eta + 1$, $\kappa \in h(\eta+1)$ iff $\eta \geq \kappa$. Hence, $\kappa \in h(\kappa+1) \setminus h(\{\kappa\}) = h(\kappa)$ and $\kappa \in \mathcal{S}$. By the Boolean operations, $\mathcal{S}$ is upward closed, $\varnothing \notin \mathcal{S}$ and $S \setminus (\eta+1) \in \mathcal{S}$ for all $S \in \mathcal{S}$, $\eta < \kappa$. Thus, κ is a limit and every $S \in \mathcal{S}$ is unbounded.

For each $S \in \mathcal{S}$, extend a ground S-list to $\vec{A}$ by empty sets. Pick $q \leq p$ forcing $\kappa \in \dot{h}(S)$. Since $\dot{h}(\vec{A})$ names a ground list, strengthen q to decide its κth entry B. Then, $q \Vdash \kappa \in \dot{h}(S \cap e(\vec{A}, B))$, so $S \cap e(\vec{A}, B) \in \mathcal{S}$.

In V, let $f : S \to \kappa$ be regressive with $0 \notin S \in \mathcal{S}$. Pick $B \subseteq \kappa$ and $H \in \mathcal{S}$ with $B \cap \xi = \{f(\xi)\}$ for all $\xi \in H \subseteq S$. The singletons are nested, so f has an unbounded fiber. If $\lambda = \mathrm{cf}(\kappa) < \kappa$, pick a cofinal $(d_i)_{i<\lambda}$ in κ. On $\kappa \setminus (\lambda+1) \in \mathcal{S}$, $f(\xi) = \min\{i : \xi \leq d_i\} < \xi$ has $f^{-1}\{i\} \subseteq d_i + 1$. If $S \in \mathcal{S}$ misses a club D, then $g(\xi) = \sup(D \cap \xi) < \xi$ on $S \setminus \{0\} \in \mathcal{S}$, with $g^{-1}\{\eta\} \subseteq d+1$ whenever $\eta < d \in D$. Both contradict the unbounded fiber. Hence, $\kappa \in I$.

By the named constants, p forces $\dot{h}(I \cap \kappa) = (I \cap (\alpha+1)) \cap \dot{h}(\kappa) \ni \kappa$ and $\dot{h}(\vec{U}_\beta)_\kappa = U_\kappa$. Thus, p witnesses $I \cap \kappa \cap e(\vec{U}_\beta, U_\kappa) \in \mathcal{S}$. □

4 Grothendieck categories and preservation of CPC by set forcing

A class $\mathcal{A}$ is *generating* if every X admits $A \twoheadrightarrow X$ with $A \in \mathcal{A}$. An object U is a generator if every X admits $U^{(J)} \twoheadrightarrow X$ for some set J. An abelian category $\mathcal{G}$ is Grothendieck if $\mathcal{G}$ has a generator and all small colimits, and filtered colimits preserve short exact sequences. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete if every X has exact sequences

$$0 \to B \to A \to X \to 0, \quad 0 \to X \to B' \to A' \to 0$$

with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$.

Theorem 4.1. CPC *holds iff, in every Grothendieck abelian category, every cotorsion pair with generating left class is complete. For every set forcing* $\mathbb{P}$*,*

$$\text{CPC} \implies \Vdash_{\mathbb{P}} \text{CPC}.$$

In $\Vdash_{\mathbb{P}}$ CPC, the classes of modules in $V[G]$ are interpretations of ground-model class names. We use the forcing theorem with class parameters in GBC [8, Theorem 4.3] and the maximum principle [11, Ch. VII, Theorem 8.2].

Lemma 4.2. *Assume* CPC*. Let* $L : \text{Mod-}R \to \mathcal{E}$ *be an exact functor to an abelian category with right adjoint* Γ*. For each* Y *let* $\varepsilon_Y : L\Gamma Y \to Y$ *correspond to* $\mathrm{id}_{\Gamma Y}$ *under adjunction. Suppose each* ε_Y *is an isomorphism. Let* $(\mathcal{A}, \mathcal{B})$ *be a cotorsion pair in* $\mathcal{E}$*. If* Γ *preserves epimorphisms with kernel in* $\mathcal{B}$*, then* $(\mathcal{A}, \mathcal{B})$ *is complete.*

Proof. We apply CPC to complexes of three modules. Let T be the upper triangular 3×3 matrix ring over R modulo Re_{02}. On a right T-module M, $M^i = Me_{ii}$ and $d^i(x) = xe_{i,i+1}$ define a complex since $e_{01}e_{12} = 0$. Conversely, on $M^0 \oplus M^1 \oplus M^2$ define

$$(x_0, x_1, x_2)(r_{ij}) = (x_0 r_{00},\ d^0(x_0)r_{01} + x_1 r_{11},\ d^1(x_1)r_{12} + x_2 r_{22}).$$

The T-maps are the R-linear triples f with $f^{j+1}d^j_M = d^j_N f^j$ $(j < 2)$. Exactness is degreewise.

Let $GY = (0 \to 0 \to \Gamma Y)$, $FM = L(\operatorname{coker} d^1_M)$ and $Ff = L([x] \mapsto [f^2(x)])$. Identify FGY with Y by ε_Y. Since each $M \to GY$ is $(0, 0, h)$ with $hd^1_M = 0$, $f \mapsto Ff$ is the adjunction

$$\operatorname{Hom}_T(M, GY) \cong \operatorname{Hom}_R(\operatorname{coker} d^1_M, \Gamma Y) \cong \operatorname{Hom}_{\mathcal{E}}(FM, Y).$$

Let $\mathcal{C} = {}^{\perp}\{GB : B \in \mathcal{B}\}$. By CPC, for each X pick the first exact sequence

$$0 \to K \to P \to GX \to 0, \quad 0 \to FK \to FP \to X \to 0$$

with $P \in \mathcal{C}$, $K \in \mathcal{C}^{\perp}$. The second sequence is exact since $K^1 = P^1$. Indeed, quotient K^2, P^2 by the common image $d^1_K[K^1] = d^1_P[P^1]$ and apply L.

Let $0 \to B \to Z \xrightarrow{v} FP \to 0$ with $B \in \mathcal{B}$. Then, Gv is onto by hypothesis and $\ker(Gv) = GB$ by left exactness. By $P \perp GB$, the adjoint of id_{FP} lifts to $\ell : P \to GZ$. Thus, $v(F\ell) = \mathrm{id}_{FP}$ and $FP \in \mathcal{A}$.

Let $d = d^1_K$ and $q : LK^2 \twoheadrightarrow FK$ be the cokernel of Ld. For each extension $0 \to FK \xrightarrow{i} Z \to A \to 0$ with $A \in \mathcal{A}$, let $(0, 0, h) : K \to GZ$ be adjoint to i. Then, $hd = 0$ and $Lh = iq$.

Let $\pi_2 : E \to V$ be the projection between the rows and $\iota : K \hookrightarrow E$ the inclusion of the first summands.

$$\begin{array}{ccccc} K^0 \oplus K^1 & \xrightarrow{d_E^0} & K^1 \oplus K^2 & \xrightarrow{d_E^1} & K^2 \oplus \Gamma Z \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\ K^1 & \xrightarrow[d]{} & K^2 & \xrightarrow[h]{} & \Gamma Z \end{array}$$

Let $d_E^0(x_0, x_1) = (d_K^0 x_0 - x_1, dx_1)$ and $d_E^1(x_1, x_2) = (dx_1 + x_2, hx_2)$. Since $dd_K^0 = hd = 0$, $d_E^1 d_E^0 = 0$ and $\iota[K] = \ker \pi_2$. The image of d_E^1 is $\{(u, hu) : u \in K^2\}$, so $(u, z) \mapsto hu - z$ identifies $\operatorname{coker} d_E^1$ with ΓZ. Under $FE \cong Z$, $F\iota = i$ by $Lh = iq$, and $FV \cong \operatorname{coker} i = A$ since F preserves cokernels.

Let $0 \to GB \xrightarrow{j} W \xrightarrow{v} V \to 0$ with $B \in \mathcal{B}$. Identify $W^0 = K^1$, $W^1 = K^2$ and $\Gamma B = \ker v^2$. By $v^2 d_W^1 = h$ and exactness of L,

$$\ker(Fj) = L(\Gamma B \cap \operatorname{im} d_W^1) = L(d_W^1[\ker h]) = (Ld_W^1)[\ker(iq)] = 0$$

since $\ker(iq) = \operatorname{im} Ld$ and $d_W^1 d = 0$. Since $\operatorname{coker}(Fj) = FV \cong A$ and $A \perp B$, pick a retraction $r : FW \to B$ of Fj. Its adjoint retracts j, so $V \in \mathcal{C}$.

By $V \perp K$, ι splits. Thus, $i = F\iota$ splits and $FK \in \mathcal{B}$.

For each X, applying L to $R^{(J)} \twoheadrightarrow \Gamma X$ shows that $L(R)$ is a generator. By adjunction, $\operatorname{colim} D \cong L(\operatorname{colim} \Gamma D)$ for each small diagram D. For each filtered D, the three functors on the right preserve monomorphisms. Colimits preserve cokernels, so $\mathcal{E}$ is Grothendieck.

Pick $i : X \hookrightarrow I$ with I injective [15, Tag 079H] and a special $\mathcal{A}$-precover $v : A \to I/X$ with kernel $b : B \to A$. Let $Y = I \times_{I/X} A$. The rows and columns are exact [7, § 6.3].

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & X & = & X & & \\ & & & & {\scriptstyle (i,0)}\downarrow & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & B & \xrightarrow{(0,b)} & Y & \xrightarrow{\pi_I} & I & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \pi_A}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \xrightarrow[b]{} & A & \xrightarrow[v]{} & I/X & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

For each $A' \in \mathcal{A}$, $0 = \operatorname{Ext}^1(A', B) \to \operatorname{Ext}^1(A', Y) \to \operatorname{Ext}^1(A', I) = 0$ is exact by the row through Y and injectivity of I [15, Tag 05E2]. Thus, $0 \to X \to Y \to A \to 0$ with $Y \in \mathcal{B}$ proves completeness. □

Proof of Theorem 4.1. Assume CPC. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a Grothendieck category $\mathcal{G}$ with $\mathcal{A}$ generating. Pick $U \twoheadrightarrow U_0$ with $U \in \mathcal{A}$ and U_0 a generator. For each X, $U^{(I)} \twoheadrightarrow U_0^{(I)} \twoheadrightarrow X$ for some set I, so U is a generator. Let $R = \operatorname{End}(U)$ and $\Gamma X = \operatorname{Hom}(U, X)$ with $fr = f \circ r$. By the Gabriel–Popescu theorem, Γ has an exact left adjoint with invertible counit [15, Tag 0F5U]. If $v : Y \twoheadrightarrow X$ has kernel in $\mathcal{B}$, each pullback along $U \to X$ splits by $U \perp \ker v$, so Γv is onto. By Lemma 4.2, $(\mathcal{A}, \mathcal{B})$ is complete. Conversely, Mod-R is Grothendieck, $R^{(X)} \twoheadrightarrow X$ by $e_x \mapsto x$, and free modules are in $\mathcal{A}$ by lifting a basis.

Let $\Vdash$ mean $p \Vdash$, where $p \in \mathbb{P}$ forces that $(\dot{\mathcal{A}}, \dot{\mathcal{B}})$ is a cotorsion pair over $\dot{R}$ and $\dot{M}$ is a right $\dot{R}$-module. In V, let the objects of $\mathcal{E}$ be all names X with $\Vdash X \in \text{Mod-}\dot{R}$. For each X, Y, choose U naming the product of their underlying sets. The arrows $X \to Y$ are the subsets

$$[f] = \{(\sigma, q) \in \operatorname{dom} U \times \mathbb{P} : q \leq p,\ q \Vdash \sigma \in f\} \quad (\Vdash f \in \operatorname{Hom}_{\dot{R}}(X, Y)).$$

For every set name U, $U^G \subseteq \{\sigma^G : \sigma \in \operatorname{dom} U\}$. Since $\Vdash f \subseteq U$, $\Vdash [f] = f$ by the truth lemma. Thus, $[f] = [g]$ iff $\Vdash f = g$. Addition and composition are $[f + g]$ and $[gf]$.

The equations of each small diagram $D : J \to \mathcal{E}$ are forced. Let (Q, ι_i) name its $\check{J}$-colimit. For each cocone $(t_i : D_i \to T)$,

$$\Vdash (\exists! k : Q \to T)(\forall i \in \operatorname{Ob}(\check{J}))\ k\iota_i = t_i.$$

By the maximum principle, $[k]$ is unique, so Q is the colimit in $\mathcal{E}$. Dually we obtain limits. Sections, retractions and inverses exist iff forced. Since $X/\ker f \to \operatorname{im} f$ is forced invertible, $\mathcal{E}$ is abelian. Exactness of $X \xrightarrow{f} Y \xrightarrow{g} Z$ at Y means $\Vdash \operatorname{im} f = \ker g$. Each filtered J remains filtered since finite tuples of its arrows are in V, so filtered colimits are exact as in modules.

Let $\mathcal{A}_p = \{X : \Vdash X \in \dot{\mathcal{A}}\}$ and $\mathcal{B}_p = \{X : \Vdash X \in \dot{\mathcal{B}}\}$ in $\mathcal{E}$. For each X, let U name its underlying set and $J = \operatorname{dom} U$. By the bound on U^G, $\Vdash |X| \leq |\check{J}|$. Pick $\dot{R}^{(\check{J})} \twoheadrightarrow X$ by the maximum principle. Thus, $\dot{R}$ generates $\mathcal{E}$, and $\mathcal{A}_p$ is generating since free modules are in $\dot{\mathcal{A}}$ whenever $p \in G$. Hence, $\mathcal{E}$ is Grothendieck.

For all $A \in \mathcal{A}_p$ and $B \in \mathcal{B}_p$, $A \perp B$ by forced splitting. In the forcing language, let $u = \operatorname{id}_X$ if $X \in \dot{\mathcal{A}}$. Otherwise pick a nonsplit $u : Z \twoheadrightarrow X$ with $\ker u \in \dot{\mathcal{B}}$ by $\dot{\mathcal{A}} = {}^{\perp}\dot{\mathcal{B}}$. By the maximum principle, $\ker u \in \mathcal{B}_p$ and u has a section in $\mathcal{E}$ iff $\Vdash X \in \dot{\mathcal{A}}$. Hence, $\mathcal{A}_p = {}^{\perp}\mathcal{B}_p$. Dually, $\mathcal{B}_p = \mathcal{A}_p^{\perp}$ using $X \hookrightarrow W$ with cokernel in $\dot{\mathcal{A}}$. By the first assertion, pick $0 \to B \to A \to \dot{M} \to 0$ with $B \in \mathcal{B}_p$, $A \in \mathcal{A}_p$. Then, p forces a special $\dot{\mathcal{A}}$-precover of $\dot{M}$. Hence, $\Vdash_{\mathbb{P}}$ CPC since p and the names were arbitrary. □

A Precovers of a cyclic module over a right coherent ring

Corollary A.1. *Let κ satisfy the hypothesis of Theorem 2.1. For some right coherent ring S and some cyclic right S-module N with a free resolution*

$$0 \longrightarrow S \longrightarrow S \longrightarrow N \longrightarrow 0,$$

WVP *and* CPC *are each equivalent to the assertion that N has an $\mathcal{X}$-precover for every cotorsion pair $(\mathcal{X}, \mathcal{Y})$ over S.*

Proof. Let $R = R_\kappa$ and $J = \ker u_0$. Under $\delta_f \mapsto f$, $M_0 = \operatorname{id}_{H_0} R$. Since $\varnothing \to H_0$ is unique, $J(\varnothing) = 0$. For each $h : J \to \underline{k}$ and $x \in J(C)$, $h(x)|_\varnothing = h(x|_\varnothing) = 0$. Thus, $h = 0$ since $\delta_C|_\varnothing = \delta_\varnothing$ and $J = \bigoplus_C J(C)$. Let $U = R \oplus J$ and define

$$\theta = 2^{|U|},\ \ G = U^{(\theta)},\ \ S = \operatorname{End}_R(G),\ \Phi(X) = \operatorname{Hom}_R(G, X),\ \Phi(v)(h) = vh,$$

with $hs = h \circ s$ and $\ker \Phi(v) = \Phi(\ker v)$. For each $v : Y \twoheadrightarrow \underline{k}$, $\Phi(v)$ is onto. Each $G \to \underline{k}$ is $(r, z) \mapsto x_i r$ on summand i. Choose $y_i \in v^{-1}(x_i)$ and lift by $(r, z) \mapsto y_i r$.

Let $\pi = (u_0, 0) : M_0 \oplus G \twoheadrightarrow \underline{k}$. Since M_0, J are summands of U and $1 + \theta = \theta$, $M_0 \oplus G \cong G \cong J \oplus G = \ker \pi$. Thus, $N = \Phi(\underline{k})$ has the stated resolution.

Let $f : G^m \to G$ have components f_i ($m < \omega$). By finite support, $|G| = \theta$. Choose $g : G \to \ker f$ whose summand restrictions g_i enumerate $\mathrm{Hom}_R(U, \ker f)$ since $\theta^{|U|} = \theta$. Each $h : G \to \ker f$ is gs, where s sends summand j identically to the first i with $g_i = h_j$. Thus, S is right coherent by

$$S \xrightarrow{s \mapsto gs} S^m \xrightarrow{h \mapsto fh} \sum_{i<m} f_i S \to 0 \quad (\Phi(G^m) = S^m).$$

In $G = R \oplus G'$, let $e(r, y) = (r, 0)$. Evaluation at $(1, 0)$ has inverse $x \mapsto [(r, y) \mapsto xr]$. Thus, $Se \cong G$, $eSe \cong eG = R$, $\Phi(X)e \cong X$ and $\Phi(X) \cong \mathrm{Hom}_R(Se, X)$.

Suppose $\neg$WVP and let $\mathcal{X} = {}^{\perp}\{\Phi(K_\alpha) : \alpha > 0\}$ as in Theorem 2.1. For each extension in the upper row ($\alpha, \beta > 0$), the lower row is exact since $\ker f_e = (\ker f)e$ and $f(Ze) = f(Z)e$ ($f_e = f|_{Ze}$).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi(K_\alpha) & \xrightarrow{i} & Z & \xrightarrow{q} & \Phi(M_\beta) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h \mapsto h(e)} & & \downarrow{\scriptstyle z \mapsto ze} & & \downarrow{\scriptstyle h \mapsto h(e)} & & \\ 0 & \longrightarrow & K_\alpha & \xrightarrow[i_e]{} & Ze & \xrightarrow[q_e]{} & M_\beta & \longrightarrow & 0 \end{array}$$

By the claim in Theorem 2.1, pick $r : Ze \to K_\alpha$ with $ri_e = \mathrm{id}$. Define $\widetilde{r} : Z \to \Phi(K_\alpha)$ by $\widetilde{r}(z)(u) = r(zu)$ since $zu = zue \in Ze$ for all $u \in Se$. For all $t \in R$, $s \in S$, $r(zut) = r(zu)t$ and $(zs)u = z(su)$ give linearity. Since $\widetilde{r}(i(h))(u) = r(i(hu)) = (hu)(e) = h(u)$, $\Phi(M_\beta) \in \mathcal{X}$.

By hypothesis, pick an $\mathcal{X}$-precover $p : P \to N$. For each $\beta > 0$, factor $\Phi(u_\beta) = pa$ by the precover property and $p = \Phi(\varepsilon_\beta)b$ by $P \perp \Phi(K_\beta)$. Then, $p_e a_e = u_\beta$ and $\varepsilon_\beta b_e = p_e$. By the final diagram in Theorem 2.1, $|H_\beta| \leq |Pe|$ for every $\beta > 0$, a contradiction. Conversely, CPC gives special precovers and WVP $\Rightarrow$ CPC by Corollary 1.2. $\square$

B Subadditive colorings on $[X]^{\leq\kappa}$

Proposition B.1. *Let $\kappa \geq \aleph_1$ be a cardinal. Either of the following implies the coloring hypothesis of Theorem 2.1.*

(i) *For every set X of size $> \kappa$, there is a proper countably complete fine ideal J on $[X]^{\leq\kappa}$ with $\mathbb{P} \times \mathbb{P}$ satisfying the κ^+-chain condition, where $\mathbb{P} = (\mathcal{P}([X]^{\leq\kappa})/J)^+$.*

(ii) *Some forcing $\mathbb{P}$ with $\mathbb{P} \times \mathbb{P}$ satisfying the κ^+-chain condition forces $|\check{\kappa}| \geq \aleph_1$ and either the P-ideal dichotomy or $\mathrm{ISP}(\mu)$ for some regular μ with $\aleph_2 \leq \mu \leq |\check{\kappa}|^+$.*

If κ^+ is ω_1-strongly compact, the κ^+-complete filter generated by the cones $\{a \in [X]^{\leq\kappa} : b \subseteq a\}$ ($b \in [X]^{\leq\kappa}$) extends to a countably complete ultrafilter [1, Definition 4.6]. The dual ideal satisfies (i) with $|\mathbb{P}| = 1$. If $\mathbb{P}$ forces the P-ideal dichotomy, let $\kappa = \max(\aleph_1, |\mathbb{P}|^+)$. If $p \Vdash_{\mathbb{P}} \mathrm{ISP}(\check{\mu})$ for some p, μ, let $\kappa = \max(\aleph_1, |\mathbb{P}|^+, |\mu|^+)$ and use $\mathbb{P}$ below p. In both cases, CPC $\iff$ WVP by (ii) and Theorem 2.1.

Proof. Let $|X| > \kappa$, $D = [X]^{\leq\kappa}$ and let $c : D^{[2]} \to \omega$ be subadditive. For all $C' \in D$, $n < \omega$, let $B_n(C') = \{C \subseteq C' : c(C, C') \leq n\}$. By subadditivity,

$$B_n(C'') \cap \mathcal{P}(C') = B_n(C') \quad (C' \in B_n(C'')), \quad c[B_n(C'')^{[2]}] \subseteq n + 1. \tag{2}$$

In (i), pick J on $S = [D]^{\leq\kappa}$ with quotient forcing $\mathbb{P}$, since $|D| > \kappa$. In both cases, let G be $\mathbb{P}$-generic over V. If $p \in G$ forces $\dot{f} : \check{\kappa} \to \check{D}$, each coordinate has $\leq \kappa$ possible values below p by the κ^+-c.c. The union of all possible values bounds $\mathrm{ran}(\dot{f}^G)$ in D. Every countable cover of a countably directed poset has a cofinal member since countably many cones intersect.

In (i), let $S_n(C) = \{a \in S : C \in B_n(\bigcup a)\}$. By fineness and countable completeness, $\bigvee_n [S_n(C)]_J \geq [\{a \in S : C \in a\}]_J = 1$. By genericity, $D = \bigcup_n H_n$ with $H_n = \{C \in D : [S_n(C)]_J \in G\}$. Pick a cofinal $H = H_n$. For all $C \subseteq C'$ in H, $S_n(C) \cap S_n(C') \neq \varnothing$, so $c(C, C') \leq n$ by (2).

If $\mathrm{ISP}(\mu)$ holds in (ii), pick an internally unbounded guessing $N \prec H(\chi)$ of size $< \mu$ with $D, c \in N$ for some sufficiently large regular χ [10, Corollary 1.5]. Let $E = D \cap N$. By internal unboundedness, $N \cap [D]^{\leq\omega}$ is cofinal in $[E]^{\leq\omega}$. Each member has a bound in E by elementarity. Pick $C'' \in D$ above E since $|E| \leq |\kappa|$, and n with $H_0 = E \cap B_n(C'')$ cofinal in E. For each $b \in N \cap [D]^{\leq\omega}$, pick $C' \in H_0$ above $b \subseteq E$. By (2), $b \cap H_0 = b \cap B_n(C') \in N$. By guessing, $H_0 = H \cap N$ for some $H \subseteq D$ in N. By elementarity and (2), H is cofinal in D and $c[H^{[2]}] \subseteq n + 1$.

In the P-ideal case of (ii), define the ideal $\mathcal{I}$ of countable $a \subseteq D$ with $a \cap B_n(C)$ finite for all $C \in D$ and $n < \omega$. At each bound E of a, $a \in \mathcal{I}$ iff $c(-, E)|_a$ is finite-to-one. Indeed, $a \cap B_n(C) \subseteq a \cap B_m(E)$ by (2) with $m = \max\{n, c(C, C \cup E), c(E, C \cup E)\}$. For each $(a_j)_{j<\omega}$ in $\mathcal{I}$, pick C' above $\bigcup_j a_j$ and let $a = \bigcup_j (a_j \setminus B_j(C'))$. Then,

$$a_j \setminus a \subseteq a_j \cap B_j(C'),\ a \cap B_n(C') \subseteq \bigcup_{j<n} (a_j \cap B_n(C')).$$

Both right sides are finite, so $\mathcal{I}$ is a P-ideal. Each $Z \in [D]^{\aleph_1}$ has a bound E since $|\kappa| \geq \aleph_1$, and $c(-, E)$ is constant on some $a \in [Z]^{\aleph_0} \setminus \mathcal{I}$. By the P-ideal dichotomy [16], $D = \bigcup_{i<\omega} Z_i$ with $\mathcal{I} \cap \mathcal{P}(Z_i) = [Z_i]^{<\omega}$. Let $H_{i,n} = \{E \in Z_i : Z_i \cap \mathcal{P}(E) \subseteq B_n(E)\}$. At each $E \in Z_i$, choose one point of each color of $c(-, E)$ on $Z_i \cap \mathcal{P}(E)$. The chosen set is in $\mathcal{I} \cap \mathcal{P}(Z_i)$, hence finite. Thus, $Z_i = \bigcup_n H_{i,n}$. Pick a cofinal $H = H_{i,n}$. Then, $c[H^{[2]}] \subseteq n + 1$.

Let $A = \bigcup_{C \in H} B_n(C) \supseteq H$. By (2) and directedness of H, $A \cap \mathcal{P}(C) = B_n(C)$ for each $C \in H$. Thus, $c[A^{[2]}] \subseteq n + 1$. For each $b \in ([D]^{\leq\kappa})^V$, pick $C \in H$ bounding b. Then, $A \cap b = B_n(C) \cap b \in V$. Since $\mathbb{P}^2$ is κ^+-c.c., $A \in V$ by κ^+-approximation [18, Lemma 1.6]. □

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