STATIONARY COMMON NEIGHBORS AND PARTITION HYPOTHESES

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Abstract. From an ω_1-Ramsey cardinal we force that every countable coloring of $[\omega_2]^2$ has a stationary set X and a color i such that every finite subset of X has stationarily many color-i common neighbors in X. The resulting monochromatic graph has diameter at most two after any nonstationary deletion, answering a question of Hrušák–Shelah–Zhang. From a Ramsey cardinal, we force the partition hypothesis $\mathrm{PH}_1(\omega_2)$, improving the huge-cardinal upper bound recorded by Bannister–Bergfalk–Moore–Todorcevic. The proofs share a local normal-ideal construction. The stationary argument uses countably closed models, whereas the Ramsey argument uses pointwise fusion and external countable completeness of the ultrafilter. Finally, for nonempty directed quasi-orders $P \leq_T Q$, every $n < \omega$, and every cardinal λ, we prove that $\mathrm{PH}_n(Q, \lambda)$ implies $\mathrm{PH}_n(P, \lambda)$, answering the Tukey-transfer question of the same authors.

1. Introduction

A graph of cardinality κ is *highly connected* if it remains connected after deleting fewer than κ vertices. The relation $\kappa \to_{\mathrm{hc}} (\kappa)^2_\omega$, introduced by Bergfalk–Hrušák–Shelah [2], asks for a monochromatic highly connected graph of size κ in every countable coloring of $[\kappa]^2$. For an integer $n \geq 2$, the refinement $\kappa \to_{\mathrm{hc},<n} (\kappa)^2_\omega$ requires that, after each such deletion, any two remaining vertices be joined inside the remaining graph by a path of fewer than n edges.

Hrušák–Shelah–Zhang [7, Theorem 5.1] obtained the three-edge relation from a countably complete uniform ideal with a countably closed dense family of positive sets. They asked whether $\omega_2 \to_{\mathrm{hc},<3} (\omega_2)^2_\omega$ is consistent [7, Question 8.2]. Our first result gives a stronger stationary common-neighbor principle.

Throughout, κ is a regular uncountable cardinal unless stated otherwise. We write $[X]^r$ for the r-element subsets of X and $[X]^{<\omega}$ for its finite subsets. A set is *club* in κ if it is closed and unbounded, and *stationary* if it meets every club. For a coloring $c : [\kappa]^2 \to \omega$ and $i < \omega$, let $G_i = (\kappa, c^{-1}\{i\})$ be its color-i graph, and write $G_i[X]$ for its induced subgraph on $X \subseteq \kappa$. For $x < \kappa$, put

$$N_i(x) = \{y < \kappa : y \neq x \text{ and } c(\{x, y\}) = i\}, \qquad \Gamma^X_i(F) = X \cap \bigcap_{x \in F} N_i(x).$$

Here $X \subseteq \kappa$ and $F \subseteq \kappa$ is finite, with $\Gamma^X_i(\varnothing) = X$. The *finite common-neighbor property* is

$$\forall F \in [X]^{<\omega} \quad |\Gamma^X_i(F)| = \kappa. \tag{1.1}$$

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Its *stationary* version is

$$\forall F \in [X]^{<\omega} \quad \Gamma_i^X(F) \text{ is stationary in } \kappa. \tag{1.2}$$

Write $\mathsf{FCN}_\omega(\kappa)$ and $\mathsf{SFCN}_\omega(\kappa)$ when every countable coloring has a set and a color satisfying the corresponding property. The empty-set clause requires $|X| = \kappa$, respectively that X is stationary. Choosing a common neighbor outside a small deleted set gives

$$\mathsf{SFCN}_\omega(\kappa) \implies \mathsf{FCN}_\omega(\kappa) \implies \kappa \to_{\mathrm{hc},<3} (\kappa)^2_\omega. \tag{1.3}$$

Theorem 1.1 (Stationary common neighbors)**.** *Suppose Λ is ω_1-Ramsey and $G \subseteq \mathrm{Coll}(\omega_1, <\Lambda)$ is generic over V. Then*

$$V[G] \models \mathsf{SFCN}_\omega(\omega_2).$$

In particular, every countable coloring of $[\omega_2]^2$ has a monochromatic graph on a stationary set which has diameter at most two after deleting any nonstationary subset of ω_2.

The hypothesis is Holy–Schlicht's ω_1-Ramsey property, recalled in Section 3.3, and is below measurability [6, Proposition 5.11]. Theorem 3.5 and corollary 3.6 give the corresponding results at successors of uncountable regular cardinals and at a weakly inaccessible cardinal which is not weakly compact.

The partition hypotheses $\mathrm{PH}_n(P, \lambda)$ for directed quasi-orders were introduced by Bannister–Bergfalk–Moore–Todorcevic in their study of higher derived limits [1]. We recall the definitions in Section 4.

Theorem 1.2 (Partition hypotheses)**.** *If Λ is Ramsey and $G \subseteq \mathrm{Coll}(\omega_1, <\Lambda)$ is generic over V, then $V[G] \models \mathrm{PH}_1(\omega_2)$. Moreover, for all nonempty directed quasi-orders $P \leq_T Q$, all $n < \omega$, and all cardinals λ,*

$$\mathrm{PH}_n(Q, \lambda) \implies \mathrm{PH}_n(P, \lambda).$$

The first assertion lowers the huge-cardinal upper bound in [1, Theorem 8.13 and Remark 8.14] to a Ramsey cardinal. Our stationary proof uses the stronger ω_1-Ramsey hypothesis. Every ω_1-Ramsey cardinal is Ramsey and is a limit of Ramsey cardinals [6, Proposition 5.4]. The weakly compact lower bound for $\mathrm{PH}_1(\omega_2)$ remains [1, Proposition 8.19], so its exact consistency strength, asked for in [1, Question 9.4], is not determined here. The second assertion answers [1, Question 9.11].

For the stationary result, a finite-extension lemma strengthens the frequent-sequence method of Hrušák–Shelah–Zhang. Normality keeps the witness positive, and the argument localizes to countably closed weak models. For PH_1, one frequent pair suffices. A two-sequence intersection argument replaces model closure by pointwise fusion in the collapse. Its common neighbors need not belong to the selected set, so this argument does not by itself give the two-edge highly connected relation. The Tukey-transfer proof is independent of the ideal construction.

Hrušák–Shelah–Zhang already discuss a strongly Ramsey version of their three-edge argument [7, Section 8]. Common-neighbor constructions occur in [10, proof of Theorem 2.2] and [1, proof of Theorem 8.13]. The local extraction and fusion arguments give the two upper bounds proved here.

2. Common-Neighbor Extraction

An ideal I on κ is *proper* if $\kappa \notin I$, *uniform* if $[\kappa]^{<\kappa} \subseteq I$, and *$\kappa$-complete* if it is closed under unions of fewer than κ members. Countable completeness means closure under countable unions. Normality means closure under diagonal unions

$$\nabla_{\alpha<\kappa} A_\alpha = \{\beta < \kappa : \exists \alpha < \beta \ (\beta \in A_\alpha)\}.$$

Write NS_κ for the ideal of nonstationary subsets of κ. A uniform normal ideal contains NS_κ. Write $I^+ = \mathcal{P}(\kappa) \setminus I$ and $[A]_I$ for the class of A in $\mathcal{P}(\kappa)/I$, where $\mathcal{P}$ denotes the power set. Thus $[A]_I = [B]_I$ exactly when the symmetric difference $A \triangle B$ belongs to I, and $[A]_I \leq [B]_I$ exactly when $A \setminus B \in I$.

A family $\mathcal{D} \subseteq I^+$ is *dense* if every positive set contains a member of $\mathcal{D}$. It is *countably closed* if every decreasing ω-sequence in $\mathcal{D}$ has a lower bound in $\mathcal{D}$. Unless a quotient order is stated, the order is actual inclusion.

Theorem 2.1 (Ideal criteria)**.** *Suppose I is a proper uniform κ-complete ideal on κ and $(I^+, \subseteq)$ has a countably closed dense subset. For every $c : [\kappa]^2 \to \omega$, there are $i < \omega$ and $X \subseteq \kappa$ satisfying* (1.1)*. If I is normal, they can be chosen so that*

$$\forall F \in [X]^{<\omega} \quad \Gamma_i^X(F) \in I^+. \tag{2.1}$$

Thus $\mathsf{FCN}_\omega(\kappa)$ holds, and normality gives $\mathsf{SFCN}_\omega(\kappa)$.

2.1. **Frequent sequences and finite extensions.** Fix an ideal I on κ and a coloring $c : [\kappa]^2 \to \omega$. For $i < \omega$ and $B_0, B_1 \in I^+$, write $\mathrm{Freq}_i(B_0, B_1)$ if

$$\forall D \in I^+ \quad (D \subseteq B_1 \implies \{z \in B_0 : N_i(z) \cap D \in I\} \in I)\,. \tag{2.2}$$

This is the null-set form of [7, Remark 5.2]. Frequency is preserved by positive restriction of either coordinate: the exceptional set shrinks in the first coordinate, and there are fewer positive subsets to test in the second. The following construction is from [7, Theorem 5.1].

Lemma 2.2 (Hrušák–Shelah–Zhang)**.** *Suppose I is proper, uniform and countably complete, and $(I^+, \subseteq)$ has a countably closed dense subset. For every $c : [\kappa]^2 \to \omega$, there are $i < \omega$ and $B_n \in I^+$, $n < \omega$, such that*

$$\forall n < m < \omega \quad \mathrm{Freq}_i(B_n, B_m). \tag{2.3}$$

Proof. Fix a countably closed dense family $\mathcal{D} \subseteq I^+$. We first show that every positive A contains a frequent pair in some color. Otherwise, construct decreasing sequences $C_n, D_n \in \mathcal{D}$, both contained in A, such that

$$N_n(x) \cap D_n \in I \qquad (x \in C_n).$$

At stage n, use $C = D = A$ if $n = 0$, and $C = C_{n-1}$, $D = D_{n-1}$ otherwise. Apply the failure of frequency to this positive pair. It gives a positive $D' \subseteq D$ for which $E = \{x \in C : N_n(x) \cap D' \in I\}$ is positive. Choose $C_n \in \mathcal{D}$ below E and $D_n \in \mathcal{D}$ below D'. Countable closure gives positive lower bounds C_*, D_* for the two sequences. For $x \in C_*$,

$$D_* = (D_* \cap \{x\}) \cup \bigcup_{n<\omega} (D_* \cap N_n(x)) \in I,$$

contrary to the positivity of D_*. Here uniformity handles the singleton, and countable completeness handles the union.

Next, some positive A and color i have the property that every positive subset of A contains an i-frequent pair. Otherwise, put $A_{-1} = \kappa$ and recursively choose $A_n \in \mathcal{D}$ below A_{n-1} such that no positive pair contained in A_n is n-frequent. The negation of the asserted property gives a positive set with this feature, and any dense refinement keeps it. A lower bound $A_* \in \mathcal{D}$ for the sequence contains no frequent pair in any color, contrary to the preceding paragraph.

Set $A_0 = A$. Recursively choose $B_n, A_{n+1} \in I^+$, both contained in A_n, with $\mathrm{Freq}_i(B_n, A_{n+1})$. For $n < m$, we have $B_m \subseteq A_{n+1}$. Frequency is preserved under positive restriction of its second coordinate, giving (2.3). □

Fix a countably complete ideal I, a coloring c, a color i, and a sequence of positive sets satisfying (2.3). Put $B = \bigcup_{n<\omega} B_n$ and, for finite $F \subseteq \kappa$, let

$$D(F, m) = B_m \cap \bigcap_{x \in F} N_i(x) \qquad (m < \omega).$$

Call F *good* if

$$\exists m_F < \omega \; \forall m \geq m_F \quad D(F, m) \in I^+. \tag{2.4}$$

The empty set is good. For good F, put

$$E_F = \{z \in B : F \cup \{z\} \text{ is not good}\}.$$

For nongood F, put $E_F = \varnothing$.

Lemma 2.3 (Finite extensions)**.** *For every finite $F \subseteq \kappa$, $E_F \in I$.*

Proof. If F is not good, then $E_F = \varnothing \in I$. Suppose that F is good and fix m_F as in (2.4). For each $n < \omega$, put $k_n = \max\{m_F, n+1\}$. If $m \geq k_n$, then $D(F, m)$ is a positive subset of B_m and $n < m$. Applying $\mathrm{Freq}_i(B_n, B_m)$ to this particular positive subset gives

$$E_{F,n,m} = \{z \in B_n : N_i(z) \cap D(F, m) \in I\} \in I.$$

For every $z < \kappa$ we have the identity

$$D(F \cup \{z\}, m) = D(F, m) \cap N_i(z).$$

Consequently, if $z \in B_n$ avoids $E_{F,n,m}$ for all $m \geq k_n$, then $D(F \cup \{z\}, m)$ is positive for all such m. This makes $F \cup \{z\}$ good, with threshold k_n, so $z \notin E_F$. Taking the contrapositive gives

$$E_F \cap B_n \subseteq \bigcup_{m \geq k_n} E_{F,n,m} \in I.$$

Countable completeness gives $E_F \cap B_n \in I$. Since $E_F \subseteq B = \bigcup_n B_n$, another countable union gives $E_F = \bigcup_n (E_F \cap B_n) \in I$. □

2.2. **Normal ideals.**

Theorem 2.4 (Positive common-neighbor sets)**.** *Suppose I is a proper uniform normal κ-complete ideal and $\langle B_n : n < \omega \rangle$ satisfies* (2.3)*. There is $X \subseteq B = \bigcup_{n<\omega} B_n$ such that*

$$B \setminus X \in I, \qquad \forall F \in [X]^{<\omega} \quad \Gamma_i^X(F) \in I^+. \tag{2.5}$$

Proof. Fix a bijection $q : [\kappa]^{<\omega} \to \kappa$ and a club $C \subseteq \kappa$ such that

$$\beta \in C, \quad F \in [\beta]^{<\omega} \implies q(F) < \beta. \tag{2.6}$$

To obtain such a club, take the limit ordinals $\beta < \kappa$ closed under the finite-set coding q. This set is closed. To see it is unbounded, start above any prescribed ordinal and choose increasing $\delta_n < \kappa$ so that

$$\delta_{n+1} > \sup\{q(F) : F \in [\delta_n]^{<\omega}\}.$$

Regularity bounds each supremum and $\sup_n \delta_n$ below κ. Every finite subset of this limit is contained in some δ_n, so the limit satisfies (2.6). By Lemma 2.3 and normality,

$$Z = \nabla_{\xi<\kappa} E_{q^{-1}(\xi)} \in I.$$

Set $X = (B \cap C) \setminus Z$. Since $NS_\kappa \subseteq I$,

$$B \setminus X = (B \setminus C) \cup (B \cap Z) \subseteq (\kappa \setminus C) \cup Z \in I.$$

Since $B_0 \subseteq B$ is positive and $B \setminus X \in I$, we have $X \in I^+$.

Every finite subset of X is good. Indeed, start with the empty set and proceed along the increasing enumeration of a finite subset of X. If $F \subseteq X \cap \beta$ is good and $\beta \in X$, then $q(F) < \beta$ by (2.6). Since $\beta \notin Z$, we have $\beta \notin E_F$, so $F \cup \{\beta\}$ is good.

For finite $F \subseteq X$, choose m with $D(F, m) \in I^+$. As $D(F, m) \subseteq B$,

$$D(F, m) \setminus X \subseteq B \setminus X \in I, \qquad D(F, m) \cap X \subseteq \Gamma_i^X(F).$$

The first inclusion makes $D(F, m) \cap X$ positive, and the second makes $\Gamma_i^X(F)$ positive. ☐

2.3. **Without normality.**

Theorem 2.5 (Common-neighbor sets of full cardinality)**.** *Suppose I is a proper uniform κ-complete ideal and $\langle B_n : n < \omega\rangle$ satisfies* (2.3). *There is $X \subseteq B$ satisfying* (1.1) *in color i.*

Proof. Choose a bijection $b : \kappa \to [\kappa]^{<\omega} \times \kappa$ and let $h(\alpha)$ be its first coordinate. Such a bijection exists because $|[\kappa]^{<\omega}| = \kappa$. Each finite set has exactly κ preimages under h. For $a \in [\kappa]^{<\omega}$, let

$$T_a = \{\alpha < \kappa : h(\alpha) = a \text{ and } a \subseteq \alpha\}.$$

Only boundedly many members of $h^{-1}(\{a\})$ are outside T_a, so $|T_a| = \kappa$.

Construct distinct $x_\alpha \in B$, $\alpha < \kappa$, so that every finite subset of $X_\alpha = \{x_\xi : \xi < \alpha\}$ is good. At stage α, let

$$F_\alpha = \{x_\xi : \xi \in h(\alpha) \cap \alpha\}.$$

Choose m with $R = D(F_\alpha, m) \in I^+$. By Lemma 2.3, uniformity and κ-completeness,

$$E = X_\alpha \cup \bigcup_{F \in [X_\alpha]^{<\omega}} E_F \in I, \tag{2.7}$$

since $|[X_\alpha]^{<\omega}| < \kappa$. Choose $x_\alpha \in R \setminus E$. This vertex is new, is adjacent in color i to every member of F_α, and makes $F \cup \{x_\alpha\}$ good for every finite $F \subseteq X_\alpha$. To check the successor stage, a finite subset of $X_{\alpha+1} = X_\alpha \cup \{x_\alpha\}$ either lies in X_α or has the form $F \cup \{x_\alpha\}$ with $F \in [X_\alpha]^{<\omega}$. The induction hypothesis handles the first case and avoidance of E_F handles the second. At a limit stage, the finitely many indices of any finite subset are bounded below that stage, so the same requirement follows from an earlier stage.

Put $X = \{x_\alpha : \alpha < \kappa\}$. Fix a finite $F \subseteq X$ and let $a = \{\xi < \kappa : x_\xi \in F\}$. The vertices are distinct, so a is finite and $F = \{x_\xi : \xi \in a\}$. For $\alpha \in T_a$, we have $h(\alpha) = a \subseteq \alpha$, and hence

$$F_\alpha = \{x_\xi : \xi \in h(\alpha) \cap \alpha\} = \{x_\xi : \xi \in a\} = F.$$

Therefore $\{x_\alpha : \alpha \in T_a\} \subseteq \Gamma_i^X(F)$. Distinctness and $|T_a| = \kappa$ give $|\Gamma_i^X(F)| = \kappa$. □

Proof of Theorem 2.1. Apply Lemma 2.2 and then Theorem 2.5. If I is normal, use Theorem 2.4 instead. Since $NS_\kappa \subseteq I$, its positive sets are stationary. □

Lemma 2.3 uses only countable completeness. Full κ-completeness is used in (2.7).

3. Local Ideals and the Consistency Theorem

3.1. **From quotient order to inclusion.** All forcing notions have a largest condition, denoted by 1, and $p \leq q$ means that p is stronger than q. For a separative forcing $\mathbb{R}$, let $\mathrm{RO}(\mathbb{R})$ be its regular-open Boolean completion. We identify conditions with their nonzero Boolean values. For a Boolean algebra $\mathbb{B}$, write $\mathbb{B}^+ = \mathbb{B} \setminus \{0\}$. A Boolean embedding is an injective map preserving the Boolean operations and $0, 1$.

Lemma 3.1. *Let $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ be a Boolean algebra closed under countable unions, let I be a countably complete ideal on $\mathcal{A}$, and let $\Phi : \mathcal{A}/I \to \mathbb{B}$ be a Boolean embedding into a complete Boolean algebra. Suppose $E \subseteq \mathbb{B}^+$ is countably closed and dense and $E \subseteq \mathrm{ran}(\Phi)$. Then*

$$\mathcal{D} = \{A \in I^+ : \Phi([A]_I) \in E\}$$

is countably closed and dense in $(I^+, \subseteq)$, where $I^+ = \mathcal{A} \setminus I$.

Proof. For $A \in I^+$, choose $e \in E$ below $\Phi([A]_I)$, and choose B with $\Phi([B]_I) = e$. Then $\Phi([A \cap B]_I) = e$, so $A \cap B \in \mathcal{D}$ and $A \cap B \subseteq A$.

Given decreasing $A_n \in \mathcal{D}$, choose $e \in E$ below every $\Phi([A_n]_I)$, and choose B with $\Phi([B]_I) = e$. For every n,

$$\Phi([B \setminus A_n]_I) = e \wedge \neg\Phi([A_n]_I) = 0.$$

Thus $N = \bigcup_{n<\omega}(B \setminus A_n) \in I$, and

$$B \setminus N \subseteq \bigcap_{n<\omega} A_n, \qquad \Phi([B \setminus N]_I) = e.$$

Therefore $B \setminus N \in \mathcal{D}$ is a lower bound. □

3.2. **Local normal extraction.** Here ZFC^- means ZFC without Power Set, with Collection in place of Replacement and the well-ordering form of Choice. The notation $W^\omega \subseteq W$ means external closure: every countable sequence of elements of W in the ambient universe belongs to W. An internal sequence is a sequence which is an element of W. An ideal on $\mathcal{P}(\kappa) \cap W$ need not belong to W. We write $\mathrm{tc}(x)$ for the transitive closure of a set x, V_α for the αth level of the cumulative hierarchy, and $H(\theta) = \{x : |\,\mathrm{tc}(\{x\})| < \theta\}$ for an infinite cardinal θ.

Lemma 3.2 (Local normal extraction)**.** *Let W be a transitive model of ZFC^- containing κ and closed under countable sequences. Put $\mathcal{A} = \mathcal{P}(\kappa) \cap W$. Suppose I is a proper ideal on $\mathcal{A}$ such that:*

(1) *I contains the bounded members of $\mathcal{A}$ and the complements of clubs in κ belonging to W.*
(2) *I is closed under countable unions and under diagonal unions of κ-sequences of null sets belonging to W.*
(3) *For each $\langle A_\xi : \xi < \kappa \rangle \in W$ of members of $\mathcal{A}$, the set $\{\xi < \kappa : A_\xi \in I\}$ belongs to W.*
(4) *$(\mathcal{A} \setminus I, \subseteq)$ has an externally countably closed dense subset.*

For every $c : [\kappa]^2 \to \omega$ in W, there are $i < \omega$ and $X \in \mathcal{A} \setminus I$ such that

$$\Gamma_i^X(F) \in \mathcal{A} \setminus I \qquad (F \in [X]^{<\omega}).$$

Proof. All positive sets and all quantifiers over positive subsets are now restricted to $\mathcal{A} \setminus I$. The algebra $\mathcal{A}$ is closed under external countable unions: a countable sequence of its members belongs to W, and its union is computed correctly by transitivity. The model W regards κ as a regular uncountable cardinal, since an internal counterexample to cardinality or regularity would be an actual counterexample. Thus the finite-set codings used below exist inside W. Since W is transitive and contains κ, it contains every finite subset of κ and computes $[\kappa]^{<\omega}$ correctly. For $C, D \in \mathcal{A}$ and $i < \omega$, condition (3), applied to $\langle N_i(x) \cap D : x < \kappa \rangle$, puts the exceptional set

$$\{x \in C : N_i(x) \cap D \in I\}$$

in W. We can now run both decreasing constructions from Lemma 2.2 using only this algebra. At a stage where frequency fails, condition (3) puts the exceptional set in $\mathcal{A}$, so it can be refined into the dense family from (4). That family supplies a positive lower bound after countably many stages. For a point x in the first lower bound, the other lower bound is partitioned into its singleton part and the countably many sets in the colors. Conditions (1) and (2) give the same contradiction as in Lemma 2.2. The second construction then finds one color which works in every positive subset of a fixed positive set. Recursively choosing frequent pairs gives $B_n \in \mathcal{A} \setminus I$ with $\mathrm{Freq}_i(B_n, B_m)$ for $n < m$. These choices are made externally. Countable closure puts the resulting sequence $\langle B_n : n < \omega \rangle$ in W.

Put $B = \bigcup_n B_n$ and $D(F, m) = B_m \cap \bigcap_{x \in F} N_i(x)$. Choose in W a bijection $h : \kappa \to [\kappa]^{<\omega} \times \omega$ and define $R_\xi = D(F, m)$ when $h(\xi) = (F, m)$. The sequence $\langle R_\xi : \xi < \kappa \rangle$ belongs to W. By (3), $T = \{\xi < \kappa : R_\xi \in I\}$ belongs to W. Thus the family

$$\mathcal{H} = \{F \in [\kappa]^{<\omega} : \exists m_0 < \omega \ \forall m \geq m_0 \quad h^{-1}(F, m) \notin T\}$$

belongs to W by Separation. Since $R_{h^{-1}(F,m)} = D(F, m)$, this is exactly the family of good finite sets. Consequently, the whole family

$$E_F = \{z \in B : F \cup \{z\} \notin \mathcal{H}\} \quad (F \in \mathcal{H}), \qquad E_F = \varnothing \quad (F \notin \mathcal{H})$$

is internal to W. More precisely, Separation gives each E_F from B and $\mathcal{H}$, and Replacement gives the function $F \mapsto E_F$ on the internal set $[\kappa]^{<\omega}$. For a good F, local frequency applied to $D(F, m)$ gives

$$\{z \in B_n : N_i(z) \cap D(F, m) \in I\} \in I \quad (m \geq \max\{m_F, n+1\}).$$

Condition (3) makes these exceptional sets members of $\mathcal{A}$. The same containment as in Lemma 2.3 bounds $E_F \cap B_n$ by their countable union. Condition (2), first over m and then over n, gives $E_F \in I$. This also holds for nongood F by the definition $E_F = \varnothing$.

Choose in W a bijection $q : [\kappa]^{<\omega} \to \kappa$ and a club $C \subseteq \kappa$ with $q(F) < \beta$ whenever $\beta \in C$ and $F \subseteq \beta$ is finite. The sequence $\langle E_{q^{-1}(\xi)} : \xi < \kappa \rangle$ belongs to W, so (2) gives

$$Z = \nabla_{\xi<\kappa} E_{q^{-1}(\xi)} \in I.$$

Set $X = (B \cap C) \setminus Z$. Then $X \in \mathcal{A}$ and

$$B \setminus X = (B \setminus C) \cup (B \cap Z) \in I.$$

Since $B_0 \subseteq B$ is positive, this identity implies $X \notin I$. For a finite $F = \{\beta_0 < \cdots < \beta_{r-1}\} \subseteq X$, put $F_j = \{\beta_\ell : \ell < j\}$ for $j \leq r$. We prove by induction that $F_j \in \mathcal{H}$. This holds for $j = 0$. Suppose $j < r$ and $F_j \in \mathcal{H}$. We have $q(F_j) < \beta_j$ because $\beta_j \in C$. If $\beta_j \in E_{F_j}$, this inequality would put β_j in the diagonal union Z. Since $\beta_j \in X$ avoids Z, we obtain $F_{j+1} \in \mathcal{H}$. Hence $F \in \mathcal{H}$. Choose m with $D(F, m) \notin I$. Then

$$D(F, m) \cap X \notin I, \qquad D(F, m) \cap X \subseteq \Gamma_i^X(F) \in \mathcal{A}.$$

This proves the conclusion. □

3.3. **Ramsey-like models.** A *weak κ-model* is a set M of cardinality κ with $\kappa+1 \subseteq M$ and $(M, \in) \models \mathsf{ZFC}^-$. It need not be transitive. We pass to its transitive collapse when taking an ultrapower. For transitive M, N, an elementary embedding $j : M \to N$ is *κ-powerset preserving* if its critical point is κ and

$$\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N.$$

Following [6, Definition 5.1], a cardinal κ is *ω_1-Ramsey* if, for arbitrarily large regular θ, every $A \subseteq \kappa$ belongs to a weak κ-model $M^* \prec H(\theta)$ with $(M^*)^\omega \subseteq M^*$ whose collapse admits a κ-powerset preserving embedding. By [6, Theorem 5.5], the prescribed parameter may be any element of $H(\theta)$. Such a cardinal is inaccessible [6, Proposition 5.4].

An M-ultrafilter U is a uniform ultrafilter on $\mathcal{P}(\kappa) \cap M$ which is κ-complete and normal with respect to sequences in M. It is *weakly amenable* if $U \cap a \in M$ whenever $a \in M$ has cardinality at most κ in M. A κ-powerset preserving embedding gives a weakly amenable M-ultrafilter whose normal ultrapower is well founded and remains κ-powerset preserving [6, Lemma 2.12]. We use this ultrapower representation. For background on these embeddings, see [5, Section 2].

Lemma 3.3 (Powerset agreement and names)**.** *Suppose M, N are transitive models of ZFC^- containing the infinite cardinal κ, and $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$. Then M and N have the same sets of hereditary size at most κ, as computed in the respective models. If a forcing partial order P, including its order relation, belongs to this common collection and G is P-generic over both models, then*

$$\mathcal{P}(\kappa) \cap M[G] = \mathcal{P}(\kappa) \cap N[G].$$

Proof. If $x \in M$ has hereditary size at most κ in M, code $\mathrm{tc}(\{x\})$, with a distinguished point for x, by a well-founded extensional relation on a subset of κ. The canonical pairing of ordinals is absolute between transitive models and maps $\kappa \times \kappa$ into κ. It therefore codes this relation, its domain, and the distinguished point by a subset of κ in M, hence in N. The relation is actually well founded, so its

Mostowski collapse in N is the same transitive set. Thus $x \in N$ with the same size bound. Interchanging M and N proves the first assertion.

For the second assertion, let $A \subseteq \kappa$ belong to $M[G]$ and choose a P-name $\tau \in M$ with $\tau_G = A$. In M, form

$$\sigma = \{(\check{\alpha}, p) : \alpha < \kappa,\ p \in P,\ p \Vdash_P \check{\alpha} \in \tau\}.$$

If $\alpha \in \sigma_G$, some $p \in G$ forces $\check{\alpha} \in \tau$, so $\alpha \in \tau_G = A$. Conversely, if $\alpha \in A$, the truth lemma in M gives such a $p \in G$, so $\alpha \in \sigma_G$. Thus $\sigma_G = A$. This name has hereditary size at most κ in M: it has at most κ pairs, the check names for ordinals below κ have total hereditary size κ, and P has hereditary size at most κ. The first assertion gives $\sigma \in N$, so $A \in N[G]$. The reverse inclusion is identical. □

Lemma (Atomic forcing absoluteness). *Let T_0, T_1 be transitive models of ZFC^- containing the same forcing partial order $R = (R, \leq_R)$, including its entire underlying set and order relation. If $\sigma, \tau \in T_0 \cap T_1$ are R-names and $p \in R$, then*

$$T_0 \models p \Vdash_R \varphi(\sigma, \tau) \iff T_1 \models p \Vdash_R \varphi(\sigma, \tau)$$

for $\varphi(\sigma, \tau)$ equal to $\sigma \in \tau$, $\sigma = \tau$, $\sigma \notin \tau$, or $\sigma \neq \tau$. Either model may be the ambient universe. No claim is made for arbitrary formulas.

Proof. The recursive membership clause says that $p \Vdash_R \sigma \in \tau$ exactly when the conditions $r \leq p$ for which

$$\exists (\rho, s) \in \tau \quad r \leq s \text{ and } r \Vdash_R \sigma = \rho$$

are dense below p. Equality is mutual inclusion, with

$$p \Vdash_R \sigma \subseteq \tau \iff \forall (\rho, s) \in \sigma\ \forall r \leq p, s \quad r \Vdash_R \rho \in \tau.$$

Transitivity gives the same subnames in both models. Since the whole forcing and its order are shared, the density tests and the ranges of all displayed quantifiers also agree. Simultaneous induction on pairs of name ranks therefore gives agreement for membership and equality. Finally, $p \Vdash_R \neg\varphi$ means that no $r \leq p$ forces φ. This gives agreement for the two negated atomic formulas. □

3.4. **The seed ideal.** Both consistency proofs use the following construction. No external countable closure of either model is assumed.

Lemma 3.4 (Seed ideals). *Let $j : M \to N$ be a κ-powerset preserving normal ultrapower embedding between transitive set models of ZFC^-. Let $P \in M \cap N$ be a separative forcing of hereditary size at most κ in both models. Suppose that in N there are a separative forcing Q and an order isomorphism*

$$\iota : j(P) \longrightarrow P \times Q, \qquad \iota(j(p)) = (p, 1_Q) \quad (p \in P).$$

In this lemma and its applications, product notation uses this identification, with forcing names transported recursively along ι. Let $G \subseteq P$ be generic over V, and put $W = M[G]$, $\mathcal{A} = \mathcal{P}(\kappa) \cap W$ and $\mathbb{B} = \mathrm{RO}(Q)^{V[G]}$. For $H \subseteq Q$ generic over $V[G]$, let $j_H : W \to N[G \times H]$ be the induced lift. These lifts define in $V[G]$ a Boolean homomorphism

$$\mu(A) = \left\|\kappa \in j_{\dot{H}}(A)\right\|_{\mathbb{B}}, \qquad I = \{A \in \mathcal{A} : \mu(A) = 0\}, \tag{3.1}$$

with the following properties.

(1) *I is proper and contains the bounded members of $\mathcal{A}$ and complements of clubs in κ belonging to W. It is closed under unions of null sets indexed by internal sequences of length less than κ, and under diagonal unions of internal κ-sequences of null sets.*
(2) *For every $\langle A_\xi : \xi < \kappa\rangle \in W$ from $\mathcal{A}$, its null-index set $\{\xi < \kappa : A_\xi \in I\}$ belongs to W.*
(3) *Every condition of Q belongs to the range of μ. More precisely, if $B \in \mathcal{A}$ and $q \in Q$ satisfies $q \leq \mu(B)$, then some $A \in \mathcal{A}$ satisfies $A \subseteq B$ and $\mu(A) = q$.*

Proof. Normality and powerset preservation give

$$\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N, \qquad N = \{j(f)(\kappa) : f \in M,\ \mathrm{dom}(f) = \kappa\}. \tag{3.2}$$

The forcing P and its order are the same in both models, so G is generic over both. By Lemma 3.3,

$$\mathcal{P}(\kappa) \cap M[G] = \mathcal{P}(\kappa) \cap N[G]. \tag{3.3}$$

For $H \subseteq Q$ generic over $V[G]$, the product $G \times H$ is $P \times Q$-generic over N. Its pullback $K_H = \iota^{-1}[G \times H]$ is $j(P)$-generic and contains $j``G$. We write $N[G \times H]$ for $N[K_H]$. The lifting criterion gives

$$j_H : M[G] \longrightarrow N[G \times H], \qquad j_H(\dot{x}_G) = j(\dot{x})_{K_H}.$$

Equation (3.3) concerns the two initial P-extensions, not powerset preservation by this lift.

To define (3.1), choose an M-name for A, apply j, and evaluate its initial P-coordinate at G, leaving a Q-name. If two M-names have the same G-value, some $p \in G$ forces their equality over M. Since $\iota(j(p)) = (p, 1_Q)$, the two tail names are forced equal. Thus μ is well defined. The models and j are sets, so these name operations define a set map in $V[G]$.

Each lift preserves finite unions and complements relative to κ. Membership of κ in their images therefore gives a Boolean homomorphism, with $\mu(\kappa) = 1$. If $A \subseteq \beta < \kappa$, then $j_H(A) \subseteq \beta$, giving $\mu(A) = 0$. For an internal sequence of length $\tau < \kappa$, j_H fixes its index set, so

$$\mu\Big(\bigcup_{\xi<\tau} A_\xi\Big) = \bigvee_{\xi<\tau} \mu(A_\xi).$$

For an internal κ-sequence of null sets, every lift satisfies

$$\kappa \in j_H\Big(\nabla_{\xi<\kappa} A_\xi\Big) \quad\Longleftrightarrow\quad \exists \xi < \kappa\ \big(\kappa \in j_H(A_\xi)\big).$$

The Boolean value is zero. If $C \in W$ is club in κ, then $j_H(C)$ contains C and is closed at $\kappa < j(\kappa)$, so $\mu(C) = 1$. This proves (1).

For (2), take $\vec{A} = \langle A_\xi : \xi < \kappa\rangle \in W$. Choose an M-name $\dot{\vec{A}}$ and $p \in G$ forcing over M that it is a κ-sequence of subsets of κ. Apply j and evaluate the initial coordinate at G. As $\iota(j(p)) = (p, 1_Q)$, the resulting tail name is forced by 1_Q over $N[G]$ to be a sequence of length $j(\kappa)$. The maximum principle and Collection in $N[G]$ give Q-names $\langle \dot{B}_\xi : \xi < \kappa\rangle$ for its first κ coordinates. Thus $(\dot{B}_\xi)_H = j_H(A_\xi)$. Separation gives

$$E = \{\xi < \kappa : 1_Q \Vdash_Q \check{\kappa} \notin \dot{B}_\xi\} \in N[G]. \tag{3.4}$$

By the atomic forcing absoluteness lemma, $N[G]$ and $V[G]$ compute the negated atomic statement in (3.4) identically: they share the entire forcing Q, its order, and the indicated names. Hence

$$E = \{\xi < \kappa : \mu(A_\xi) = 0\} = \{\xi < \kappa : A_\xi \in I\}.$$

By (3.3), $E \in W$.

For (3), fix $q \in Q$. Represent the actual $j(P)$-condition $\iota^{-1}(1_P, q)$ as $j(f_0)(\kappa)$ for some $f_0 \in M$. By Łoś's theorem, f_0 is P-valued on a set in the M-ultrafilter. Replacing its other values by 1_P gives $f : \kappa \to P$ in M with the same ultrapower value. Put $B_f = \{\alpha < \kappa : f(\alpha) \in G\} \in \mathcal{A}$. Then

$$\kappa \in j_H(B_f) \iff \iota(j(f)(\kappa)) \in G \times H \iff q \in H,$$

so $\mu(B_f) = q$. If $q \leq \mu(B)$, take $A = B \cap B_f$. Then $\mu(A) = \mu(B) \wedge q = q$. □

3.5. **The stationary collapse theorem.**

Claim (Ambient-stationarity reflection)**.** *Let $\theta > \kappa$ be regular, let $E \prec H(\theta)$ with $\kappa + 1 \subseteq E$, and let W be the transitive collapse of E. Suppose I is an ideal on $\mathcal{A} = \mathcal{P}(\kappa) \cap W$ containing the complements of all clubs in κ belonging to W. Then every $A \in \mathcal{A} \setminus I$ is stationary in the ambient universe.*

Proof. The collapse fixes κ and all subsets of κ belonging to E, so $\mathcal{P}(\kappa) \cap E = \mathcal{A}$. If $A \in \mathcal{A}$ is nonstationary, a club disjoint from A belongs to $H(\theta)$. Elementarity gives such a club $C \in E$. Its collapse is C itself, so $C \in W$ is an actual club disjoint from A. Thus $A \subseteq \kappa \setminus C \in I$, and $A \in I$. □

Theorem 3.5. *Suppose κ is ω_1-Ramsey and $\nu < \kappa$ is uncountable regular. If $G \subseteq \mathrm{Coll}(\nu, {<}\kappa)$ is generic over V, then*

$$V[G] \models \mathsf{SFCN}_\omega(\kappa), \qquad \kappa = \nu^+.$$

Proof. Let $P = \mathrm{Coll}(\nu, {<}\kappa)$, with conditions partial functions on $[\nu, \kappa) \times \nu$ of size less than ν satisfying $p(\alpha, \xi) < \alpha$, ordered by reverse inclusion. For such a condition, write

$$\mathrm{supp}(p) = \{\alpha : \exists \xi\ ((\alpha, \xi) \in \mathrm{dom}(p))\}.$$

Fix a nice P-name $\dot{c}$ for a coloring $[\kappa]^2 \to \omega$.

Models and closure. Choose a sufficiently large regular θ and a countably closed weak κ-model $M^* \prec H(\theta)$ as above containing $\dot{c}$, P, V_κ, and an enumeration of V_κ of length κ. Since $\kappa \subseteq M^*$, the enumeration gives $V_\kappa \subseteq M^*$. Let $\pi : M^* \to M$ be the transitive collapse. It fixes κ, P and $\dot{c}$, since every condition and every member of the nice name has rank below κ. Let $j : M \to N$ be the κ-powerset preserving normal ultrapower. An induction on rank shows that j fixes V_κ pointwise: each $x \in V_\kappa$ has in M an enumeration of length less than κ, and j fixes the length and the previously fixed members. Thus both models contain V_κ and compute the initial collapse P correctly.

The collapse of M^* is externally countably closed. For N, given a sequence $\langle y_n : n < \omega \rangle$ of members of N in V, choose $f_n \in M$ with $y_n = j(f_n)(\kappa)$. Countable closure puts $\langle f_n : n < \omega \rangle$ in M, and the function

$$h(\alpha) = \langle f_n(\alpha) : n < \omega \rangle \qquad (\alpha < \kappa)$$

belongs to M. Then $j(h)(\kappa) = \langle y_n : n < \omega \rangle$, proving external countable closure of N.

Unions give $<\nu$-closure of P, and the Δ-system argument at the inaccessible κ gives the κ-chain condition. The hereditary size of P, including its order, is κ in M: a witnessing enumeration can be chosen in M^* and collapsed. Lemma 3.3 gives the same bound in N. Since P adds no countable sequences of ground-model objects, $M[G]$ and $N[G]$ remain externally countably closed. Indeed, choose ground-model names for the terms of such a sequence. The sequence of names belongs to V, then to the relevant model by countable closure, and evaluates there to the given sequence.

The local ideal. The coordinate split in N gives the isomorphism ι of Lemma 3.4, with tail Q and $\iota(j(p)) = (p, 1_Q)$ for $p \in P$. A decreasing countable sequence of conditions of the fixed tail Q in $V[G]$ is already in V, then in N. Its union is a condition of Q, so Q is externally countably closed. Apply Lemma 3.4, writing $W = M[G]$ and $\mathcal{A} = \mathcal{P}(\kappa) \cap W$. Its ideal I satisfies conditions (1) and (3) of Lemma 3.2 and is normal for internal sequences. Every external countable sequence from $\mathcal{A}$ belongs to W, so internal completeness gives external countable completeness as well. This verifies condition (2).

The induced Boolean embedding

$$\mathcal{A}/I \longrightarrow \mathrm{RO}(Q)^{V[G]}, \qquad [A]_I \longmapsto \mu(A)$$

has Q in its range. The algebra $\mathcal{A}$ is closed under external countable unions. Thus Lemma 3.1, with $E = Q$, gives a countably closed dense family under actual inclusion, verifying condition (4). Lemma 3.2 now supplies $i < \omega$ and $X \in \mathcal{A}$ with X and every $\Gamma_i^X(F)$ locally positive.

Ambient stationarity. We have

$$M^*[G] \prec H(\theta)^{V[G]}.$$

Here $P \in M^*$, $P \subseteq M^*$, and $|P| = \kappa < \theta$. Small forcing preserves the regularity of θ and gives $H(\theta)^V[G] = H(\theta)^{V[G]}$: every member of the right-hand side has a name of hereditary size less than θ in V. For the Tarski–Vaught test, take parameter names in M^* and $p \in G$ forcing an existential statement over $H(\theta)^V$. Then $p \in M^*$, and the maximum principle supplies a witness name in $H(\theta)^V$. Elementarity supplies one in M^* with the same forcing property. Its value is the required witness in $M^*[G]$. The collapse of this extension is $M[G]$ and fixes its subsets of κ.

Apply the ambient-stationarity claim in $V[G]$ with $E = M^*[G]$ and $W = M[G]$. Its ideal hypothesis is Lemma 3.4(1). Thus X and its finite common-neighbor sets are stationary in $V[G]$. Finally, P makes $\kappa = \nu^+$. $\square$

Proof of Theorem 1.1. Apply Theorem 3.5 with $\nu = \omega_1$. If X, i witness $\mathsf{SFCN}_\omega(\omega_2)$ and Z is nonstationary, then for every finite $F \subseteq X \setminus Z$,

$$\Gamma_i^{X \setminus Z}(F) = \Gamma_i^X(F) \setminus Z$$

is stationary. The empty-set and pair cases give a stationary remaining graph of diameter at most two. $\square$

3.6. **The Cohen theorem.**

Corollary 3.6 (Cohen version)**.** *Assume CH and let κ be ω_1-Ramsey. If $G \subseteq \mathrm{Add}(\omega_1, \kappa)$ is generic over V, then*

$$V[G] \models \mathsf{SFCN}_\omega(\kappa).$$

In this extension, $\kappa = 2^{\omega_1}$ is weakly inaccessible and is not weakly compact.

Proof. Use countable partial functions from $\kappa \times \omega_1$ to 2 for $P = \mathrm{Add}(\omega_1, \kappa)$. Unions give countable closure. Under CH, the generalized Δ-system lemma gives a Δ-system of size ω_2 from any ω_2 countable domains. There are at most $2^\omega = \omega_1$ restrictions on its root, so two conditions agree there and are compatible. Thus P is ω_2-c.c. and preserves cardinals and cofinalities.

For a nice coloring name, choose M^* and $j : M \to N$ as in Theorem 3.5. Every condition is fixed by j, and the coordinate split gives the identification of Lemma 3.4, with

$$Q = \mathrm{Add}(\omega_1, j(\kappa) \setminus \kappa)^N.$$

The ground-model inaccessibility of κ gives $\kappa^{\omega_1} = \kappa$, hence P has hereditary size κ. Choose its witnessing enumeration in M^* as before. Both models and their initial extensions are externally countably closed, and the same countable-sequence argument makes the fixed tail Q externally countably closed. Lemmas 3.1 and 3.4 therefore give all four hypotheses of Lemma 3.2. Since $P \subseteq M^*$ and $|P| = \kappa < \theta$, the Tarski–Vaught argument in Theorem 3.5 gives $M^*[G] \prec H(\theta)^{V[G]}$. The ambient-stationarity claim now makes the locally positive sets stationary in $V[G]$. Thus $\mathsf{SFCN}_\omega(\kappa)$ holds.

For the cardinal-arithmetic claim, every ground-model function $\omega_1 \to \kappa$ has bounded range. For each $\beta < \kappa$,

$$|\beta|^{\omega_1} \leq 2^{\max\{|\beta|, \omega_1\}} < \kappa,$$

so $\kappa^{\omega_1} = \kappa$. A nice name for a subset of ω_1 uses ω_1 antichains, each of size at most ω_1. As $|P| = \kappa$, there are at most $\kappa^{\omega_1} = \kappa$ such names. Distinct Cohen coordinates give κ distinct subsets, so $2^{\omega_1} = \kappa$. Cardinal preservation keeps κ weakly inaccessible, while this equality prevents strong inaccessibility and hence weak compactness. $\square$

4. Partition Hypotheses and Tukey Transfer

4.1. **Definitions and the common-neighbor implication.** Let $P = (P, \leq_P)$ be a nonempty *directed quasi-order*. Thus $\leq_P$ is reflexive and transitive, and every finite subset of P has a common upper bound. Antisymmetry is not required. We omit the subscript on $\leq_P$ when the quasi-order is clear.

For an integer $r \geq 1$, let

$$P^{\leq r} = \bigcup_{1 \leq j \leq r} P^j.$$

Members are ordered tuples, and repeated coordinates are allowed. The length of a tuple s is $|s|$. We also use $P^0 = \{\varnothing\}$ for the singleton consisting of the empty tuple. If a is a tuple and $x \in P$, $a \frown (x)$ is obtained by appending x to a. For $s \in P^k$ and $t \in P^\ell$, write $s \trianglelefteq t$ if there is a strictly increasing map $u : k \to \ell$ with $s(j) = t(u(j))$ for every $j < k$. Thus s is obtained by deleting coordinates of t. This relation concerns positions in tuples, not the quasi-order on P.

A *full r-flag* is a tuple $(s_1, \ldots, s_r)$ of nonempty tuples such that $|s_j| = j$ and $s_1 \trianglelefteq \cdots \trianglelefteq s_r$. A map $F : P^{\leq r} \to P$ is *r-cofinal* if

$$p \leq_P F((p)) \quad (p \in P), \qquad s \trianglelefteq t \implies F(s) \leq_P F(t).$$

We abbreviate $F((p))$ by $F(p)$ and $F((p,q))$ by $F(p,q)$. The induced map on full flags is

$$F^*(s_1, \ldots, s_r) = (F(s_1), \ldots, F(s_r)) \in P^r.$$

For $n < \omega$ and a cardinal λ, the assertion $\mathrm{PH}_n(P, \lambda)$ says that every coloring $c : P^{n+1} \to \lambda$ admits an $(n+1)$-cofinal F for which $c \circ F^*$ is constant on all full $(n+1)$-flags. These are the conventions of [1, Definitions 3.1–3.2]. The tuples $F^*(s_1, \dots, s_r)$ are weakly increasing in $\leq_P$, even though the input tuples need not be. No definability or measurability is required of c or F.

For an ordinal κ, we write $\mathrm{PH}_n(\kappa)$ for $\mathrm{PH}_n(\kappa, \omega)$ with the usual ordinal order.

Proposition 4.1 (Upper common neighbors)**.** *Suppose that every coloring $c : [\kappa]^2 \to \omega$ has an unbounded $X \subseteq \kappa$ and a color $i < \omega$ such that*

$$\alpha < \beta \text{ in } X \quad \Longrightarrow \quad \exists \gamma \in (\beta, \kappa) \quad c(\{\alpha, \gamma\}) = c(\{\beta, \gamma\}) = i. \tag{4.1}$$

Then $\mathrm{PH}_1(\kappa)$ holds. In particular, $\mathsf{FCN}_\omega(\kappa)$ implies $\mathrm{PH}_1(\kappa)$.

Proof. Given $d : \kappa^2 \to \omega$, let $c(\{\xi, \eta\}) = d(\min\{\xi, \eta\}, \max\{\xi, \eta\})$. Choose X, i as in (4.1) and enumerate X increasingly as $\langle b_\alpha : \alpha < \kappa \rangle$. Set $F(\alpha) = b_\alpha$. For $\alpha \neq \beta$, choose $F(\alpha, \beta)$ above both b_α, b_β and joined to them in color i. For $F(\alpha, \alpha)$, choose a later point of X and a common upper neighbor of it and b_α.

Since $\alpha \leq b_\alpha$ and $b_\alpha, b_\beta < F(\alpha, \beta)$, the map F is 2-cofinal. The full flags ending in (α, β) begin with (α) or (β), and

$$d(F(\alpha), F(\alpha, \beta)) = d(F(\beta), F(\alpha, \beta)) = i.$$

Thus $d \circ F^*$ is constant, including on repeated-coordinate flags. An FCN witness satisfies (4.1), since its pair common-neighbor sets have size κ and are unbounded. □

The proof uses the common-neighbor construction in [1, proof of Theorem 8.13], without its dense-ideal hypothesis. Unlike (1.1), condition (4.1) does not require the common neighbors to belong to X.

4.2. **A Ramsey-cardinal upper bound.** For (4.1), one frequent pair is enough. The next lemma uses frequency as defined in (2.2), with all positive sets and quantifiers restricted to the indicated algebra.

Lemma 4.2 (One frequent pair)**.** *Let W be a transitive model of ZFC^- containing κ and $c : [\kappa]^2 \to \omega$, and put $\mathcal{A} = \mathcal{P}(\kappa) \cap W$. Let I be a proper ideal on $\mathcal{A}$ containing its bounded members, normal for internal κ-sequences, and such that $\{\xi < \kappa : A_\xi \in I\} \in W$ for every $\langle A_\xi : \xi < \kappa \rangle \in W$ from $\mathcal{A}$. For $A, B \in \mathcal{A} \setminus I$ and $i < \omega$, the following hold.*

(1) *If $\mathrm{Freq}_i(A, B)$, there is $X \in \mathcal{A} \setminus I$, with $X \subseteq A$, such that $\Gamma_i^B(\{\alpha, \beta\}) \notin I$ for all $\alpha < \beta$ in X.*
(2) *If $\mathrm{Freq}_i(A, B)$ fails, there are $E, D' \in \mathcal{A} \setminus I$ with $E \subseteq A$ and $D' \subseteq B$ such that $c(\{\alpha, \beta\}) \neq i$ whenever $\alpha \in E$, $\beta \in D'$ and $\alpha < \beta$.*

Proof. For (1), put $R_\alpha = N_i(\alpha) \cap B$ and $A_0 = \{\alpha \in A : R_\alpha \notin I\}$. Null-index amenability gives $A_0 \in W$. Frequency applied to B gives $A \setminus A_0 \in I$, so $A_0 \notin I$. Define

$$E_\alpha = \{\beta \in A_0 : R_\alpha \cap R_\beta \in I\} \quad (\alpha \in A_0), \qquad E_\alpha = \varnothing \quad (\alpha \notin A_0).$$

For $\alpha \in A_0$, apply frequency to the positive subset $R_\alpha \subseteq B$. Since $N_i(\beta) \cap R_\alpha = R_\beta \cap R_\alpha$, it gives $E_\alpha \in I$. The whole sequence is internal: code the family $\langle R_\alpha \cap R_\beta :$

$\alpha, \beta < \kappa\rangle \in W$ by a κ-sequence, obtain its null-index relation in W, and take its rows restricted to A_0. Thus

$$Z = \nabla_{\alpha<\kappa} E_\alpha \in I, \qquad X = A_0 \setminus Z \notin I.$$

If $\alpha < \beta$ lie in X, avoidance of Z gives $\beta \notin E_\alpha$, proving (1).

For (2), choose a positive $D \subseteq B$ such that $E = \{\alpha \in A : N_i(\alpha) \cap D \in I\}$ is positive. The null-index hypothesis makes E internal. The sequence

$$T_\alpha = \begin{cases} N_i(\alpha) \cap D, & \alpha \in E, \\ \varnothing, & \alpha \notin E \end{cases}$$

is internal and consists of null sets. Set $D' = D \setminus \nabla_{\alpha<\kappa} T_\alpha$. This is positive. If $\alpha \in E$, $\beta \in D'$ and $\alpha < \beta$ had color i, then $\beta \in T_\alpha$ would put β in the removed diagonal union. □

We use the following characterization of a Ramsey cardinal [5, Propositions 2.8(3) and 4.6]. If κ is Ramsey, every prescribed parameter of hereditary size at most κ belongs to a transitive weak κ-model $M \models \mathsf{ZFC}$ with a weakly amenable, externally countably complete normal M-ultrafilter U. External countable completeness means

$$\bigcap_{n<\omega} U_n \neq \varnothing \quad \text{whenever } \langle U_n : n < \omega\rangle \in V \text{ and each } U_n \in U.$$

The intersection need not belong to M or to U. Adding a cobounded member of U shows that every such intersection is unbounded in κ.

Fix a Ramsey κ, $P = \mathrm{Coll}(\omega_1, {<}\kappa)$, and a nice name $\dot{c}$ for a coloring. Code $\dot{c}$, P, V_κ, and enumerations witnessing their hereditary size bounds by one subset of κ. Choose M, U for this code. Decoding the well-founded membership relation in M recovers the parameters and their enumerations. In particular, $V_\kappa \subseteq M$, and P, including its order, has hereditary size κ as computed in M. External countable completeness of U makes $\mathrm{Ult}(M, U)$ well founded. Identifying it with its transitive collapse N, weak amenability makes the ultrapower embedding $j : M \to N$ κ-powerset preserving [5, Proposition 2.6(1)]. Both models compute P correctly. The coordinate split gives the isomorphism ι of Lemma 3.4, with tail Q. For $G \subseteq P$ generic over V, let $\mathcal{A}, \mu, I$ be given by that lemma. The symbol Q always denotes the fixed tail poset computed in N. Neither M, N, nor Q is assumed externally countably closed. No elementary preimage in an ambient $H(\theta)$ is assumed here, so we do not invoke the ambient-stationarity claim.

Lemma 4.3 (Two-sequence intersection)**.** *In this setup, suppose that in $V[G]$ we have $A^e_n \in \mathcal{A}$ and $q^e_n \in Q$ for $n < \omega$ and $e < 2$, with*

$$q^e_{n+1} \leq q^e_n, \qquad q^e_n \leq \mu(A^e_n).$$

There are $\alpha < \beta < \kappa$ such that

$$\alpha \in \bigcap_{n<\omega} A^0_n, \qquad \beta \in \bigcap_{n<\omega} A^1_n.$$

Proof. Representatives and a common initial condition. Choose M-names $\dot{A}^e_n$ for A^e_n and P-valued functions $f^e_n \in M$ such that $j(f^e_n)(\kappa) = \iota^{-1}(1_P, q^e_n)$. We may arrange

$$\mathrm{supp}(f^e_n(\xi)) \subseteq [\xi, \kappa) \qquad (\xi < \kappa). \tag{4.2}$$

The represented condition has support in $[\kappa, j(\kappa))$. By Łoś's theorem, (4.2) holds on a set in U. Replacing all other values by the empty condition normalizes f^e_n without changing its ultrapower value.

Let $j(\dot{A}^e_n)_G$ denote the Q-name obtained by transporting $j(\dot{A}^e_n)$ along ι and evaluating the initial P-coordinate. From $q^e_n \leq \mu(A^e_n)$ and the atomic forcing absoluteness lemma, applied to $N[G]$ and $V[G]$ with their common Q, we obtain

$$N[G] \models q^e_n \Vdash_Q \check{\kappa} \in j(\dot{A}^e_n)_G.$$

Apply the truth lemma for P over N to this statement, with the ground-model parameters q^e_n and $j(\dot{A}^e_n)$. Some $p^e_n \in G$ forces it over N. The product forcing theorem, computed inside N, then gives

$$N \models \iota^{-1}(p^e_n, q^e_n) \Vdash_{j(P)} \check{\kappa} \in j(\dot{A}^e_n).$$

This use of the truth lemma takes place in N and does not require external absoluteness of the composite forcing formula. The selected names, representatives and conditions all belong to fixed ground-model sets. Since P adds no countable sequences of ground-model objects, their countable sequences belong to V, though not necessarily to M.

There is $p \in G$ below every p^e_n. Indeed, for any ground-model sequence $\langle r_n : n < \omega\rangle$ from G, the set

$$\{s \in P : \exists n\ (s \perp r_n)\}\ \cup\ \{s \in P : \forall n\ (s \leq r_n)\}$$

is dense: successively extend below the r_n when possible, and use countable closure if all steps succeed. Genericity supplies a member of this dense set in G, which must belong to its second part. Apply this to the sequence of all p^e_n. Since $P \subseteq V_\kappa$, we have $p \in M$ and $\iota(j(p)) = (p, 1_Q)$.

Pointwise fusion. By Łoś's theorem, for each n, e there is $U^e_n \in U$ such that, for $\xi \in U^e_n$, the conditions $p, f^e_n(\xi)$ are compatible and

$$p \cup f^e_n(\xi) \Vdash_P \check{\xi} \in \dot{A}^e_n.$$

These truth sets belong to M by Separation. The atomic forcing absoluteness lemma, applied to M and V with their common P, identifies this membership forcing with the ambient one. Likewise, Łoś's theorem gives

$$V^e_n = \{\xi < \kappa : f^e_{n+1}(\xi) \leq f^e_n(\xi)\} \in U.$$

The countable sequences of these truth sets belong to V. Choose $\delta < \kappa$ above $\operatorname{supp}(p)$ and put

$$S_e = (\delta, \kappa) \cap \bigcap_{n<\omega} U^e_n \cap \bigcap_{n<\omega} V^e_n.$$

For every $\eta < \kappa$, external countable completeness applies to this countable family together with the cobounded set (η, κ). It gives a member of S_e above η. Thus $S_e \in V$ is unbounded. For $\xi \in S_e$, form in V

$$h^e_\xi = \bigcup_{n<\omega} f^e_n(\xi).$$

This is a condition of P, with bounded support contained in $[\xi, \kappa)$. It is compatible with p, and $p \cup h^e_\xi \Vdash_P \check{\xi} \in \dot{A}^e_n$ for every n. Neither the family of representatives nor this fusion is required to belong to M.

Return to the given generic. The selected sequences belong to V, so the parameter package

$$\big(P, \kappa, p, \delta, \langle f_n^e, U_n^e, V_n^e : n < \omega,\ e < 2\rangle\big) \in V.$$

Consequently S_e and $\langle h_\xi^e : \xi \in S_e\rangle$ belong to V for $e < 2$. For any $r \leq p$, choose $\alpha \in S_0$ above $\sup \operatorname{supp}(r)$, then $\beta \in S_1$ above both α and $\sup \operatorname{supp}(h_\alpha^0)$. By (4.2), the three supports are pairwise disjoint. Their union $r \cup h_\alpha^0 \cup h_\beta^1$ is a condition extending r. Consequently the set

$$D = \{t \leq p : \exists \alpha \in S_0\ \exists \beta \in S_1\ (\alpha < \beta \text{ and } h_\alpha^0 \cup h_\beta^1 \subseteq t)\}$$

belongs to V and is dense below p. Since $p \in G$, choose $t \in G \cap D$ and its witnesses $\alpha < \beta$. For every n,

$$t \leq p \cup f_n^0(\alpha), \qquad t \leq p \cup f_n^1(\beta).$$

Thus t forces each membership $\alpha \in \dot{A}_n^0$ and $\beta \in \dot{A}_n^1$. Evaluating these individual assertions in the given G proves the conclusion. No lower bound for the sequences $\langle q_n^e : n < \omega\rangle$ in Q is used. □

Theorem 4.4 (Ramsey collapse)**.** *Suppose κ is Ramsey and $G \subseteq \operatorname{Coll}(\omega_1, {<}\kappa)$ is generic over V. Then*

$$V[G] \models \mathrm{PH}_1(\kappa), \qquad \kappa = \omega_2^{V[G]}.$$

Proof. Fix $c : [\kappa]^2 \to \omega$ in $V[G]$. Choose a nice name $\dot{c}$ with $\dot{c}_G = c$ which is forced to be a total coloring. This can be arranged by mixing the given name with a fixed coloring outside a condition in G. Choose M, U as above for this name. The ideal from Lemma 3.4 satisfies the hypotheses of Lemma 4.2. Suppose there is no frequent positive pair in any color. Starting with (κ, κ), recursively apply Lemma 4.2(2) to omit color n below the previously chosen positive pair. This gives positive E^0, E^1 whose ordered rectangle has no color-n edge. For each $e < 2$, choose $q_n^e \in Q$ below $\mu(E^e)$ and use Lemma 3.4(3) to choose $A_n^e \subseteq E^e$ with $\mu(A_n^e) = q_n^e$. At every stage after the first,

$$A_n^e \subseteq A_{n-1}^e, \qquad q_n^e \leq \mu(E^e) \leq \mu(A_{n-1}^e) = q_{n-1}^e.$$

The rectangle $A_n^0 \times A_n^1$, restricted to increasing pairs, still omits color n.

By Lemma 4.3, there are $\alpha < \beta$ with $\alpha \in \bigcap_n A_n^0$ and $\beta \in \bigcap_n A_n^1$. Taking $n = c(\{\alpha, \beta\})$ contradicts color omission. Thus there are $i < \omega$ and $A, B \in \mathcal{A} \setminus I$ with $\mathrm{Freq}_i(A, B)$. By Lemma 4.2(1), some positive $X \subseteq A$ satisfies

$$\Gamma_i^B(\{\alpha, \beta\}) \notin I \qquad (\alpha < \beta \text{ in } X).$$

The set X and these common-neighbor sets are unbounded, since they belong to $\mathcal{A}$ and bounded members of $\mathcal{A}$ are null. Removing $\beta + 1$ from each common-neighbor set gives (4.1). The coloring was arbitrary, so Proposition 4.1 gives $\mathrm{PH}_1(\kappa)$. The collapse makes $\kappa = \omega_2$. □

Proposition 4.1 also applies to the Cohen model of Corollary 3.6. For comparison, Todorcevic–Zhang [11, Corollary 4.12] obtain $\mathrm{PH}_n(\kappa)$ for all $n < \omega$ from a uniform normal ideal on κ whose quotient is isomorphic to a nontrivial measure algebra. They obtain the same simultaneous conclusion after a length-κ finite-support iteration of σ-centered forcing over a ground model in which κ is measurable [11, Corollary 4.13]. That iteration is c.c.c., so κ remains a regular limit cardinal. In contrast, Theorem 4.4 gives PH_1 at ω_2 from a Ramsey cardinal.

Write $\square(\kappa)$ for the existence of a coherent sequence $\langle C_\alpha : \alpha < \kappa,\ \alpha \text{ limit}\rangle$ of clubs $C_\alpha \subseteq \alpha$ with no club thread. Coherence means $C_\beta \cap \alpha = C_\alpha$ whenever α is a limit point of C_β. A thread would be a club $D \subseteq \kappa$ with $D \cap \alpha = C_\alpha$ at every limit point α of D. With this convention, Proposition 4.1 gives the first implication below, and [1, Proposition 8.19] gives the second:

$$\mathsf{SFCN}_\omega(\kappa) \implies \mathrm{PH}_1(\kappa) \implies \neg\square(\kappa).$$

4.3. **Tukey transfer.** Let $P = (P, \leq_P)$ and $Q = (Q, \leq_Q)$ be nonempty directed quasi-orders. A subset of P is *bounded* if it has a common upper bound in P, and *cofinal* if it contains an element above every point of P. We use the convention $P \leq_T Q$ if there is a map $g : P \to Q$ such that

$$A_q = \{p \in P : g(p) \leq_Q q\}$$

is bounded in P for every $q \in Q$. Equivalently, g sends unbounded subsets to unbounded subsets. Indeed, an unbounded set whose image is bounded would be contained in some A_q, and an unbounded A_q itself violates the latter property. Such a g is a *Tukey map*. This is the convention of [1, Section 8].

We first handle finite colors.

Lemma 4.5 (Finite colors)**.** *For every directed quasi-order P, every $n < \omega$, and every positive integer k, $\mathrm{PH}_n(P, k)$ holds in ZFC.*

Proof. Put $m = n + 1$. For $p \in P$, let $\uparrow p = \{q \in P : p \leq_P q\}$. Any finitely many upper cones have a point in common, by directedness. They therefore generate a proper filter, which extends to an ultrafilter U on P.

Let $c : P^m \to k$ be given. Starting with $c_m = c$, define the functions $c_r : P^r \to k$ for $r = m - 1, \ldots, 0$ by

$$c_r(a) = i \iff \{x \in P : c_{r+1}(a^\frown(x)) = i\} \in U.$$

For each a, these k fibers partition P. An ultrafilter contains exactly one cell of a finite partition, so c_r is defined uniquely. As $P^0 = \{\varnothing\}$, there is one value $i = c_0(\varnothing)$. We define F by tuple length, maintaining

$$c_r(F(s_1), \ldots, F(s_r)) = i \qquad \text{for every full } r\text{-flag.} \tag{4.3}$$

For each $p \in P$, choose

$$F(p) \in \uparrow p \cap \{x : c_1(x) = i\}.$$

The first set belongs to U by its choice, and the second belongs to U by the definition of c_0. This gives singleton domination and the invariant for $r = 1$.

Suppose $2 \leq r \leq m$ and F has been defined on all shorter tuples. Fix $t \in P^r$. For every full $(r-1)$-flag $(s_1, \ldots, s_{r-1})$ whose last tuple is a subtuple of t, put

$$D_{s_1,\ldots,s_{r-1}} = \{x \in P : c_r(F(s_1), \ldots, F(s_{r-1}), x) = i\}.$$

The invariant gives $c_{r-1}(F(s_1), \ldots, F(s_{r-1})) = i$, so the backward definition of c_{r-1} puts this set in U. Also impose $F(t) \in \uparrow F(s)$ for each nonempty proper subtuple $s \trianglelefteq t$. There are finitely many subtuples of the finite tuple t and finitely many flags among them. Hence all these requirements form a finite family of members of U, whose intersection is nonempty. Choose $F(t)$ in that intersection.

The upper-cone requirements make F monotone under subtuples. The sets $D_{s_1,\ldots,s_{r-1}}$ give (4.3) for every flag ending at t. Each choice uses only shorter tuples, so the choices on P^r are independent. At $r = m$, the invariant gives homogeneity of $c \circ F^*$. Singleton domination completes the proof. When $m = 1$, the singleton step suffices. $\square$

As in [1, Lemma 8.7], the next proof uses a two-valued refinement, here to control the chosen Tukey upper bounds.

Theorem 4.6. *Let $P \leq_T Q$ be directed quasi-orders. For every cardinal λ and every $n < \omega$,*

$$\mathrm{PH}_n(Q, \lambda) \implies \mathrm{PH}_n(P, \lambda).$$

Proof. For $\lambda = 0$ the assertions are vacuous. For positive finite λ, Lemma 4.5 proves the conclusion without any assumption on Q. Suppose that λ is infinite.

Fix a Tukey map $g : P \to Q$. For each $q \in Q$, choose an upper bound $r(q) \in P$ of A_q. If A_q is empty, choose any element of P. Given $c : P^{n+1} \to \lambda$, define

$$d(q_0, \dots, q_n) = \big(c(r(q_0), \dots, r(q_n)),\ b(q_0, \dots, q_n)\big),$$

where

$$b(q_0, \dots, q_n) = 1 \quad \iff \quad \forall j < n \ \ g(r(q_j)) \leq_Q q_{j+1}.$$

The empty conjunction is true when $n = 0$. Identify $\lambda \times 2$ with λ and apply $\mathrm{PH}_n(Q, \lambda)$. There is an $(n + 1)$-cofinal $F : Q^{\leq n+1} \to Q$ such that $d \circ F^*$ has a constant value (i, ε).

We have $\varepsilon = 1$. To see this, start with any singleton t_1. Given t_j, append the coordinate $q = g(r(F(t_j)))$ to obtain t_{j+1}. Since $(q) \trianglelefteq t_{j+1}$,

$$g(r(F(t_j))) = q \leq_Q F(q) \leq_Q F(t_{j+1}).$$

Thus one full flag has second coordinate 1, and so every full flag does.

Suppose $n \geq 1$. Every immediate subtuple inclusion $s \trianglelefteq t$, with $|t| = |s|+1 \leq n+1$, occurs consecutively in a full flag. To construct one, delete one coordinate at a time from s down to a singleton, reverse this list, follow s by t, and append arbitrary coordinates to t until the length is $n + 1$. The common second coordinate of this flag is 1, so its clause at the positions of s, t gives

$$g(r(F(s))) \leq_Q F(t).$$

By the definition of $A_{F(t)}$ and its chosen upper bound, we obtain

$$r(F(s)) \in A_{F(t)}, \qquad r(F(s)) \leq_P r(F(t)).$$

Any proper inclusion of nonempty subtuples factors into immediate inclusions by inserting the deleted coordinates one at a time. Transitivity of $\leq_P$ therefore makes $r \circ F$ monotone under subtuples. For equal tuples this follows from reflexivity. When $n = 0$, only singleton tuples occur, so this requirement is automatic.

For a nonempty tuple s from P, apply g coordinatewise and put

$$H(s) = r(F(g \circ s)).$$

For a singleton, $g(p) \leq_Q F(g(p))$ implies $p \in A_{F(g(p))}$, hence $p \leq_P H(p)$. Coordinatewise application of g preserves tuple lengths and subtuple relations, whether or not g is injective or monotone. Thus H is $(n+1)$-cofinal. For a full flag $(s_1, \dots, s_{n+1})$ from P, the tuples $(g \circ s_1, \dots, g \circ s_{n+1})$ form a full flag from Q. The first coordinate of its $d \circ F^*$-value gives

$$c(H(s_1), \dots, H(s_{n+1})) = i.$$

For $n = 0$, these singleton and homogeneity checks complete the proof. $\square$

Corollary 4.7. *For fixed n and λ, $\mathrm{PH}_n(-,\lambda)$ is invariant under Tukey equivalence and under passage to a cofinal suborder.*

Proof. Tukey equivalence means reducibility in both directions. Applying Theorem 4.6 to both reductions gives the first assertion.

For the second assertion, let $P \subseteq Q$ be cofinal with the inherited order. Every finite subset of P has an upper bound in Q and hence one in P, so P is directed. Choose $p_q \in P$ with $q \leq_Q p_q$ for each $q \in Q$. For the inclusion map $\iota : P \to Q$,

$$\{p \in P : \iota(p) \leq_Q q\} \subseteq \{p \in P : p \leq_P p_q\}.$$

Thus ι is Tukey, with an upper bound belonging to P, as required. Conversely, the map $q \mapsto p_q$ is Tukey because for each $p \in P$,

$$\{q \in Q : p_q \leq_P p\} \subseteq \{q \in Q : q \leq_Q p\}.$$

Thus P and Q are Tukey equivalent, and the first assertion applies. $\square$

Theorems 4.4 and 4.6 prove Theorem 1.2.

5. Further Consequences

In the first two subsections, fix a coloring $c : [\kappa]^2 \to \omega$ and use G_i, N_i and Γ_i^X for this coloring.

5.1. **Ideal-sensitive connectivity.** Let $\mathcal{J}$ be a proper ideal on κ containing $[\kappa]^{<\kappa}$. For $X \in [\kappa]^\kappa$, let $e_X : \kappa \to X$ be its increasing enumeration and set

$$\mathcal{J}_X = \{e_X[A] : A \in \mathcal{J}\}.$$

In the diagonal case of [7, Problem 8.6], the relation $\kappa \to_{\mathcal{J}\text{-hc}} (\kappa)^2_\omega$ asserts that every $c : [\kappa]^2 \to \omega$ has $i < \omega$ and $X \in [\kappa]^\kappa$ such that $G_i[X \setminus Z]$ is connected for every $Z \in \mathcal{J}_X$. We add $< n$ when all required paths have fewer than n edges. Here $(NS_\kappa)_X$ is transported through the increasing enumeration, not obtained by restricting the ambient nonstationary ideal to X. The witness X need not be stationary.

Proposition 5.1. *Let I be an ideal on κ containing NS_κ. Suppose $X \in I^+$ and $\Gamma_i^X(F) \in I^+$ for every $F \in [X]^{<\omega}$. For every $Z \in I$,*

$$X \setminus Z \in I^+, \qquad \forall F \in [X \setminus Z]^{<\omega} \quad \Gamma_i^{X\setminus Z}(F) \in I^+. \tag{5.1}$$

Hence $G_i[X \setminus Z]$ has diameter at most two. The same X is a $\mathcal{J}$-hc, < 3 witness for every ideal $\mathcal{J}$ with $[\kappa]^{<\kappa} \subseteq \mathcal{J} \subseteq I$.

Proof. For finite $F \subseteq X \setminus Z$,

$$\Gamma_i^{X\setminus Z}(F) = \Gamma_i^X(F) \setminus Z \in I^+.$$

The empty-set case gives $X \setminus Z \in I^+$. For two distinct vertices $x, y \in X \setminus Z$, a point of $\Gamma_i^{X\setminus Z}(\{x, y\})$ is different from both and gives a two-edge color-i path inside $X \setminus Z$. This proves the diameter bound.

Since $NS_\kappa \subseteq I$, positivity gives $|X| = \kappa$. Let $e = e_X$ and $C_e = \{\delta < \kappa : e``\delta \subseteq \delta\}$, where $e``\delta = \{e(\alpha) : \alpha < \delta\}$. The set C_e is closed. It is unbounded because, above any prescribed ordinal, one can iterate $\delta \mapsto \max\{\delta, \sup(e``\delta)\} + 1$ countably many times and take the supremum. Regularity keeps this supremum below κ and makes it a closure point of e. If $e(\alpha) = \delta \in C_e$, then $\alpha \leq \delta$ because e is increasing. If

$\alpha < \delta$, the definition of C_e would give $e(\alpha) < \delta$, a contradiction. Thus $\alpha = \delta$, and for every $A \subseteq \kappa$,

$$e[A] \cap C_e \subseteq A, \qquad e[A] \subseteq A \cup (\kappa \setminus C_e).$$

If $A \in \mathcal{J} \subseteq I$, it follows that $e_X[A] \in I$. Apply the ambient conclusion with $Z = e_X[A]$. $\square$

Consequently, the normal case of Theorem 2.1 gives

$$\kappa \to_{I\text{-hc},<3} (\kappa)^2_\omega, \qquad \kappa \to_{NS_\kappa\text{-hc},<3} (\kappa)^2_\omega,$$

with the same witnesses for every ideal between $[\kappa]^{<\kappa}$ and I. Taking $I = NS_\kappa$ in the proposition also gives

$$\mathsf{SFCN}_\omega(\kappa) \implies \kappa \to_{NS_\kappa\text{-hc},<3} (\kappa)^2_\omega.$$

5.2. **Stationary subdivisions.** A < 2-*subdivision* of K_κ has κ branch vertices and replaces each edge by a path with zero or one internal vertex. The internal vertices are distinct and lie outside the branch set. Thus < 2 counts internal vertices, unlike the path-length bound in the partition relation. If the branch set is enumerated increasingly as $\langle b_\alpha : \alpha < \kappa \rangle$, the subdivision is *order-rigid* when the internal vertex for $b_\alpha < b_\beta$ lies strictly between b_β and $b_{\beta+1}$.

Common-neighbor constructions of topological complete graphs occur in Erdős–Hajnal [3, Theorem 3] and Komjáth–Shelah [9, Lemma 4].

Theorem 5.2 (Stationary < 2-subdivisions)**.** *Suppose $X \subseteq \kappa$ and $i < \omega$ satisfy* (1.2)*. There is a club $C \subseteq \kappa$ such that $B = X \cap C$ still satisfies* (1.2) *and is the branch set of an order-rigid color-i < 2-subdivision of K_κ with the following property: every color-i edge on B is left unsubdivided, and every other pair is represented by a two-edge path whose internal vertex belongs to $X \setminus C$.*

In particular, every finite $F \subseteq B$ has stationarily many common unsubdivided neighbors in B. For every $b \in B$, the later branches joined to b by unsubdivided edges form a stationary subset of κ.

Proof. Construct a strictly increasing continuous sequence $\langle \gamma_\xi : \xi < \kappa \rangle$. Set $\gamma_0 = 0$. At stage ξ, if $\gamma_\xi \in X$, choose distinct vertices

$$w_{\xi,x} \in \Gamma^X_i(\{x, \gamma_\xi\}) \setminus (\gamma_\xi + 1) \qquad (x \in X \cap \gamma_\xi).$$

Enumerate $X \cap \gamma_\xi$ in order type less than κ. At each step, the required common-neighbor set has size κ, whereas $\gamma_\xi + 1$ and the previously chosen vertices together have size less than κ. Thus the choices can be made distinctly. The resulting family has size less than κ. Choose $\gamma_{\xi+1} < \kappa$ above γ_ξ and all these vertices. If $\gamma_\xi \notin X$, choose any $\gamma_{\xi+1} > \gamma_\xi$. At a nonzero limit $\xi < \kappa$, let $\gamma_\xi = \sup_{\eta<\xi} \gamma_\eta$. Regularity keeps every stage below κ.

The range $C = \{\gamma_\xi : \xi < \kappa\}$ is club. Put $B = X \cap C$, and enumerate it increasingly as $\langle b_\alpha : \alpha < \kappa \rangle$. For every finite $F \subseteq B$,

$$\Gamma^B_i(F) = \Gamma^X_i(F) \cap C$$

is stationary. For $b_\alpha < b_\beta$, keep the direct edge if it has color i. Otherwise, write $b_\beta = \gamma_\xi$ and use the path

$$b_\alpha - w_{\xi,b_\alpha} - b_\beta.$$

Its edges have color i, and

$$b_\beta < w_{\xi,b_\alpha} < \gamma_{\xi+1} \le b_{\beta+1}.$$

The internal vertices are distinct within each stage, lie in disjoint intervals at different stages, and avoid C, hence B. Since every color-i edge on B was kept, the stationary sets $\Gamma_i^B(F)$ consist of common unsubdivided neighbors. Removing the bounded initial segment $b+1$ gives the final assertion. $\square$

Under Theorem 2.4, the same construction keeps $B, \Gamma_i^B(F) \in I^+$, since intersection with a club preserves positivity when $NS_\kappa \subseteq I$.

5.3. **Global ideal applications.** For the following countably closed formulation of the collapse-ideal theorem and its history, see [8, Definition 3.1 and Lemma 3.2].

Fact 5.3 (The collapse ideal)**.** *Suppose κ is measurable and $G \subseteq \mathrm{Coll}(\omega_1, {<}\kappa)$ is generic over V. In $V[G]$, κ carries a proper uniform normal κ-complete ideal I such that $(I^+, \subseteq)$ has a countably closed dense subset.*

In this model one global ideal works for every coloring. Apply Theorem 2.1 for positive common-neighbor witnesses, and Proposition 5.1 and theorem 5.2 for the deletion and subdivision conclusions.

This conclusion concerns the collapse extension before the additional forcing used in the separation model of [7, Theorem 7.2]. We do not assert that this additional forcing preserves the full closed-dense ideal hypothesis of Theorem 2.1.

Eskew–Hayut [4, Theorem 56] prove that, from a huge cardinal θ, there is a forcing extension in which θ remains inaccessible and V_θ satisfies the following: for every regular μ, there is a proper uniform normal μ^+-complete ideal I_μ on μ^+ with

$$\mathcal{P}(\mu^+)/I_\mu \cong \mathrm{RO}(\mathrm{Coll}(\mu, \mu^+)).$$

Here $\mathrm{Coll}(\mu, \mu^+)$ is the standard $< \mu$-closed collapse.

Corollary 5.4. *Relative to a huge cardinal, there is a model of* ZFC *such that, for every regular $\mu \geq \omega_1$,*

$$\mathsf{SFCN}_\omega(\mu^+), \qquad \mu^+ \to_{NS_{\mu^+}\text{-}hc, <3} (\mu^+)^2_\omega.$$

At each target, every countable coloring also has the stationary < 2-subdivision of Theorem 5.2.

Proof. Work inside the model V_θ just described. All ideals, completions and closure properties in this paragraph are computed in that model. Fix a regular $\mu \geq \omega_1$ there. The standard partial-function order $\mathrm{Coll}(\mu, \mu^+)$ is countably closed and dense in its Boolean completion. The displayed isomorphism has the whole completion as its range, so Lemma 3.1 gives a countably closed dense family in $(I_\mu^+, \subseteq)$.

Theorem 2.1 now gives an I_μ-positive common-neighbor witness for each countable coloring. Normality makes its common-neighbor sets stationary. Proposition 5.1 gives the NS_{μ^+}-sensitive arrow, and Theorem 5.2 gives the asserted subdivision. The same model supplies I_μ for every regular μ, so these conclusions hold simultaneously. $\square$

The simultaneous consistency of $\mathrm{PH}_n(\omega_m)$ for $n < m < \omega$ is due to Eskew–Hayut [4, Theorem 75].

6. Open Problems

Question 6.1. *What is the exact consistency strength of* $\mathsf{SFCN}_\omega(\omega_2)$*? Does a Ramsey cardinal suffice for the consistency of* $\omega_2 \to_{\mathrm{hc},<3} (\omega_2)^2_\omega$*? Can the Ramsey upper bound for* $\mathrm{PH}_1(\omega_2)$ *be lowered to a weakly compact cardinal?*

The last question is the remaining upper-bound direction of [1, Question 9.4].

Question 6.2. *Is it consistent that a regular uncountable* κ *satisfies*

$$\kappa \to_{\mathrm{hc},<3} (\kappa)^2_\omega \qquad \textit{and} \qquad \neg\mathsf{FCN}_\omega(\kappa)?$$

The global ideal application covers successors of uncountable regular cardinals, while the Cohen forcing preserves smaller weakly inaccessible cardinals. Neither construction settles the following targets.

Question 6.3. *Is* $\aleph_{\omega+1} \to_{\mathrm{hc},<3} (\aleph_{\omega+1})^2_\omega$ *consistent, and can it be strengthened to* $\mathsf{SFCN}_\omega(\aleph_{\omega+1})$*? Can* $\kappa \to_{\mathrm{hc},<3} (\kappa)^2_\omega$ *hold when* κ *is the least weakly inaccessible cardinal, or the least strongly inaccessible cardinal?*

With < 4 in place of < 3, these target instances have positive answers by [7, Theorems 5.1 and 6.4]. For the least-inaccessible instances, start with a strongly inaccessible cardinal above the supercompact so that the target cardinals exist after the collapse.

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